Chapter 4

Optimal Trajectory Tracking Using Newton and Levenberg-Marquardt-Like Algorithms for Constrained Time-Varying Optimization

4.1 Introduction

This chapter introduces a continuous-time dynamics for tracking the optimal trajectory of time-varying optimization problems with time-varying constraints in a predefined time, where the time of convergence is selected a priori. A robust Newton-like approach is developed for the cases, where the exact knowledge of the rate of change of the gradient of objective function is unknown. Levenberg- Marquardt-like algorithm is proposed for the cases when the Hessian of the objective function is singular or near singular. Lyapunov-based convergence analysis is discussed for the proposed algorithms. Simulation results for TVO problems show the efficacy of the proposed approach.

The rest of the chapter is structured as follows: In section 4.2, problem statement is discussed. Section 4.3 presents the main result for unconstrained time-varying optimization problems. In section 4.4, the results discussed in section 4.3 are extended for constrained TVO. Numerical examples are illustrated in section 4.5 based on the results discussed in section 4.3 and 4.4.

4.2 Problem Statement

Consider the following TVO problem

$$x^*(t) := \underset{x\in\mathbb{R}^n}{\text{argmin}}\ \mathcal{F}(t,x)$$
$$\text{subject to} \quad g_i(t,x) \le 0, \quad i \in \{1,2,...,r\} \tag{4.1}$$

where $\mathcal{F} : \mathbb{R}_{\ge 0} \times \mathbb{R}^n \to \mathbb{R}$, $g = [g_1, g_2, ..., g_r]^\top$ with all g_i being convex functions in $x \in \mathbb{R}^n$, $\forall t \ge 0$ and fulfills following Assumptions.

Assumption 4.1 *1. The objective function $\mathcal{F} : \mathbb{R}_{\ge 0} \times \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable with respect to x, continuously differentiable with respect to t, and convex with respect to x.*

2. A unique optimal solution $x^*(t)$ exists.

Assumption 4.2 *(Polyak-Łojasiewicz inequality) (see [9])*

$$w(\mathcal{F}(t,x) - \mathcal{F}^*(t))^p \le \|\nabla\mathcal{F}(t,x)\| \tag{4.2}$$

where $\mathcal{F}^* := \min_{z\in\mathbb{R}^n} \mathcal{F}(t,x), 0 < p < 1$ and $w > 0$.

Assumption 4.3 *The inequality constraint functions $g_i(t,x)$ are twice continuously differentiable with respect to x and continuously differentiable with respect to t.*

Assumption 4.4 *Constraint Qualification: There exists atleast one $x(t) \in \mathbb{R}^n$ such that $g_i(t,x) < 0$, $\forall i \in \{1,2,...r\}$ and $\forall t \ge 0$.*

4.3 Time-Varying Unconstrained Convex Optimization

Consider the following unconstrained TVO problem

$$x^*(t) := \underset{x\in\mathbb{R}^n}{\text{argmin}}\ \mathcal{G}(t,x), \tag{4.3}$$

Remark 4.1 *For TVO problem* (4.3)*, Assumption 4.1 is an important condition for the convergence of gradient-based methods to an optimal solution [13]. In Karimi et al. [9], it is noted that Assumption 4.2 is the weakest condition among similar conditions commonly used in the literature to demonstrate convergence in gradient-based algorithms.*

Lemma 4.2 *(see [13]) The continuously differentiable convex function* $\mathcal{G} : \mathbb{R}_{\geq 0} \times \mathbb{R}^p \to \mathbb{R}$ *is minimized at solution* $x^*(t)$ *if and only if* $\nabla \mathcal{G}(t, x^*) = 0, \ \forall t$.

The optimal trajectory $x^*(t)$ satisfies the first order optimality condition $\nabla \mathcal{G}(t, x^*) = 0, \quad \forall t \geq 0$, then time derivative of this condition will also be zero which results in $\nabla^2 \mathcal{G}(t, x)\dot{x}^*(t) + \frac{\partial \nabla \mathcal{G}(t,x)}{\partial t} = 0$. Solving for $\dot{x}^*(t)$ leads to following dynamical system:

$$\dot{x}^*(t) = -\nabla^2 \mathcal{G}(t, x^*))^{-1} \frac{\partial \nabla \mathcal{G}(t, x^*)}{\partial t}, \tag{4.4}$$

If the optimal solution $x^*(t)$ was known for some $t_0 > 0$, the system in (4.4) could be used to track the evolution of $x^*(t)$. As we do not have access to $x^*(t)$ at any point in time, we propose a predefined-time convergent dynamics to solve the TVO problem in a priori chosen time, utilizing the idea of prediction and correction.

In the next section, we discuss predefined-time convergent dynamics to solve TVO problems. The main result of this chapter discusses TVO problems for which $\lim\limits_{t \to t_f} \|x(t) - x^*(t)\| = 0$, where $t_f < \infty$, that is chosen a priori. The convergence of the trajectories based on the predefined-time dynamics to the optimal solution $x^*(t)$ also signifies $\lim\limits_{t \to t_f} \|\nabla \mathcal{G}(t, x)\| = 0$. First, we consider TVO problems (4.3), where $\mathcal{G}(t, x)$ is given so $\nabla \mathcal{G}(t, x)$ is known $\forall t$, and aim is to converge to optimal solution $x^*(t)$ within priori chosen time t_f.

4.3.1 Predefined-Time Convergent Dynamics for Time-Varying Optimization (PTC-TVO)

In this subsection, we present two predefined-time convergent dynamical systems to solve unconstrained TVO problems as (4.3). One is based on the assumption that $\nabla^2 \mathcal{G}$ is invertible and second if we do not consider $\nabla^2 \mathcal{G}$ to be invertible. We revise equation (4.4) by adding a correction term to propose dynamics that track the optimal trajectory in a predefined time.

Newton-like PTC-TVO

Let $\nabla^2\mathcal{G}$ is invertible then in order to track $x^*(t)$ of (4.3) within predefined-time, we propose the following dynamics

$$\dot{x}(t) = \begin{cases} -(\nabla^2\mathcal{G}(t,x))^{-1}\left(\vartheta\psi(t,x) + \frac{\partial\nabla\mathcal{G}(t,x)}{\partial t}\right), \text{for } t_0 \le t < t_f \\ -(\nabla^2\mathcal{G}(t,x))^{-1}\left(\vartheta\nabla\mathcal{G}(t,x) + \frac{\partial\nabla\mathcal{G}(t,x)}{\partial t}\right), \text{ for } \ t \ge t_f, \end{cases} \tag{4.5}$$

where $x(t) \in \mathbb{R}^n$, $\vartheta \in \mathbb{R}$ and satisfies $\vartheta > 1$, $\psi(t,x) = (\psi_1, \psi_2, \ldots, \psi_n) \in \mathbb{R}^{n\times 1}$, $\psi_i := \frac{(e^{\nabla\mathcal{G}_{x_i}(t,x)}-1)}{e^{\nabla\mathcal{G}_{x_i}(t,x)}(t_f-t)}$, for $1 \le i \le n$.

Theorem 4.3 *Consider the dynamics* (4.5)*. Suppose that* $\mathcal{G} : \mathbb{R}_{\ge 0} \times \mathbb{R}^n \to \mathbb{R}$ *in* (4.3) *has invertible Hessian, satisfies Assumption 4.1. There exist a continuously differentiable function* $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$*, satisfying* $\gamma_1(x) \le V(t,x) \le \gamma_2(x)$, $\forall x \in \mathbb{R}^n \setminus \{0\}, \forall t \in [t_0, \infty)$*,* $\gamma_1(x)$ *and* $\gamma_2(x)$ *be continuous positive definite functions,* $V(t,x^*) = 0$ *and for* $V \neq 0$*:* $\dot{V} \le 0, \forall t \ge t_f$ *and* $\dot{V} \le \frac{-\vartheta(e^V-1)}{e^V(t_f-t)}, \forall t \in [t_0, t_f)$*. Then* $x(t) \to x^*(t)$ *as* $t \to t_f$*. This signifies that* $x(t)$ *converges to the optimal solution* $x^*(t)$ *in a priori chosen time.*

Proof 1 *Let us select the Lyapunov function candidate*

$$V = \|\nabla\mathcal{G}(t,x)\|^2, \tag{4.6}$$

The derivative of the V *along the dynamics* (4.5) *for* $t \in [t_0, t_f)$

$$\begin{aligned} \dot{V}(x) =& 2(\nabla\mathcal{G}(t,x))^\top\left(\nabla^2\mathcal{G}(t,x)\dot{x} + \frac{\partial\nabla\mathcal{G}(t,x)}{\partial t}\right) \\ =& 2\nabla\mathcal{G}^\top\left(\nabla^2\mathcal{G}\left(-(\nabla^2\mathcal{G})^{-1}\left(\vartheta\psi + \frac{\partial\nabla\mathcal{G}}{\partial t}\right)\right) + \frac{\partial\nabla\mathcal{G}}{\partial t}\right) \\ =& -2\vartheta(\nabla\mathcal{G}(t,x)^\top\psi(t,x) & (4.7) \\ \le& -2\vartheta(\nabla\mathcal{G}_{x_i}(t,x))\psi_i(t,x) \qquad \textit{for any } i \\ \le& -2\vartheta\frac{|\nabla\mathcal{G}_{x_i}(t,x)|(e^{|\nabla\mathcal{G}_{x_i}(t,x)|}-1)}{e^{|\nabla\mathcal{G}_{x_i}(t,x)|}(t_f-t)} & (4.8) \end{aligned}$$

Since, $V \le n(max\{|\nabla\mathcal{G}_1|, |\nabla\mathcal{G}_2|, ..., |\nabla\mathcal{G}_n|\}^2)$ *and thus,* $\sqrt{V/n} \le max\{|\nabla\mathcal{G}_1|, |\nabla\mathcal{G}_2|, ..., |\nabla\mathcal{G}_n|\}$*. We assume that at a particular time,* $max\{|\nabla\mathcal{G}_1|, |\nabla\mathcal{G}_2|, ..., |\nabla\mathcal{G}_n|\}$ *returns* $|\nabla\mathcal{G}_i|$*, then,* $\sqrt{V/n} \le |\nabla\mathcal{G}_i|$*. Using this fact in* (4.8)

$$\dot{V}(x) \le -2\vartheta\frac{\sqrt{V/n}(e^{\sqrt{V/n}}-1)}{e^{\sqrt{V/n}}(t_f-t)}. \tag{4.9}$$

Let $\hat{V} = \sqrt{V/n}$, *hence,* $\dot{\hat{V}} = \frac{\dot{V}}{2n\hat{V}}$, *so* (4.9) *is simplified and yields,*

$$\dot{\hat{V}} \leq -\frac{\hat{\vartheta}(e^{\hat{V}} - 1)}{e^{\hat{V}}(t_f - t)} \tag{4.10}$$

where $\hat{\vartheta} = \frac{\vartheta}{n} > 1$. *Thus, inequality* (4.10) *represents PUBST based stability according to Lemma 2.9, which means that* $\|\nabla\mathcal{G}(t,x)\| \to 0$ *as* $t \to t_f$, *leading to the convergence of trajectory of* (4.5) *to the optimal solution* $x^*(t)$ *of* $\mathcal{G}(t,x)$ *in a priori chosen time. Next, the time derivative of the* V *along* (4.5) *for* $t \geq t_f$ *gives*

$$\begin{aligned}
\dot{V}(x) =& 2(\nabla\mathcal{G}(t,x))^\top \left(\nabla^2\mathcal{G}(t,x)\dot{x} + \frac{\partial\nabla\mathcal{G}(t,x)}{\partial t}\right) \\
=& 2\nabla\mathcal{G}^\top \left(\nabla^2\mathcal{G}\left(-(\nabla^2\mathcal{G})^{-1}\left(\vartheta\nabla\mathcal{G} + \frac{\partial\nabla\mathcal{G}}{\partial t}\right)\right) + \frac{\partial\nabla\mathcal{G}}{\partial t}\right) \\
=& -2\vartheta(\nabla\mathcal{G}(t,x))^\top\nabla\mathcal{G}(t,x) \\
=& -2\vartheta||\nabla\mathcal{G}(t,x)||^2 \\
=& -2\vartheta V
\end{aligned}$$

So, for $t \geq t_f$, *the equilibrium point of system* (4.5) *is asymptotically stable, therefore, trajectories of* (4.5) *will maintain* $x^*(t)$. *Hence,* $x(t) \to x^*(t)$ *as* $t \to t_f$.

Remark 4.4 *In* (4.5), *it is required to know the true information of* $\frac{\partial\nabla\mathcal{G}(t,x)}{\partial t}$.

Robust Newton-like PTC-TVO

In the previous subsection, we require the exact knowledge of $\frac{\partial\nabla\mathcal{G}(t,x)}{\partial t}$. Here, we assume that $\frac{\partial\nabla\mathcal{G}(t,x)}{\partial t}$ is unknown, but $\exists \ell \in \mathbb{R}_{>0}$ such that $\left\|\frac{\partial\nabla\mathcal{G}(t,x)}{\partial t}\right\| \leq \ell$, then in order to track $x^*(t)$ of (4.3) within predefined-time, we propose the following dynamics

$$\dot{x}(t) = \begin{cases} -(\nabla^2\mathcal{G}(t,x))^{-1}\left(\vartheta\psi(t,x) + \ell\frac{\nabla\mathcal{G}(t,x)}{\|\nabla\mathcal{F}(t,x)\|}\right), & t_0 \leq t < t_f \\ -(\nabla^2\mathcal{G}(t,x))^{-1}\left(\vartheta\nabla\mathcal{F}(t,x) + \ell\frac{\nabla\mathcal{G}(t,x)}{\|\nabla\mathcal{F}(t,x)\|}\right), & t \geq t_f, \end{cases} \tag{4.11}$$

where $x(t) \in \mathbb{R}^n$, $\vartheta \in \mathbb{R}$ and satisfies $\vartheta > 1$, $\psi(t,x) = (\psi_1, \psi_2, \ldots, \psi_n) \in \mathbb{R}^{n\times 1}$, $\psi_i := \frac{(e^{\nabla\mathcal{G}_{x_i}(t,x)}-1)}{e^{\nabla\mathcal{G}_{x_i}(t,x)}(t_f-t)}$, for $1 \leq i \leq n$.

Theorem 4.5 *Consider the dynamics* (4.11). *Suppose that* $\mathcal{G} : \mathbb{R}_{\geq 0}\times\mathbb{R}^n \to \mathbb{R}$ *in* (4.3) *has invertible Hessian with* $\left\|\frac{\partial\nabla\mathcal{G}}{\partial t}\right\| \leq \ell$, $\ell \in \mathbb{R}_+$ *satisfies Assumption 4.1 and* $\exists$ *a continuously differentiable function* $V : [t_0, \infty)\times\mathbb{R}^n \to \mathbb{R}$, *satisfying* $\gamma_1(x) \leq V(t,x) \leq \gamma_2(x)$, $\forall x \in \mathbb{R}^n \setminus$

$\{0\}, \forall t \in [t_0, \infty)$, $\gamma_1(x)$ *and* $\gamma_2(x)$ *be continuous positive definite functions,* $V(t, x^*) = 0$ *and for* $V \neq 0$*:* $\dot{V} \leq 0, \forall t \geq t_f$ *and* $\dot{V} \leq \frac{-\vartheta(e^V - 1)}{e^V(t_f - t)}, \forall t \in [t_0, t_f)$ *for* (4.11)*. This signifies that* $x(t)$ *converges to the optimal solution* $x^*(t)$ *in a priori chosen time.*

Proof 2 *Let us select the Lyapunov function candidate*

$$V = \|\nabla \mathcal{G}(t, x)\|^2, \tag{4.12}$$

The derivative of the V *along the dynamics* (4.11) *for* $t \in [t_0, t_f)$

$$\begin{aligned}\dot{V}(x) =& 2(\nabla \mathcal{G}(t, x))^\top \left(\nabla^2 \mathcal{G}(t, x)\dot{x} + \frac{\partial \nabla \mathcal{G}(t, x)}{\partial t}\right) \\ =& 2(\nabla \mathcal{G})^\top \left(-\vartheta \psi - \ell \frac{\nabla \mathcal{G}}{\|\nabla \mathcal{G}\|} + \frac{\partial \nabla \mathcal{G}}{\partial t}\right)\end{aligned}$$

Using Cauchy–Schwarz inequality, we can further write,

$$\leq -2\vartheta \nabla \mathcal{G}^\top \psi - 2\ell \|\nabla \mathcal{G}\| + 2\|\nabla \mathcal{G}\| \left\|\frac{\partial \nabla \mathcal{G}}{\partial t}\right\|$$

Since, $\left\|\frac{\partial \nabla \mathcal{G}}{\partial t}\right\| \leq \ell$*, then further we get,*

$$\dot{\mathcal{V}}(z) \leq -2\vartheta \nabla \mathcal{G}^\top \psi$$

The above inequality is similar to inequality 4.7. Then proceeding in a similar way as in the proof of Theorem 1, we get inequality (4.10) *which represents PUBST-based stability, implying the trajectory of* (4.11) *converges to the optimal solution of* $\mathcal{G}(t, x)$ *in a priori chosen time. Next for* $t > t_f$*, proceeding in a similar way as for* $t \in [t_0, t_f)$*, we get* $\dot{\mathcal{V}} \leq 2\vartheta \mathcal{V}$*, hence trajectories of* (4.11) *will maintain* $x^*(t)$*. Hence,* $x(t) \to x^*(t)$ *as* $t \to t_f$*.*

Computational Cost: We can discuss the convergence rates of Euler discretization to calculate complexity considering the assumption of Lipschitz smoothness. Let us consider the Euler discretized version of the proposed Robust Newton-like method for PUBSTC-TVO dynamics (4.11). To simplify the computation of complexity we can write $\psi(t, x) := \frac{\nabla \mathcal{G}(t,x)}{t_f - t}$. Let, $\mathfrak{K} = t_f / h$. Hence, the considered discretized version of the proposed dynamics, taking h as practical step size, can be written as:

$$x(k+1) = \begin{cases} x(k) - h\left(\frac{\vartheta}{\mathfrak{K} - k} + \frac{\ell}{\|\mathcal{G}(k,x)\|}\right)(\nabla^2 \mathcal{G}(k, x))^{-1} \nabla \mathcal{G}(k, x) & \text{for} \;\; 0 \leq k < \mathfrak{K} \\ x(k) - h\left(\vartheta + \frac{\ell}{\|\mathcal{G}(k,x)\|}\right)(\nabla^2 \mathcal{G}(k, x))^{-1} \nabla \mathcal{G}(k, x) & \text{for} \;\; \mathfrak{K} \leq k < \infty \end{cases} \tag{4.13}$$

Let, $\gamma_{1k} := \left(\frac{\vartheta}{\mathfrak{K}-k} + \frac{\ell}{\|\mathcal{G}(k,x)\|}\right)$, and $\gamma_{2k} := \left(\vartheta + \frac{\ell}{\|\mathcal{G}(k,x)\|}\right)$ and $\gamma_{1k}, \gamma_{2k} \in \mathbb{R}_{>0}$.

We need to introduce an extra assumption of L-Lipschitz smooth, as given below:

Assumption: ((Lipschitz Smoothness of Order q) We assume the function $\mathcal{G}$ is L-Lipschitz smooth of order $q \in (1, 2]$, i.e., for any $x, y \in \mathbb{R}^n$,

$$\|\nabla\mathcal{G}(k, y) - \nabla\mathcal{G}(k, x)\| \leq L\|y - x\|^{q-1} \tag{4.14}$$

The above assumption is also called (L, q) Holder continuity, it will lead to the following property:

$$\mathcal{G}(k, y) \leq \mathcal{G}(k, x) + \langle\nabla\mathcal{G}(k, x), y - x\rangle + \frac{L}{q}\|y - x\|^q, \tag{4.15}$$

When $q = 2$, the function will be Lipschitz smooth.

Considering the following assumptions, for our case, we can calculate iteration complexity as follows:

$$\mathcal{G}(k+1, x(k+1)) \leq \mathcal{G}(k, x(k)) + \langle\nabla\mathcal{G}(k, x(k)), x(k+1) - x(k)\rangle + \frac{L}{q}\|x(k+1) - x(k)\|^q, \tag{4.16}$$

Using algorithm (4.13), for $0 \leq k < \mathfrak{K}$ and considering the Lipschitz continuity of the gradient and for $q = 2$

$$\begin{aligned}\mathcal{G}(k+1, x(k+1)) &\leq \mathcal{G}(k, x(k)) - \frac{\gamma_{1k}h}{L}\|\nabla\mathcal{G}\|^2 + \frac{\gamma_{1k}^2h^2}{2L}\|\nabla\mathcal{G}\|^2 \\ &= \mathcal{G}(k, x(k)) - \frac{\gamma_{1k}h}{L}(1 - \gamma_{1k}h/2)\|\nabla\mathcal{G}\|^2\end{aligned}$$

Further, using the Assumption 2 (PL inequality):$\|\nabla\mathcal{G}\|^2 \geq \alpha(\mathcal{G} - \mathcal{G}^*)$, where, $\alpha > 0$, we can further write

$$\mathcal{G}(k+1, x(k+1)) \leq \mathcal{G} - \frac{\gamma_{1k}h\alpha}{L}(1 - \gamma_{1k}h/2)(\mathcal{G} - \mathcal{G}^*(k))$$

Subtracting $\mathcal{G}^*$ on both sides and $\kappa := (1 - 0.5h\gamma_{1k})$, then

$$\mathcal{G}(k+1, x(k+1)) - \mathcal{G}^*(k) \leq \left(1 - \frac{\gamma_{1k}h\alpha\kappa}{L}\right)(\mathcal{G}(k, x(k)) - \mathcal{G}^*(k))$$

Then for $\mathfrak{K}$ iterations with a fixed step size h

$$\mathcal{G}(\mathfrak{K}, x(\mathfrak{K})) - \mathcal{G}^*(k) \leq \left(1 - \frac{\gamma_{1k}h\alpha\kappa}{L}\right)^{\mathfrak{K}}(\mathcal{G}(0, x(0)) - \mathcal{G}^*(k))$$

By the inequality, $e^{-x} \geq 1 - x$ we can get that the corresponding iteration complexity is $\mathcal{O}\left(\left(\frac{\gamma_{1k}h\alpha\kappa}{L}\right)^{-1} ln\frac{1}{\epsilon}\right)$ for ϵ-closeness of $x(k)$ to $x^*(k)$. Similarly, if we run the algorithm for K iterations then for $\mathfrak{K} \leq k < K$, the corresponding iteration complexity is $\mathcal{O}\left(\left(\frac{\gamma_{2k}h\alpha\kappa}{L}\right)^{-1} ln\frac{1}{\epsilon}\right)$.

Levenberg–Marquardt–like PTC-TVO without invertible Hessian

Let inverse of $\nabla^2\mathcal{G}$ is singular or near singular then the Newton-like technique is not well defined. Suppose $(\nabla^2\mathcal{G} + \wp I_n)$ is invertible for some damping factor $\wp > 0$ then in order to track $x^*(t)$ of (4.3) in predefined-time, we propose the following dynamics:

$$\dot{x}(t) = \begin{cases} -(\nabla^2\mathcal{G}(t,x) + \wp I_n)^{-1}\left(\vartheta\psi(t,x) + \frac{\partial\nabla\mathcal{G}(t,x)}{\partial t}\right), & \text{for } \ t_0 \le t < t_f, \\ -(\nabla^2\mathcal{G}(t,x) + \wp I_n)^{-1}\left(\vartheta\nabla\mathcal{G}(t,x) + \frac{\partial\nabla\mathcal{G}(t,x)}{\partial t}\right), & \text{for } \ t \ge t_f, \end{cases} \tag{4.17}$$

where $x(t) \in \mathbb{R}^n$, $\vartheta, \wp \in \mathbb{R}$ and satisfies $\vartheta > 1$, $\psi(t,x) = (\psi_1, \psi_2, \ldots, \psi_n) \in \mathbb{R}^{n\times 1}$, $\psi_i := \frac{(e^{\nabla\mathcal{G}_{x_i}(t,x)}-1)}{e^{\nabla\mathcal{G}_{x_i}(t,x)}(t_f-t)}$, for $1 \le i \le n$ and consider the dynamics (4.17) as a perturbed version of (4.5) with input $\wp$.

Remark 4.6 *One of the methods for calculating the damping factor at a particular instant includes taking the negative of the minimum eigenvalue of the Hessian matrix at that instant.*

Theorem 4.7 *Consider the dynamics* (4.17)*. Suppose that* $\mathcal{G} : \mathbb{R}_{\ge 0} \times \mathbb{R}^n \to \mathbb{R}$ *in* (4.3) *satisfies Assumption 4.1 and Assumption 4.2 with* $p = 1/2$ *and* $\exists$ *a PUBST-ISS Lyapunov function* $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ *for* (4.17) *then dynamics* (4.17) *is PUBST-ISS.*

Proof 3 *Let us select the Lyapunov function candidate*

$$V = \|\nabla\mathcal{G}(t,x)\|^2 + 2\wp(\mathcal{G}(t,x) - \mathcal{G}^*(t)) \tag{4.18}$$

The derivative of the V *along the dynamics* (4.17) *gives*

$$\begin{aligned} \dot{V} =& 2\nabla\mathcal{G}^\top\left(\nabla^2\mathcal{G}\dot{x} + \frac{\partial\nabla\mathcal{G}}{\partial t}\right) + 2\wp\nabla\mathcal{G}^\top\dot{x} - 2\wp\dot{\mathcal{G}}^* \\ =& 2\nabla\mathcal{G}^\top\left((\nabla^2\mathcal{G} + \wp I_n)\left(-(\nabla^2\mathcal{G} + \wp I_n)^{-1}\left(\vartheta\psi(t,x) + \frac{\partial\nabla\mathcal{G}}{\partial t}\right)\right) + \frac{\partial\nabla\mathcal{G}}{\partial t}\right) - 2\wp\dot{\mathcal{G}}^* \\ =& -2\vartheta(\nabla\mathcal{G}(t,x))^\top\psi(t,x) - 2\wp\dot{\mathcal{G}}^* \end{aligned} \tag{4.19}$$

Taking the worst case, we assume $\sup_{t\ge 0}(\dot{\mathcal{G}}^*(t)) < -a,\ a \in \mathbb{R}_{>0}$*, then further we can write,*

$$\dot{V} \le -2\vartheta\frac{|\nabla\mathcal{G}_{x_i}(t,x)|(e^{|\nabla\mathcal{G}_{x_i}(t,x)|} - 1)}{e^{|\nabla\mathcal{G}_{x_i}(t,x)|}(t_f - t)} + 2a\wp, \qquad \textit{for any } i \tag{4.20}$$

Next, using PL inequality with $p = 1/2$*, we get* $V \le \left(1 + \frac{2\wp}{w}\right)\|\nabla\mathcal{G}\|^2$*. Since,* $V \le n(1 + (2\wp/w))(max\{|\nabla\mathcal{G}_1|, |\nabla\mathcal{G}_2|, ..., |\nabla\mathcal{G}_n|\}^2)$ *We assume that at a particular time,* $max\{|\nabla\mathcal{G}_1|,$

$|\nabla\mathcal{G}_2|, ..., \|\nabla\mathcal{G}_n|\}$ *returns* $|\nabla\mathcal{G}_i|$ *and* $n(1+(2\wp/w)) = c$ *then,* $\sqrt{V/c} \leq |\nabla\mathcal{G}_i|$. *Using this fact in* (4.20) *gives,* $\dot{V} \leq -2\vartheta\frac{\sqrt{V/c}(e^{\sqrt{V/c}}-1)}{e^{\sqrt{V/c}}(t_f-t)} + 2a\wp$. *Next, let* $\hat{V} = \sqrt{V/c}$, *hence,* $\dot{\hat{V}} = \frac{\dot{V}}{2c\hat{V}}$, *Further*

$$\dot{\hat{V}} \leq -\frac{\hat{\vartheta}(e^{\hat{V}}-1)}{e^{\hat{V}}(t_f-t)} + \gamma(|\wp|) \tag{4.21}$$

where γ *is a class* $\mathcal{K}$ *function and suppose* $\hat{\vartheta} = \frac{\vartheta}{c} > 1$. *Next, for* $t > t_f$, *after equation* (4.19) *along the dynamics* (4.17) *proceeding in the similar way as for* $t \in [t_0, t_f)$, *we get* $\dot{V} \leq -2\vartheta V + \gamma(|\wp|)$ *and combining it with* (4.21), *we get* $\dot{V} \leq -\beta(\|x_0\|, t_f - t) - \alpha(\|x\|) + \gamma(\|\wp\|)$, *where* β *is* $\mathcal{KL}$ *class function and* α, γ *are the class* $\mathcal{K}$ *functions. Hence* $V(t,x)$ *satisfies the PUBST-ISS Lyapunov function condition thus from Lemma 2.17, system* (4.17) *is PUBST-ISS.*

Remark 4.8 *The damping term* $\wp$ *functions as a regularizer, ensuring that* $(\nabla^2\mathcal{G}(t,x) + \wp I_n)$ *remains positive definite even when the Hessian is singular, thereby guaranteeing its invertibility. The adaptive nature of* $\wp$ *influences the algorithm's behavior:* $\wp$ *is large (when the Hessian is singular), then the algorithm resembles a Gradient-like PTC-TVO algorithm:* $\dot{x}(t) = \begin{cases} -(\wp I_n)^{-1}\left(\vartheta\psi(t,x) + \frac{\partial\nabla\mathcal{G}(t,x)}{\partial t}\right), & \text{for } t_0 \leq t < t_f, \\ -(\wp I_n)^{-1}\left(\vartheta\nabla\mathcal{G}(t,x) + \frac{\partial\nabla\mathcal{G}(t,x)}{\partial t}\right), & \text{for } t \geq t_f, \end{cases}$ *whereas* $\wp$ *is small (when the Hessian is well-conditioned), the algorithm exhibits behavior similar to the proposed Newton-like PTC-TVO algorithm* (4.5).

4.4 Time-Varying Constrained Convex Optimization

In this section, we extend the results of subsection 4.3.1 for constrained TVO problems with inequality constraints.

PTC-TVO with inequality constraints

Consider the constrained TVO problem

$$\begin{aligned} x^*(t) &:= \underset{x\in\mathbb{R}^n}{\operatorname{argmin}}\ \mathcal{F}(t,x) \\ \text{subject to } & g_i(t,x) \leq 0, \quad i \in \{1,2,...,r\} \end{aligned} \tag{4.22}$$

where $g = [g_1, g_2, \ldots, g_r]^\top$ with all g_i being convex functions in $x \in \mathbb{R}^n$, $\forall t \geq 0$ and fulfills Assumptions 4.1, 4.2, 4.3 and 4.4.

To include the constraints in the objective function, we introduce a vector of Lagrange multipliers $\lambda \in \mathbb{R}^r_{\geq 0}$ and $\mathcal{L} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^r_{\geq 0} \to \mathbb{R}$ associated with TVO in (4.22) is

$$\mathcal{L}(t, x, \lambda) := \mathcal{F}(t, x) + \sum_{i=1}^{r} \lambda_i g_i(t, x) \tag{4.23}$$

The dual function is $\mathfrak{T}^*(t) := \arg\max_{\lambda \in \mathbb{R}^r} \mathfrak{G}(t, \lambda)$, where $\mathfrak{G}(t, \lambda) := \min_{x \in \mathbb{R}^n} \mathcal{L}(t, z, \lambda)$. The necessary and sufficient KKT conditions [13] for optimality under Assumptions 4.1 4.3 and 4.4, $\forall t \geq 0$ are

$$\nabla \mathcal{F}(t, x^*) + \sum_{i=1}^{r} \lambda_i^* \nabla g_i(t, x^*) = 0 \tag{4.24}$$

$$\lambda_i^* g_i(t, x^*) = 0, \tag{4.25}$$

$$\lambda_i^* \geq 0 \tag{4.26}$$

$$g_i(t, x^*) \leq 0, \quad \forall i \in \{1, 2, ..., r\} \tag{4.27}$$

Next, we formulate perturbed KKT conditions, where we modify (4.25) as $\lambda_i^* g_i(t, \hat{x}^*) = -\frac{1}{\sigma} \implies \lambda_i^* = -\frac{1}{\sigma g_i(t, \hat{x}^*)}$, where $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is defined as a barrier parameter and $(1/\sigma) \to 0$ as $\hat{x}(t) \to x^*(t)$. Next, upon substituting the obtained λ_i^* from perturbed KKT condition in (4.24), we get

$$\nabla(\mathcal{F}(t, \hat{x}^*) - \frac{1}{\sigma} \sum_{i=1}^{r} log(-g_i(t, x^*)) = 0 \tag{4.28}$$

which also satisfies KKT conditions (4.26) and (4.27). Next, we define the approximate optimal trajectory and Lagrangian $\mathcal{L} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ associated with (4.22) using $\sigma(t)$ and $\delta(t)$, where $\delta(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is defined as a slack function. It is to be noted that the $\delta(t)$ enlarges the initial feasible set.

$$\hat{x}^*(t) := \arg\min_{x \in \tilde{D}} \mathcal{F}(t, x) - \frac{1}{\sigma(t)} \sum_{i=1}^{r} \log(\delta(t) - g_i(t, x))$$

$$\tilde{\mathcal{L}}(t, x, \sigma, \delta) := \mathcal{F}(t, x) - \frac{1}{\sigma(t)} \sum_{i=1}^{r} \log(\delta(t) - g_i(t, x)) \tag{4.29}$$

where $x \in \tilde{D}$, $\tilde{D} := \{x \in \mathbb{R}^n : g_i(t, x) < \delta(t), i \in \{1, ..., r\}\}$ is an open set such that $D \subset \tilde{D}$. In the next lemma, we indicate the approximation error.

Lemma 4.9 *(see [3]) Under the Assumption 4.3, and 4.4* $\forall\ \lambda^* \in \mathfrak{T}^*(t)$ *and* $t \geq 0$, *the following inequality holds*

$$|\mathcal{F}(t, \hat{x}^*) - \mathcal{F}(t, x^*)| \leq \frac{r}{\sigma(t)} + \delta(t) \left(\inf_{\lambda^* \in \mathfrak{T}(t)^*} \|\lambda^*\| \right) \tag{4.30}$$

Remark 4.10 *Lemma 4.9 says that if $\sigma(t)$ and $\delta(t)$ are chosen so that right-hand side of* (4.30) *converges to zero, then the approximate trajectory $\hat{x}^*(t)$ converges to $x^*(t)$.*

To track the optimal solution $\hat{x}^*(t)$ within a predefined time, we aim to design a dynamics so that the gradient of $\tilde{\mathcal{L}}$ defined in (4.29) vanishes within a predefined time.Consider the following proposed dynamics in terms of Lagrangian (4.29) as:

$$\dot{x}(t) = \begin{cases} -(\nabla^2 \tilde{\mathcal{L}})^{-1}\Big(\vartheta\psi(t,x,\sigma,\delta) + \frac{\partial\nabla\tilde{\mathcal{L}}}{\partial\sigma}\dot{\sigma} + \frac{\partial\nabla\tilde{\mathcal{L}}}{\partial\delta}\dot{\delta} + \frac{\partial\nabla\tilde{\mathcal{L}}}{\partial t}\Big) & \text{for } \ t_0 \le t < t_f, \\ -(\nabla^2 \tilde{\mathcal{L}})^{-1}\Big(\vartheta\nabla\tilde{\mathcal{L}} + \frac{\partial\nabla\tilde{\mathcal{L}}}{\partial\sigma}\dot{\sigma} + \frac{\partial\nabla\tilde{\mathcal{L}}}{\partial\delta}\dot{\delta} + \frac{\partial\nabla\tilde{\mathcal{L}}}{\partial t}\Big), & \text{for } \ t \ge t_f, \end{cases} \tag{4.31}$$

where $\vartheta \in \mathbb{R}$ and satisfies $\vartheta > 1$, $\psi(t,x,\sigma,\delta) = (\psi_1, \psi_2, ..., \psi_n) \in \mathbb{R}^{n\times 1}$, $\psi_i := \frac{(e^{\nabla\tilde{\mathcal{L}}x_i(t,x,\sigma,\delta)} - 1)}{e^{\nabla\tilde{\mathcal{L}}x_i(t,x,\sigma,\delta)}(t_f - t)}$, for $1 \le i \le n$.

Theorem 4.11 *Consider the dynamics* (4.31)*. Assume that there exist a continuously differentiable function $V : [t_0, \infty) \times D \to \mathbb{R}_{\ge 0}$ satisfying $\gamma_1(x) \le V(t,x) \le \gamma_2(x)$, $\forall x \in D \setminus \{0\}, \forall t \in [t_0, \infty)$, $\gamma_1(x)$ and $\gamma_2(x)$ be continuous positive definite functions, $V(t, \hat{x}^*) = 0$ and for $V \ne 0$: $\dot{V} \le 0, \forall t \ge t_f$ and $\dot{V} \le \frac{-\vartheta(e^V - 1)}{e^V(t_f - t)}, \forall t \in [t_0, t_f)$ for* (4.31)*. This signifies that $x(t)$ converges to the optimal solution $\hat{x}^*(t)$ in a priori chosen time.*

Proof 4 *Let us select the Lyapunov functional candidate*

$$V = \|\nabla\tilde{\mathcal{L}}(t,x,\sigma,\delta)\|^2, \tag{4.32}$$

The derivative of the V along the dynamics (4.31) *for $t \in [t_0, t_f)$*

$$\begin{aligned} \dot{V}(x) =& 2(\nabla\tilde{\mathcal{L}})^\top \left(\nabla^2\tilde{\mathcal{L}}\dot{x} + \frac{\partial\nabla\tilde{\mathcal{L}}}{\partial\sigma}\dot{\sigma} + \frac{\partial\nabla\tilde{\mathcal{L}}}{\partial\delta}\dot{\delta} + \frac{\partial\nabla\tilde{\mathcal{L}}}{\partial t}\right) \\ =& 2(\nabla\tilde{\mathcal{L}})^\top \Bigg(\nabla^2\tilde{\mathcal{L}}\Big(-(\nabla^2\tilde{\mathcal{L}})^{-1}\Big(\vartheta\psi + \frac{\partial\nabla\tilde{\mathcal{L}}}{\partial\sigma}\dot{\sigma} + \frac{\partial\nabla\tilde{\mathcal{L}}}{\partial\delta}\dot{\delta} + \frac{\partial\nabla\tilde{\mathcal{L}}}{\partial t}\Big)\Big) + \frac{\partial\nabla\tilde{\mathcal{L}}}{\partial\sigma}\dot{\sigma} + \frac{\partial\nabla\tilde{\mathcal{L}}}{\partial\delta}\dot{\delta} \\ & + \frac{\partial\nabla\tilde{\mathcal{L}}}{\partial t}\Bigg) \end{aligned} \tag{4.33}$$

$$= -2\vartheta(\nabla\tilde{\mathcal{L}})^\top\psi(t,x,\sigma,\delta) \tag{4.34}$$

(4.34) *is similar to* (4.7) *of convergence analysis performed in Theorem 1. Hence, further analysis can be performed in the same way and finally, we get a similar expression as* (4.10) *in Theorem 1,*

$$\dot{\hat{V}} \le -\frac{\hat{\vartheta}(e^{\hat{V}} - 1)}{e^{\hat{V}}(t_f - t)} \tag{4.35}$$

where $\hat{\vartheta} = \frac{\vartheta}{n} > 1$. *Thus, inequality* (4.35) *represents predefined-time stability, which means the trajectory of* (4.31) *converges to the optimal solution of* $\mathcal{F}(t, x)$ *in a priori chosen predefined-time.*

Next, convergence analysis for system (4.31) *for* $t \geq t_f$ *is similar to Theorem 1 and we get,*

$$\begin{aligned}\dot{V}(x) =& 2(\nabla\tilde{\mathcal{L}})^\top \left(\nabla^2\tilde{\mathcal{L}}\dot{x} + \frac{\partial\nabla\tilde{\mathcal{L}}}{\partial\sigma}\dot{\sigma} + \frac{\partial\nabla\tilde{\mathcal{L}}}{\partial\delta}\dot{\delta} + \frac{\partial\nabla\tilde{\mathcal{L}}}{\partial t}\right) \\ =& -2\vartheta(\nabla\tilde{\mathcal{L}}(t, x, \sigma, \delta))^\top\nabla\tilde{\mathcal{L}}(t, x, \sigma, \delta) \\ =& -2\vartheta V\end{aligned}$$

So, for $t \geq t_f$*, the equilibrium point of system* (4.31) *is asymptotically stable, therefore trajectories of* (4.31) *will remain optimal. Hence,* $x(t) \to x^*(t)$ *as* $t \to t_f$*.*

4.5 Numerical Examples

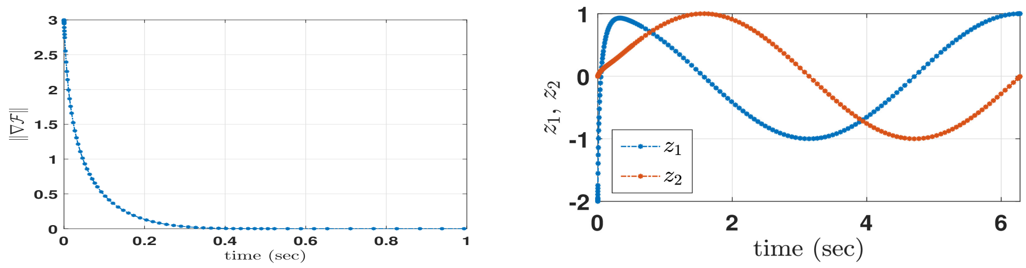
This section provides three simulation examples to illustrate the predefined-time stability for the continuous-time dynamics proposed in this note. Euler discretization is used for MATLAB implementation with a constant time step size of Δt. Here, the convergence time (t_{ac}) in seconds is equal to $t_{ac}/\Delta t$ iterations.

4.5.1 Example 1

Minimize

$$\frac{1}{2}\begin{bmatrix} z_1 - \cos(t) \\ z_2 - \sin(t)\end{bmatrix}^\top \begin{bmatrix} 3 - 2\cos(2t) & 2\sin(2t) \\ 2\sin(2t) & 3 + 2\cos(2t)\end{bmatrix}\begin{bmatrix} z_1 - \cos(t) \\ z_2 - \sin(t)\end{bmatrix}$$

First, we implemented predefined-time convergent proposed dynamics (4.5) in MATLAB to solve the above TVO problem. The initial conditions are $z_0 = [-2 \;\; 0]^\top$. Figure 4.1a shows that the norm of the gradient of objective function vanishes in predefined-time 0.6 sec and the comparison of the result with fixed-time convergent (FxTC) algorithm discussed in Hong et al. [2] is also shown. Figure 4.1b shows the evolution of states which starts tracking the optimal trajectory $z^*(t)$ in time $t_f = 0.6$sec, which is chosen a priori. In Figure 4.2, we plot $\|\mathcal{F}(t, z)\|$ versus t for different sampling periods $\Delta t \in \{10^{-2}, 10^{-3}, 10^{-4}\}$. We notice that steady state error increases as we increase Δt.



(a)

(b)

Figure 4.1: Results using PTC-TVO dynamics (4.5) in solving example 1: (a)Comparison between $||\nabla\mathcal{F}||$ with PTC-TVO with $t_f = 0.6$ sec and FxTC in [2], (b)Evolution of states using dynamics (4.5) for example 1 based on TVO problem with unconstrained case.

The log scale is used on the y axis such that the variation of $\|\nabla\mathcal{F}\|$ for values near zero is clearly shown.

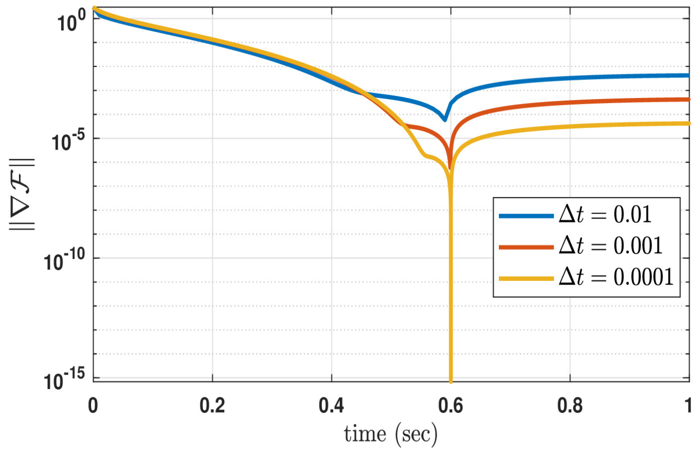
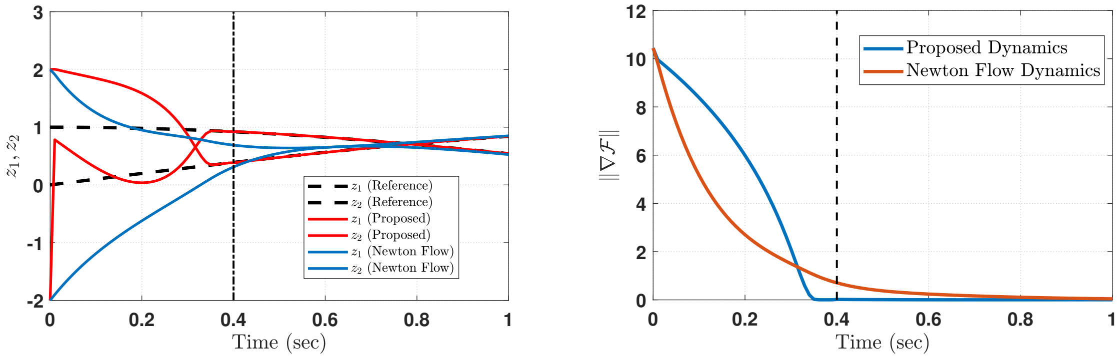


Figure 4.2: Plot of $\|\nabla\mathcal{F}(t, z(t))\|$ on semilog scale, where $z(t)$ is the solution to the discretized version of dynamics (4.5) with time for various step sizes (Δt)

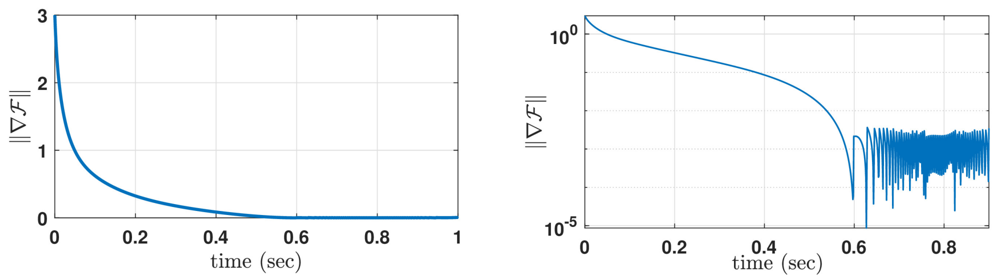
Next, we show the comparison results with the prediction-correction method based on standard Newton flow dynamics presented in Fazlyab et al.(2017) [3] as: $\dot{z}(t) = -(\nabla^2\mathcal{F}(t,z))^{-1}\left(\vartheta\nabla\mathcal{F}(t,z) + \frac{\partial\nabla\mathcal{F}(t,z)}{\partial t}\right)$. It is shown in Figure 4.3a that with the same initial condition $z_0 = [-2\ 0]^\top$ and $\vartheta = 5$, $\|\nabla\mathcal{F}\|$ vanishes within predefined time $t_f = 0.3$ sec using proposed dynamics (4.5) and using standard Newton flow dynamics discussed in Fazlyab et al. (2017) [3], $\|\nabla\mathcal{F}\|$ takes more than $t = 1.2$ second in reaching zero. In Figure 4.3b, it shown that z_1, z_2 calculated using proposed dynamics (4.5) starts tracking optimal trajectory within predefined time $t_f = 0.3$ second while using standard Newton



(a) Comparison of the evolution of states using dynamics (4.5) and Newton flow dynamics in Fazlyab et.al. (2017) [3] for example 1.

(b) Comparison between $||\nabla\mathcal{F}||$ with PTC-TVO with $t_f = 0.4$ sec and standard Newton flow dynamics in Fazlyab et.al. (2017) [3]

Figure 4.3: Comparison of results using PTC-TVO dynamics (4.5) and standard Newton flow dynamics in Fazlyab et.al. (2017) [3] in solving example 1



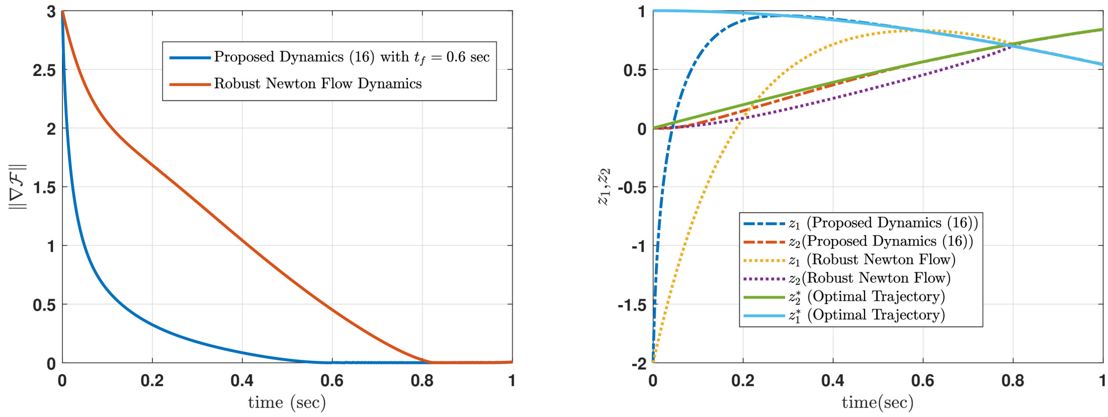
(a) Plot of $||\nabla\mathcal{F}||$ showing predefined-time convergence within $t_f = 0.6$ sec using dynamics (4.11) for Example 1 of unconstrained case.

(b) Plot of $||\nabla\mathcal{F}(t, z(t))||$ on semilog scale where $z(t)$ is the solution to the discretized version of (4.11) with sampling time $\Delta t = 0.001 sec$

Figure 4.4: Results using PTC-TVO dynamics (4.11) in solving example 1

Flow dynamics, optimal trajectory tracking starts after $t = 1.2$ second.

Next, we solved the above example using a robust dynamics proposed in (4.11). We took initial conditions as $z_0 = [-2\ 0]^\top,\ t_f = 0.6\ sec, \ell = 2,\ \vartheta = 5$. Figure 4.4a shows that the norm of the gradient of the objective function vanishes in predefined-time 0.6 second. In Figure 4.4b, log scale is used on the y axis such that the variation of $||\nabla\mathcal{F}||$ for values near zero is clearly shown. Further, we show the comparison results from Newton flow robust dynamics for solving TVO problems: $\dot{z}(t) = -(\nabla^2\mathcal{F}(t, z))^{-1}\left(\vartheta\nabla\mathcal{F}(t, z) + \ell\frac{\nabla\mathcal{F}(t,z)}{||\nabla\mathcal{F}(t,z)||}\right)$. It is shown in Figure 4.5a that with the same initial condition $z_0 = [-2\ 0]^\top,\ t_f =$



(a) (b)

Figure 4.5: Comparison of results using PTC-TVO dynamics (4.11) and standard robust Newton flow dynamics in solving example 1

0.6 $sec, \ell = 2, \ \vartheta = 5, \ \|\nabla\mathcal{F}\|$ vanishes within predefined time $t_f = 0.6$ seconds using proposed dynamics (4.11) and using standard robust Newton flow dynamics, $\|\nabla\mathcal{F}\|$ takes more than $t = 0.8$ seconds in reaching zero. In Figure 4.5b, it shown that z_1, z_2 calculated using proposed dynamics (4.11) starts tracking optimal trajectory within predefined time $t_f = 0.6$ seconds while using standard robust Newton Flow dynamics, optimal trajectory tracking starts after $t = 0.8$ seconds.

4.5.2 Example 2

Consider the following quadratic TVO problem with ill-conditioned Hessian

$$\text{minimize} \quad (tz - \sin(t))^2 \tag{4.36}$$

Since, for this example, Hessian will be ill-conditioned, when t is small. Therefore, we apply the proposed Levenberg–Marquardt-like PTC-TVO algorithm (4.17) to find an optimal trajectory in a prior chosen time. It is important to choose $\wp$ properly. Here, we chose $\wp = e^{-2t}$. The initial condition is $z(0) = 2, \ t_f = 0.5$ second. It is shown in Figure 4.6 that the state trajectory obtained by using proposed dynamics (4.17) starts tracking the optimal trajectory within chosen $t_f \leq 0.5$ second.

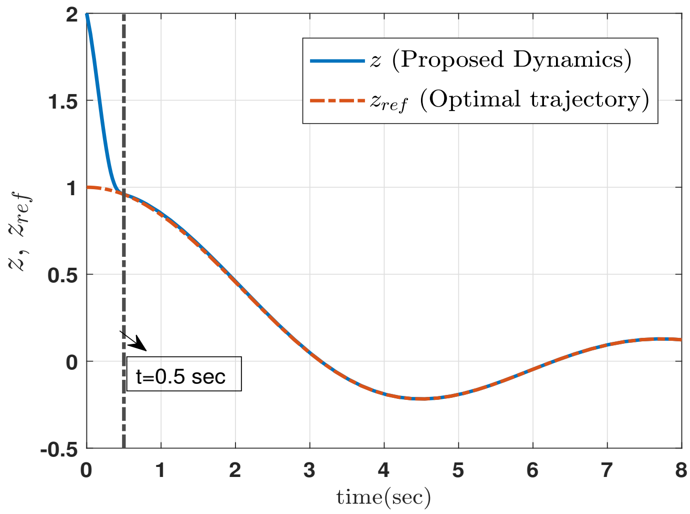


Figure 4.6: Evolution of state using dynamics (4.17) for example (4.36) with ill-conditioned Hessian and it is shown that it starts tracking optimal trajectory within $t_f = 0.5$ seconds

4.5.3 Example 3

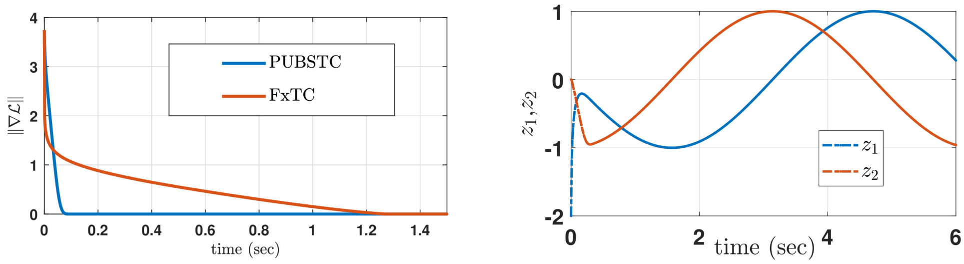
Consider the following quadratic TVO problem with inequality constraints:

$$\begin{aligned} &\text{minimize} \quad 0.5(z_1 + \sin(t))^2 + 1.5(z_2 + \cos(t))^2 \\ &\text{subject to} \quad z_2 - z_1 - \cos(t) \leq 0 \end{aligned} \tag{4.37}$$

Above example is an illustration of (4.22), so (4.29) is written as

$$\mathcal{L}(t, z, \sigma, \delta) = 0.5(z_1 + \sin(t))^2 + 0.5(z_2 + \cos(t))^2 - \frac{1}{\sigma(t)} \log(\delta(t) - (z_2 - z_1 - \cos(t))) \tag{4.38}$$

where $\sigma(t) = 10 \exp(1000 * t)$, $\delta(t) = 2 \exp(-5t)$. We solved it using predefined-time convergent proposed dynamics (4.31) with the help of forward Euler's method with constant step size in MATLAB. The initial condition is $z_0 = [-2\ 0]^\top$. In Figure 4.7a, the norm of the gradient of $\nabla\mathcal{L}$ in (4.38) goes to zero within a priori chosen time of $t_f = 0.1$ sec, which is the optimality condition in order to track the optimal trajectory and the comparison of the result with fixed-time convergent (FxTC) algorithm discussed in Hong et al. (2022) [2] is also shown. In Figure 4.7b, the evolution of time-varying states is also shown which follow their optimal trajectory as $\|\nabla\mathcal{L}(t, z(t))\|$ vanishes within predefined-time $t_f = 0.1$ sec. In Figure 4.8, we plot $\|\nabla\mathcal{L}(t, z)\|$ versus t for different $\Delta(t) \in \{10^{-2}, 10^{-3}, 10^{-4}\}$. We notice that steady state error increases as we increase Δt.



(a) Comparison between $||\nabla\mathcal{L}||$ with predefined-time convergence within $t_f =$ 0.1 sec and FxTC.

(b) Evolution of states for example based on TVO problem with inequality constraints.

Figure 4.7: Results using PTC-TVO dynamics (4.31) in solving example 2

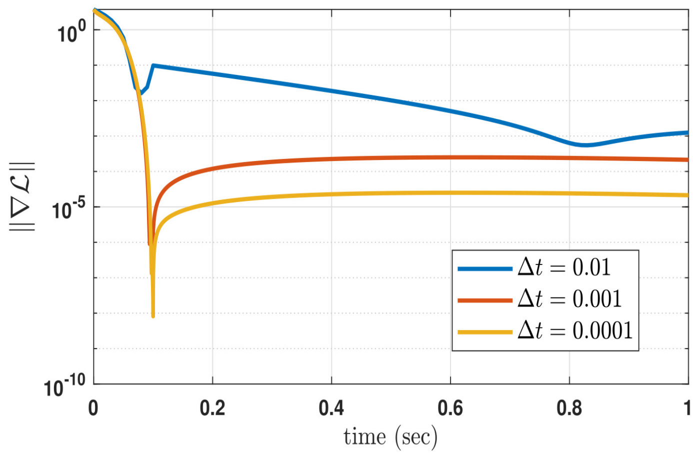


Figure 4.8: Plot of $\|\nabla\mathcal{L}(t, z(t)\|$ on semilog scale, where $z(t)$ is the solution to the discretized version of dynamics (4.31) with time for various step sizes (Δt)

4.6 Conclusion

In this chapter, predefined-time convergent dynamics are proposed and discussed to solve TVO problems. The proposed method yields a better convergence time compared to the methods available in the literature. Numerical examples show the effectiveness of proposed dynamics. Further, the proposed technique can also be extended for time-varying distributed optimization problems. In addition, one of the motivations behind the study of fast optimization methods is to generate signals for controllers at the same speed as the system evolves.

In the forthcoming chapter, our attention will be directed towards the application of the proposed approach in the present chapter in robot navigation with collision avoidance.