Chapter 7

Applications

7.1 Introduction

In this chapter, we demonstrate the implementation of vector framework based stability analysis tools in designing controls for practical applications, in particular, a 2 DOF Helicopter model [116, 117] and a two-link manipulator [118, 119]. In fact, we exploit vector control Lyapunov functions to stabilize and regulate the states to the desired position in the set arbitrary time. We first demonstrate the results experimentally on a 2 DOF Helicopter model (fourth-order system). In this, our objective is to regulate the pitch and yaw positions to the desired positions in the set arbitrary time. Then, we consider a Robotic system (two-link manipulator) to design control to stabilize the states to the origin in the set arbitrary time and provide the simulation results to validate the developed theoretical results. The aggregation procedure and the main results discussed in Chapter 3 are exploited in these applications.

7.2 2 DOF Helicopter Model

We consider the Quanser 2 DOF Helicopter Experimental set up [120] shown in Fig. 7.1. It consists of a helicopter body shown in Fig. 7.2 mounted on a fixed base with two propellers (front and back), two DC motors (pitch and yaw), two encoders, a yoke and a slipring. The front and back propellers control the elevation of the helicopter about the pitch axis and yaw axis through DC motors. The pitch and yaw encoders are used to measure the actual pitch and yaw angles of the helicopter. The measured signals of the encoders and motors are transmitted through a slipring. This reduces the wire tangling possibility on the yaw axis and allows the yaw to rotate freely about 360 degrees.

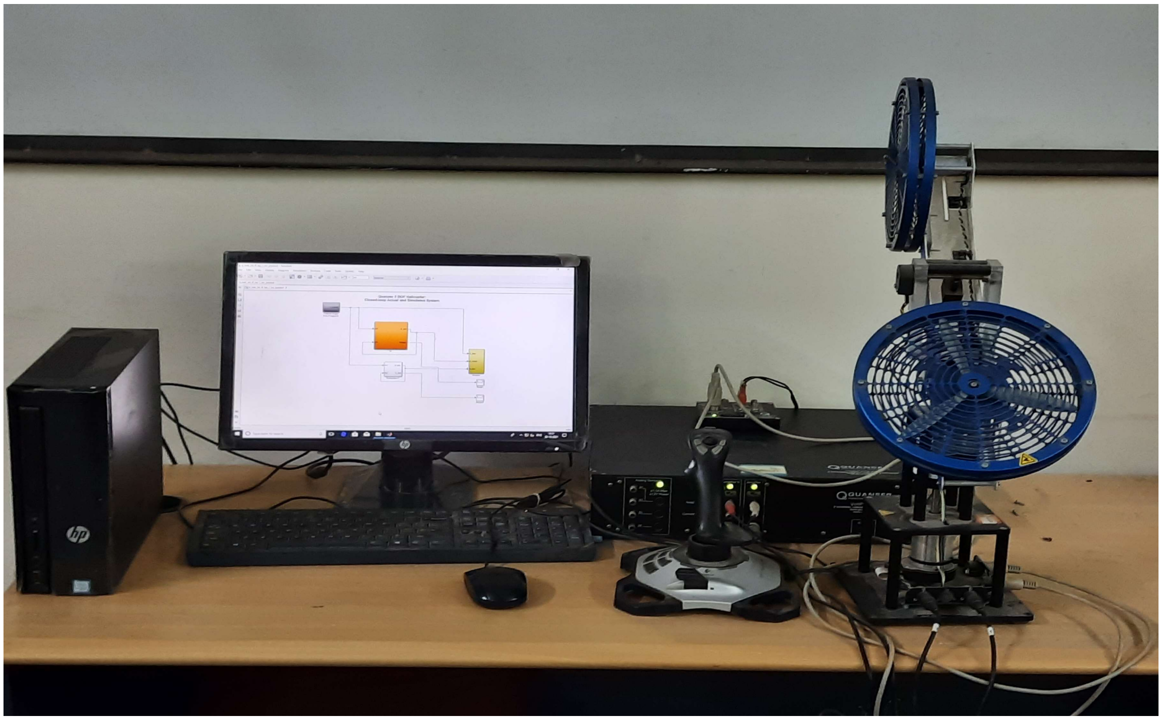

Figure 7.1: Helicopter Experimental Set up

From the free-body diagram shown in Fig. 7.3, the nonlinear dynamical equations obtained using Euler-Lagrange formula are as follows [91]:

$$
\begin{aligned}
\frac{d^2}{dt^2}\theta(t) &= \frac{K_{pp}V_{mp}}{J_p+m_hl^2}+\frac{K_{py}V_{my}}{J_p+m_hl^2}-\frac{B_p\dot{\theta}(t)+m_hl^2\dot{\psi}^2(t)\sin(\theta(t))\cos(\theta(t))+m_hgl\cos(\theta(t))}{J_p+m_hl^2}\\
\frac{d^2}{dt^2}\psi(t) &= \frac{K_{yp}V_{mp}}{J_y+m_hl^2\cos^2(\theta(t))}+\frac{K_{yy}V_{my}}{J_y+m_hl^2\cos^2(\theta(t))}+\frac{2m_hl^2\dot{\psi}(t)\sin(\theta(t))\cos(\theta(t))\dot{\theta}(t)}{J_y+m_hl^2\cos^2(\theta(t))}\\
&\quad-\frac{B_y\dot{\psi}(t)}{J_y+m_hl^2\cos^2(\theta(t))}
\end{aligned}
\tag{7.1}
$$

where $\theta(t), \dot{\theta}(t), \psi(t), \dot{\psi}(t)$ denote the pitch angle, pitch velocity, yaw angle, yaw velocity and V_{mp}, V_{my} are the input voltages to the pitch and yaw motors respectively. The parameters description and their values are specified in Table 7.1.

We can write the above equations (7.1) in the representation:

$$M(x)\ddot{x}+C(x,\dot{x})\dot{x}+D\dot{x}+g(x)=U$$

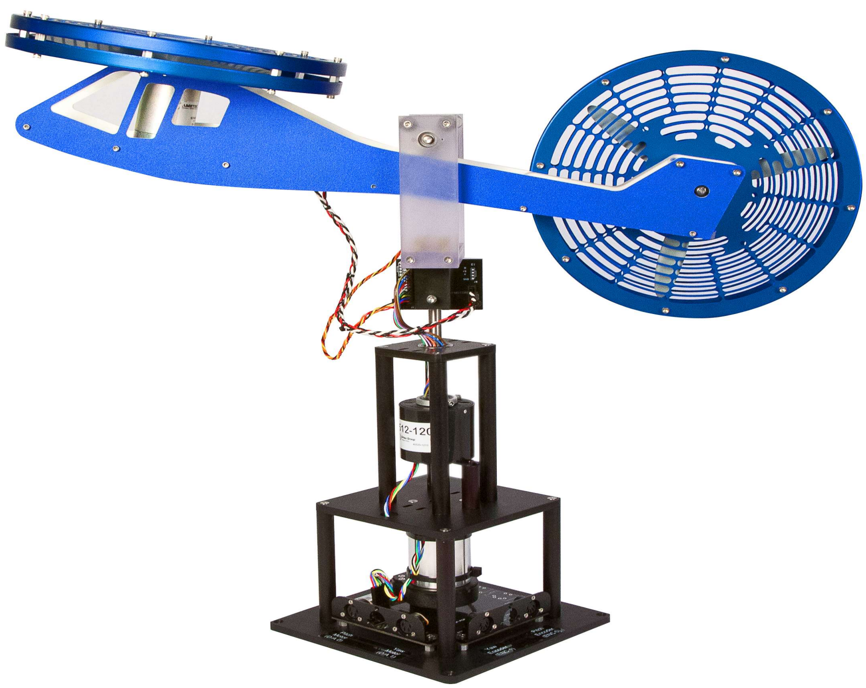

Figure 7.2: Helicopter Model

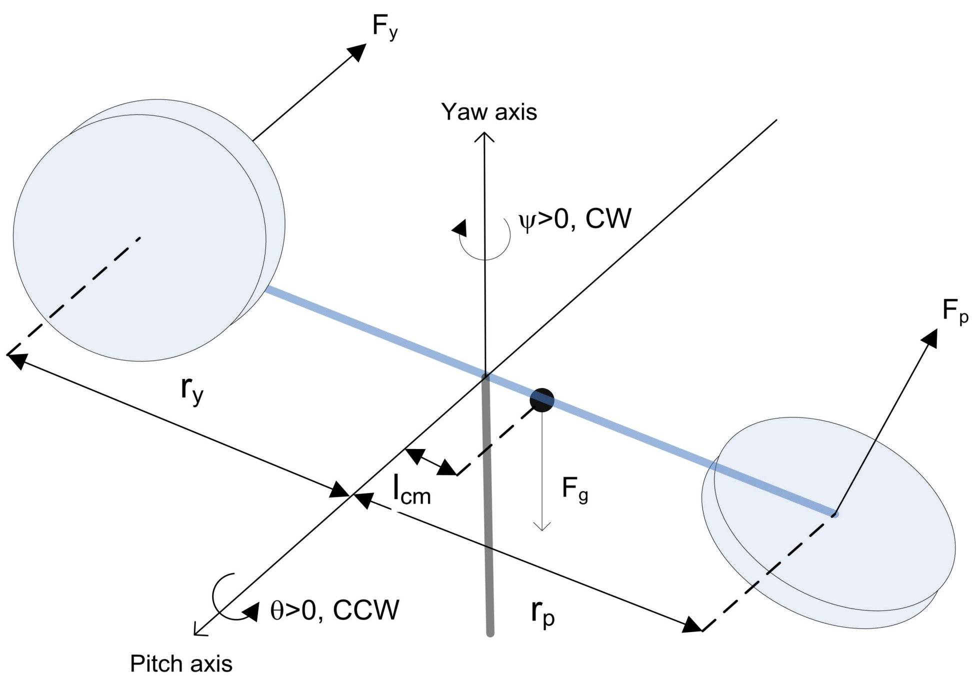

Figure 7.3: Free-body diagram of a 2 DOF Helicopter Model

where $x = [x_1, x_3]^\top = [\theta(t), \psi(t)]$ is the position vector, $\dot{x} = [x_2, x_4]^\top = [\dot{\theta}(t), \dot{\psi}(t)]^\top$ is the velocity vector, $U = [U_1, U_2]^\top = [K_{pp}V_{mp} + K_{py}V_{my}, K_{yp}V_{mp} + K_{yy}V_{my}]^\top$ is the control

Table 7.1: **System specifications**

Symbol	**Description**	**Values**	**Unit**
J_p	Moment of inertia about the pitch axis	0.0384	$kg.m^2$
J_y	Moment of inertia about the yaw axis	0.0432	$kg.m^2$
B_p	Viscous damping coefficient about the pitch axis	0.8	$\frac{N}{V}$
B_y	Viscous damping coefficient about the yaw axis	0.318	$\frac{N}{V}$
K_{pp}	thrust force constant of yaw motor	0.204	$\frac{N.m}{V}$
K_{yy}	thrust torque constant of yaw axis from pitch motor	0.072	$\frac{N.m}{V}$
K_{py}	thrust torque constant on pitch axis from yaw motor	0.0068	$\frac{N.m}{V}$
K_{yp}	thrust torque on yaw axis from pitch motor	0.0219	$\frac{N.m}{V}$
m_h	mass of the helicopter	1.3872	kg
l	length from the centre of mass to the pitch axis	0.186	m
g	acceleration due to gravity	9.81	m/s^2

input vector,

$$M = \begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} \end{bmatrix}, C = \begin{bmatrix} c_{11} & c_{12} \\ -2c_{12} & c_{22} \end{bmatrix}, \ D = 0, \ g = \begin{bmatrix} m_h g l \cos(x_1) \\ 0 \end{bmatrix}$$

where $m_{11} = J_p + m_h l^2, m_{22} = J_y + m_h l^2 \cos^2(x_1), c_{11} = B_p$, $c_{12} = m_h l^2 x_4 \sin(x_1)\cos(x_1)$, $c_{22} = B_y$. Now, consider a regulation problem to design a feedback control law so that x tracks a constant reference x_r in arbitrary time. Let $e = x - x_r$. Then e satisfies the differential equation

$$M(x)\ddot{e} + C(x, \dot{x})\dot{e} + g(x) = U$$

Our aim is to stabilize this system at $(e = 0, \dot{e} = 0)$, but this point is not an equilibrium point when $U = 0$. Let

$$U = g(x) - k_p e - v$$

where k_p is a positive definite diagonal matrix with entries k_{p1} and k_{p2}, and $v = [v_1, v_2]^\top$ is a control to be chosen appropriately. Now substituting U, we get

$$M(x)\ddot{e} + C(x, \dot{x})\dot{e} + k_p e + v = 0$$

This can be written as the set of state space equations:

$$\begin{aligned}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= -9.27e_2 - 0.5538e_4^2\sin(e_1)\cos(e_1) - 11.59k_{p1}e_1 - 11.59v_1 \\
\dot{e}_3 &= e_4 \\
\dot{e}_4 &= \frac{-2c_{12}}{m_{22}}e_2 - \frac{0.318}{m_{22}}e_4 - \frac{k_{p2}}{m_{22}}e_3 - \frac{v_2}{m_{22}}
\end{aligned} \tag{7.2}$$

In the following subsection, we discuss the design of control v using vector control Lyapunov functions and aggregation procedure developed in Chapter 3 so that x (pitch and yaw angles of the helicopter) tracks a constant reference point x_r in the set arbitrary time.

7.2.1 Controller Design

To design controls v_1 and v_2, firstly we use the aggregation procedure discussed in Section 3.3. Let us apply the transformation as $z = \mathcal{T}\zeta$ with

$$\mathcal{T} = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 0 & 0 & 3 & 2 \end{bmatrix}, \ \zeta = [e_1, e_2, e_3, e_4]^\top, \ z = [z_1, z_2]^\top$$

to transform the system (7.2) into

$$\begin{aligned}
\dot{z}_1 &= -2.3901z_1 - 0.0262z_2^2\sin(0.2308z_1)\cos(0.2308z_1) - 5.3499k_{p1}z_1 - 23.18v_1 \\
\dot{z}_2 &= 0.4614z_2 - \frac{0.3076C_2z_1}{M_2} - \frac{0.4616k_{p2}z_2}{M_2} - \frac{0.0978z_2}{M_2} - \frac{2v_2}{M_2}
\end{aligned} \tag{7.3}$$

where $C_2 = -0.0147z_2\sin(0.2308z_1)\cos(0.2308z_1)$ and $M_2 = 0.0432{+}0.0478\cos^2(0.2308z_1)$.

Let us consider the vector Lyapunov function, $V = [V_1, V_2]^\top$ with $V_1 = (z_1 - z_2)^2$ and $V_2 = (z_1 + z_2)^2$. It is easy to check that $r^\top V$ is positive definite, where $r = [1, 1]^\top$. The derivative of V_1 along the trajectories of (7.3) is given by, $\dot{V}_1 = 2(z_1 - z_2)(\dot{z}_1 - \dot{z}_2)$ for all $t \in [t_0, t_a)$. Let us choose:

$$\begin{aligned}
\frac{2v_2}{M_2} - 23.18v_1 = {} & 0.4614z_2 - \frac{0.3076C_2z_1}{M_2} - \frac{0.4616k_{p2}z_2}{M_2} - \frac{0.0978z_2}{M_2} + 2.3901z_1 \\
& + 0.0262z_2^2\sin(0.2308z_1)\cos(0.2308z_1) + 5.3499k_{p1}z_1 - \phi_1(t, z_1 - z_2) \\
& - M(z_1 - z_2),
\end{aligned}$$

where $\phi_1(t, \cdot)$ is the arbitrary time term given by: $\phi_1(t, z_1 - z_2) = \frac{\gamma_1(e^{(z_1 - z_2)} - 1)}{e^{(z_1 - z_2)}(t_a - t)}$ if $t \in [t_0, t_a)$, else $\phi_1(t, z_1 - z_2) = 0$ for $t \geq t_a$, $\gamma_1 > 1$ and $M > 0$. Then,

$$\dot{V}_1 \leq \frac{-2\gamma_1\sqrt{V_1}(e^{\sqrt{V_1}} - 1)}{e^{\sqrt{V_1}}(t_a - t)}$$

Similarly,

$$\dot{V}_2 \leq \frac{-2\gamma_2\sqrt{V_2}(e^{\sqrt{V_2}}-1)}{e^{\sqrt{V_2}}(t_a-t)}$$

with

$$\begin{aligned}\frac{2v_2}{M_2}+23.18v_1 =& -2.3901z_1-0.0262z_2^2\sin(0.2308z_1)\cos(0.2308z_1)-5.3499k_{p1}z_1\\ &+0.4614z_2-\frac{0.3076C_2z_1}{M_2}-\frac{0.4616k_{p2}z_2}{M_2}-\frac{0.0978z_2}{M_2}+\phi_2(t,z_1+z_2)\\ &+M(z_1+z_2),\end{aligned}$$

where $\phi_2(t,\cdot)$ is the arbitrary time term given by: $\phi_2(t,z_1+z_2)=\frac{\gamma_2(e^{(z_1+z_2)}-1)}{e^{(z_1+z_2)}(t_a-t)}$ if $t\in[t_0,t_a)$, else $\phi_2(t,z_1+z_2)=0$ for $t\geq t_a$, $\gamma_2>1$ and $M>0$. For all $t\geq t_a$, the designed controls v_1 and v_2 will maintain the dynamics (7.3) at the origin, hence, $\dot{V}=0$ for all $t\geq t_a$. Now, let $w_i=\sqrt{V_i}$, $i=1,2$, then $\dot{w}_i=\frac{\dot{V}_i}{2\sqrt{V_i}}\leq\frac{-\gamma_i(e^{w_i}-1)}{e^{w_i}(t_a-t)}$. Thus, the comparison system constructed over $t\in[t_0,t_a)$ is $\dot{w}_i=\frac{-\gamma_i(e^{w_i}-1)}{e^{w_i}(t_a-t)}$. For all $t\geq t_a$, $\dot{w}_i=0$, for $i=1,2$. The comparison system is quasi-monotone non-decreasing and arbitrary time stable in time t_a with $\gamma_1>2$ and $\gamma_2>2$ as $p=2$. Hence, it follows that pitch and yaw regulate at the desired position in the set arbitrary time t_a with the designed control voltages V_{mp} and V_{my} respectively.

7.2.2 Experimental Results

Experimental validation is done with the designed controls assuming the gains $\gamma_1=\gamma_2=3.5$, $M=20, k_{p1}=1.5, k_{p2}=1$ for the set arbitrary times $t_a=15,20$ and 30 sec. The desired pitch and yaw angles are assumed to be 0.2 radians. The maximum allowed value for the control input voltage to the pitch and yaw are $\pm 24V$ and $\pm 15V$ respectively. The experimental and simulation responses of the pitch and yaw along with control inputs are shown in Fig. 7.4, 7.5 and 7.6. From these figures, it is clear that pitch and yaw track the desired angles in the set arbitrary time $t\leq t_a$. However, steady-state error is present which can be reduced to zero by using Integral control [97].

In the above analysis, if the constant reference becomes zero, then our regulation problem reduces to the stabilization problem. We provide experimental and simulation results to show stabilization of the pitch and yaw to the origin in the set arbitrary time $t_a=20$ sec in Fig. 7.7 assuming the gains $\gamma_1=\gamma_2=3.5$, $M=20, k_{p1}=1.5$ and $k_{p2}=1$.

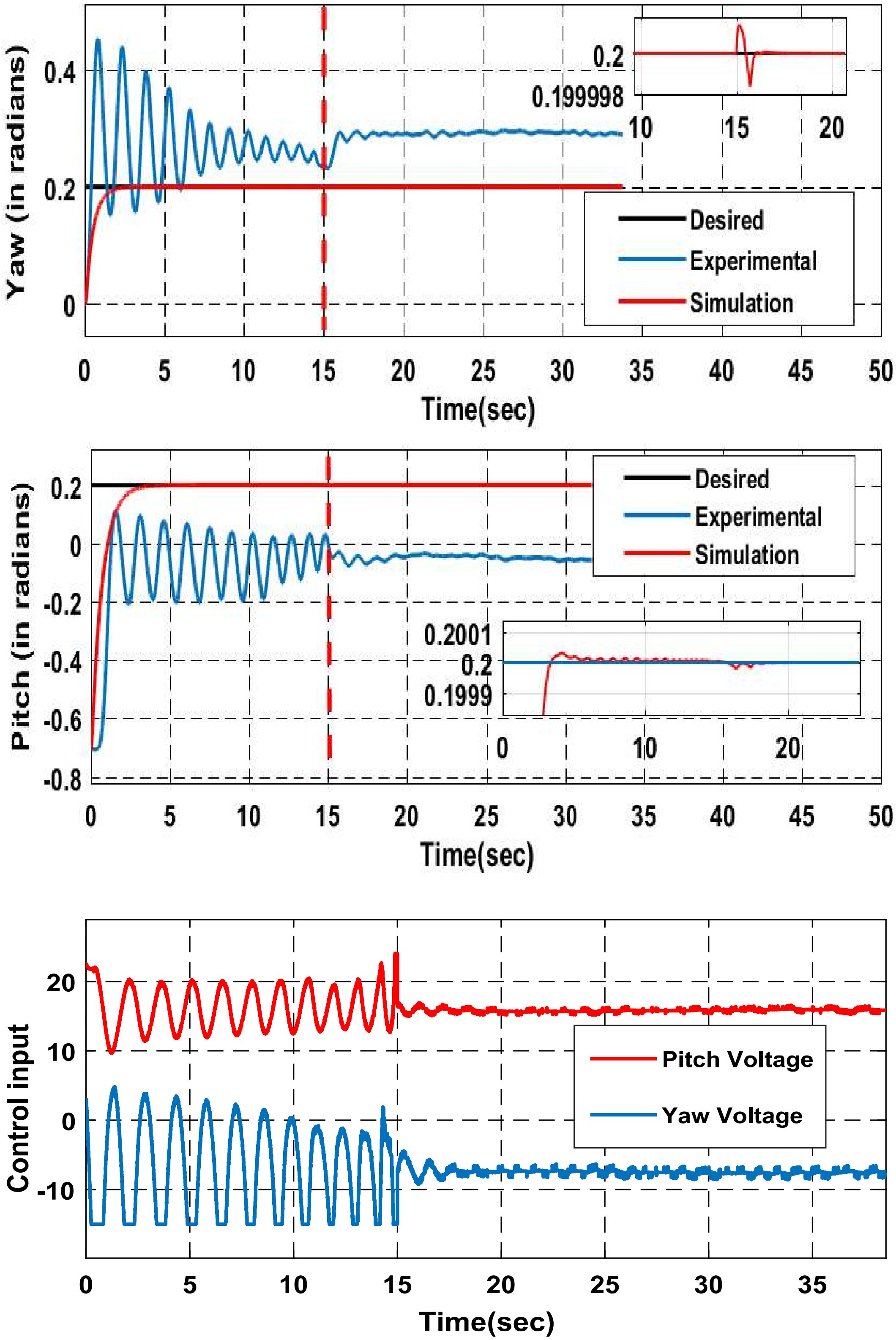

Figure 7.4: Yaw, Pitch and Control input responses of a 2 DOF Helicopter Model for the set arbitrary time $t_a = 15$ sec

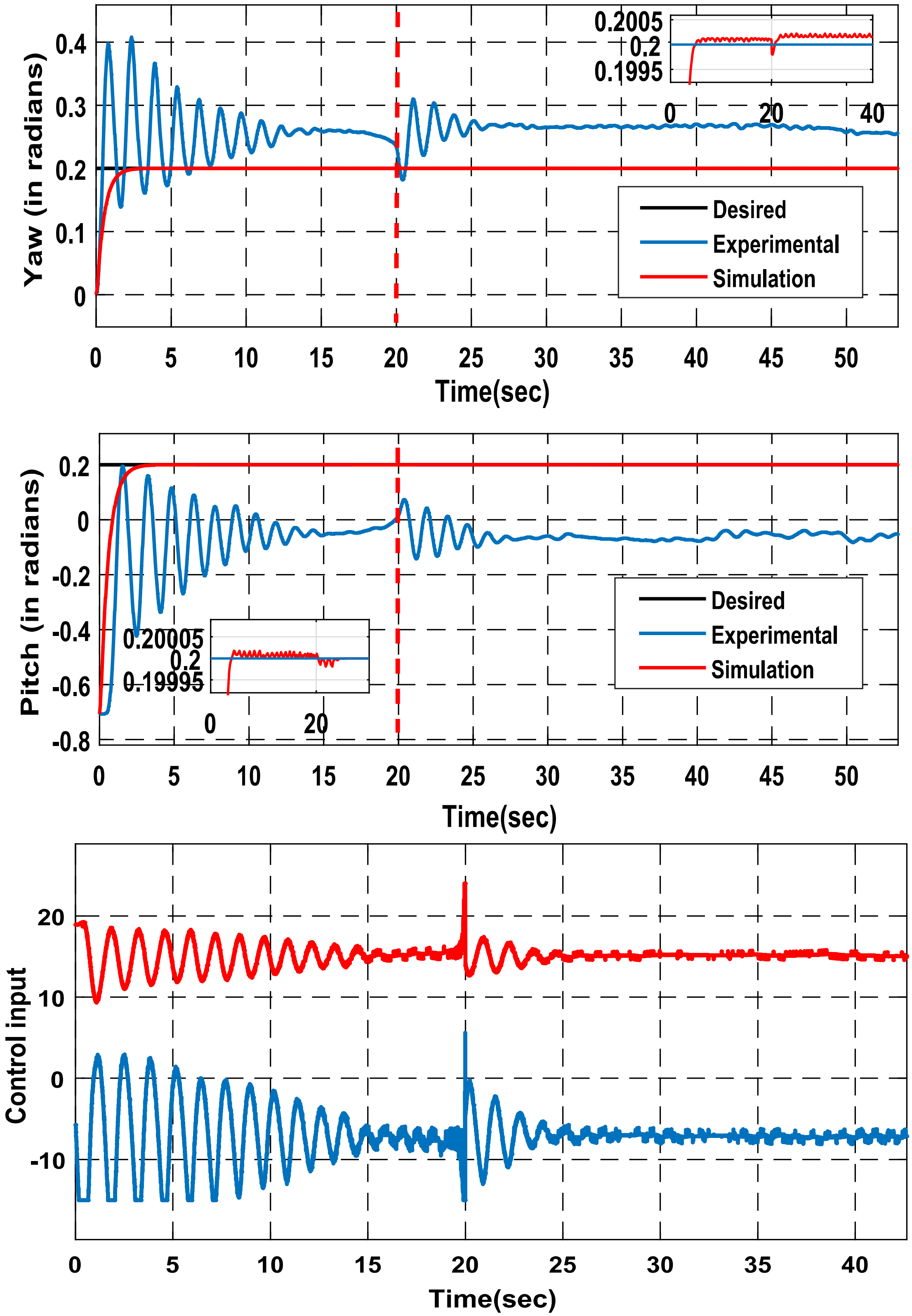

Figure 7.5: Yaw, Pitch and Control input responses of a 2 DOF Helicopter Model for the set arbitrary time $t_a = 20$ sec

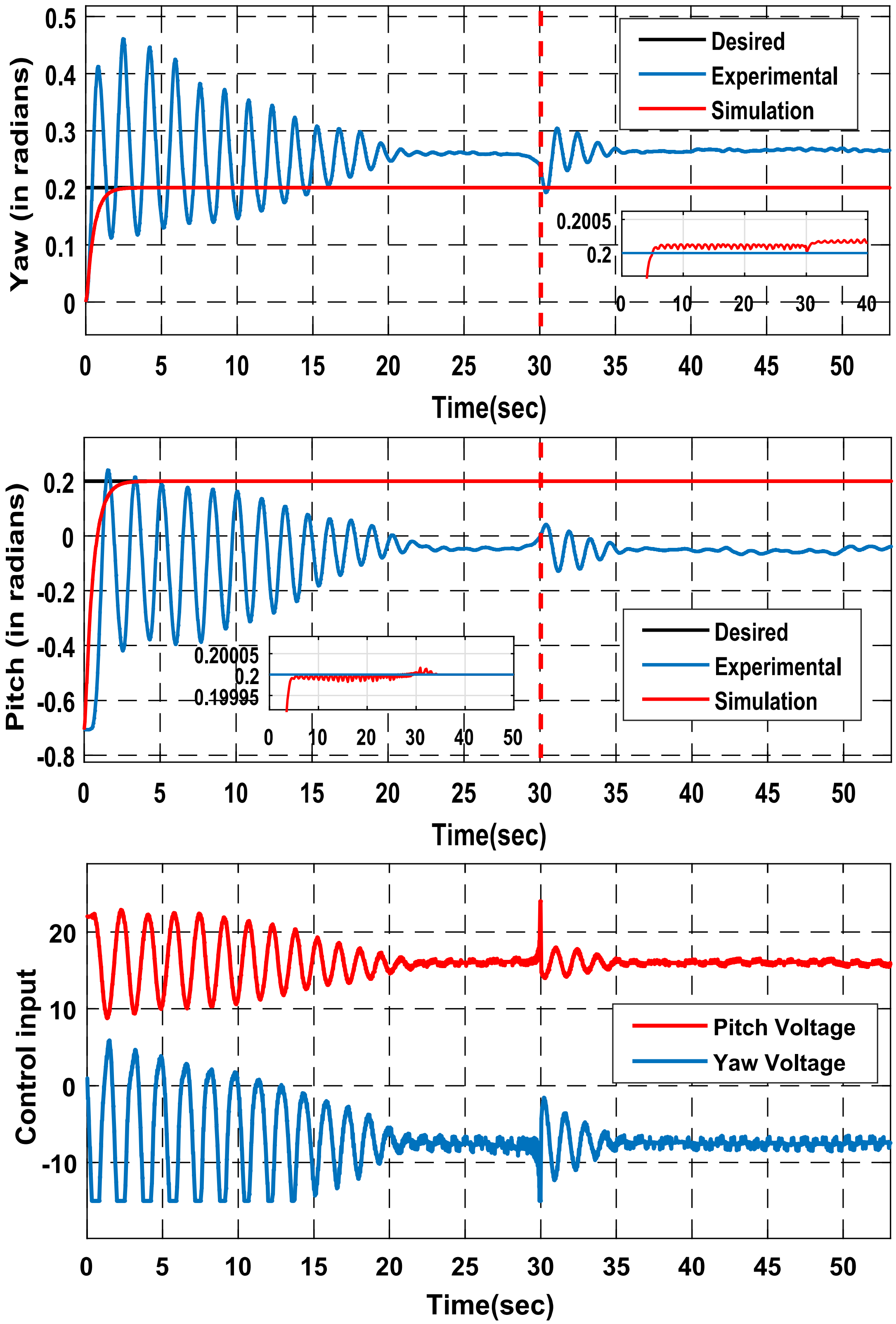

Figure 7.6: Yaw, Pitch and Control input responses of a 2 DOF Helicopter Model for the set arbitrary time $t_a = 30$ sec

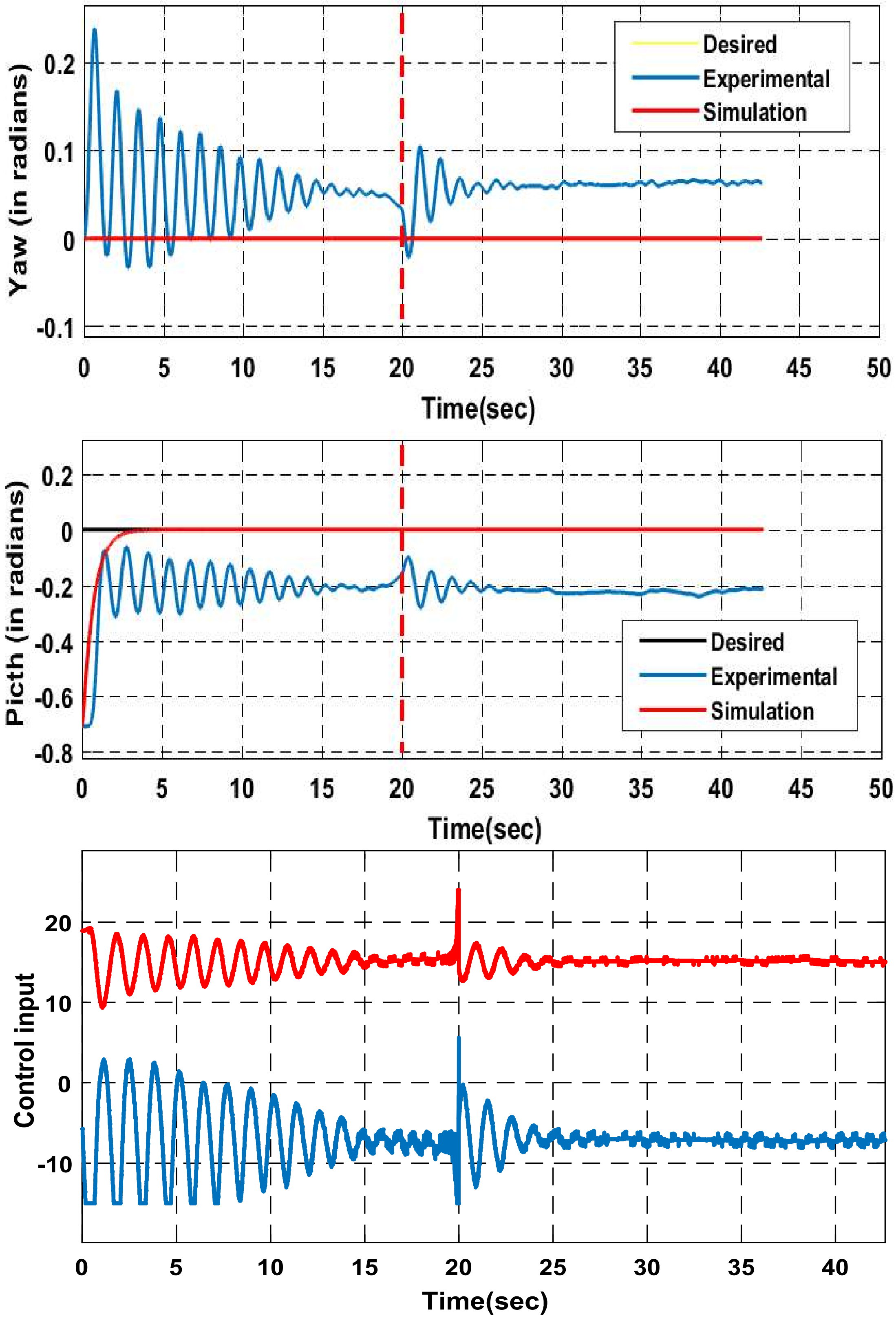

Figure 7.7: Yaw, Pitch and Control input responses of a 2 DOF Helicopter Model for the set arbitrary time $t_a = 20$ sec

7.3 Two link manipulator

Consider a nonlinear dynamical model of a two-link manipulator [97, 121] shown in Fig. 7.8:

$$M(x)\ddot{x} + C(x,\dot{x})\dot{x} + D\dot{x} + g(x) = u \tag{7.4}$$

where $x = [x_1, x_3]^\top$ denote joint positions, $\dot{x} = [x_2, x_4]^\top$ represent the joint velocities, $u = [u_1, u_2]^\top$ denote the control inputs and $M(x)$ is a positive definite symmetric matrix for all $x \in \mathbb{R}^2$. The term $C(x, \dot{x})$ denotes the centrifugal and Coriolis forces, $D\dot{x}$ is the viscous damping term and $g(x)$ denotes the gravity forces. The model parameters are taken from [92]:

$$M(x) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \; C(x,\dot{x}) = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & 0 \end{bmatrix}, \; D = 0$$

where $m_{11} = 9.77 + 2.02\cos(x_3)$, $m_{12} = m_{21} = 1.26 + 1.01\cos(x_3)$, $m_{22} = 1.12$, $c_{11} = -1.01\sin(x_3)x_4, c_{12} = -1.01\sin(x_3)(x_4 + x_2)$, and $c_{21} = 1.01\sin(x_3)x_2$. Our aim is to

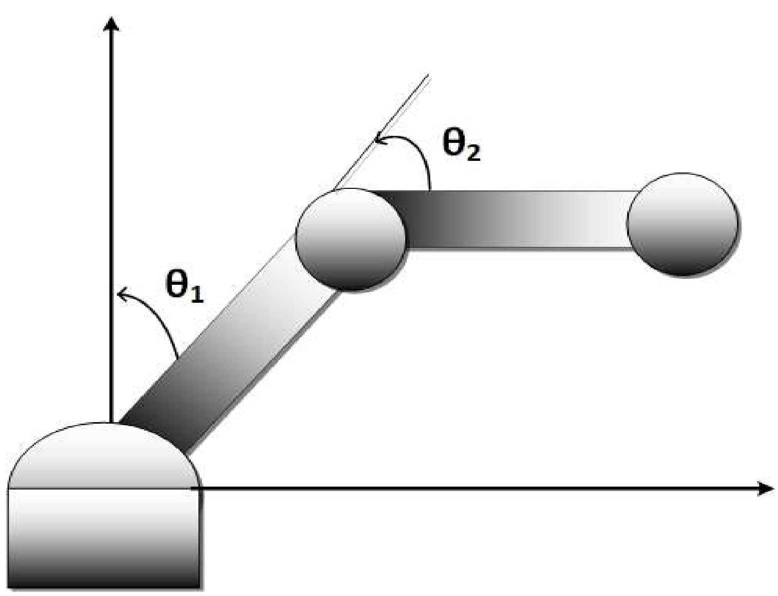

Figure 7.8: Two link manipulator

stabilize the system (7.4) at $x = 0, \dot{x} = 0$ in the set arbitrary time, but this point is not an equilibrium point when $u = 0$. Let

$$u = g(x) - k_p x + v$$

where k_p is a positive definite diagonal matrix with entries k_{p1} and k_{p2}, and $v = [v_1, v_2]^\top$ is a control to be chosen appropriately. Substituting u in the original dynamics (7.4), we get

$$M(x)\ddot{x} + C(x,\dot{x})\dot{x} + k_p x = v$$

This can be written as the set of state-space equations:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -k_{p1}L_1x_1 - c_{11}L_1x_2 - c_{12}L_1x_4 - \frac{L_1m_{12}c_{21}}{m_{22}}x_2 + \frac{L_1m_{12}k_{p2}}{m_{22}}x_3 + U_1 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -\frac{c_{21}}{m_{22}}x_2 - \frac{k_{p2}}{m_{22}}x_3 - k_{p1}L_2x_1 - c_{11}L_2x_2 - c_{12}L_2x_4 + \frac{L_2m_{12}c_{21}}{m_{22}}x_2 + \frac{L_2m_{12}k_{p2}}{m_{22}}x_3 + U_2
\end{aligned} \tag{7.5}$$

where $L_1 = \frac{m_{22}}{m_{11}m_{22}-m_{12}^2}$, $L_2 = \frac{m_{21}}{m_{12}^2-m_{11}m_{22}}$, $U_1 = -L_1v_1 + \frac{L_1m_{12}}{m_{22}}v_2$, and $U_2 = -\frac{v_2}{m_{22}} - L_2v_1 + \frac{L_2m_{12}}{m_{22}}v_2$.

Now, we discuss that how to design controls U_1 and U_2 using vector control Lyapunov functions and aggregation procedure described in Chapter 3 so that states of the system (7.4) stabilize at the origin in the set arbitrary time. Firstly we use the aggregation procedure discussed in Section 3.3. Let us apply the transformation as $z = \mathcal{T}\zeta$ with

$$\mathcal{T} = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 0 & 0 & 3 & 2 \end{bmatrix}, \ \zeta = [x_1, x_2, x_3, x_4]^\top, \ z = [z_1, z_2]^\top$$

to transform the system (7.5) into

$$\begin{aligned}
\dot{z}_1 &= E_1 + 2U_1 \\
\dot{z}_2 &= E_2 + 2U_2
\end{aligned} \tag{7.6}$$

where $E_1 = 0.4614z_1 - k_{p1}L_{11}0.4616z_1 - c_{111}L_{11}0.3076z_1 - c_{122}L_{11}0.3076z_2 + \frac{L_{11}m_{122}c_{211}0.3076}{m_{22}}z_1 - \frac{L_{11}m_{122}k_{p2}0.4616}{m_{22}}z_2$, $E_2 = 0.4614z_2 - \frac{0.1538c_{211}}{m_{22}}z_1 - \frac{0.4616k_{p2}}{m_{22}}z_2 - L_{22}k_{p1}0.4614z_1 - L_{22}c_{111}0.3076z_1 - L_{22}c_{122}0.3076z_2 + \frac{L_{22}k_{p1}0.4616z_1}{m_{22}} + \frac{L_{22}m_{122}k_{p2}0.4616z_2}{m_{22}}$, $L_{11} = \frac{m_{22}}{m_{111}m_{22}-m_{122}^2}$, $L_{22} = \frac{m_{211}}{m_{122}^2-m_{111}m_{22}}$, $m_{111} = 9.77 + 2.02\cos(0.0385z_2)$, $m_{122} = m_{211} = 1.26 + 1.01\cos(0.0385z_2)$, $c_{111} = -1.01\sin(0.0385z_2)(0.1923z_2)$, $c_{122} = -1.01\sin(0.0385z_2)(0.1923z_1 + 0.1923z_2)$, and $c_{211} = 1.01\sin(0.0385z_2)(0.1923z_1)$.

Let us consider the vector Lyapunov function, $V = [V_1, V_2]^\top$ with $V_1 = (z_1 - z_2)^2$ and $V_2 = (z_1 + z_2)^2$. It is easy to check that $r^\top V$ is positive definite, where $r = [1, 1]^\top$.

The derivative of V_1 along the trajectories of (7.6) is given by, $\dot{V}_1 = 2(z_1 - z_2)(\dot{z}_1 - \dot{z}_2)$ for all $t \in [t_0, t_a)$. Let us choose: $2U_1 - 2U_2 = \ E_2 - E_1 - \phi_1(t, z_1 - z_2) - M(z_1 - z_2)$, where $\phi_1(t, \cdot)$ is the arbitrary time term given by: $\phi_1(t, z_1 - z_2) = \frac{\gamma_1(e^{(z_1-z_2)}-1)}{e^{(z_1-z_2)}(t_a-t)}$ if $t \in [t_0, t_a)$, else $\phi_1(t, z_1 - z_2) = 0$ for $t \geq t_a$, $\gamma_1 > 1$ and $M > 0$. Then,

$$\dot{V}_1 \leq \frac{-2\gamma_1\sqrt{V_1}(e^{\sqrt{V_1}} - 1)}{e^{\sqrt{V_1}}(t_a - t)}$$

Similarly,

$$\dot{V}_2 \leq \frac{-2\gamma_2\sqrt{V_2}(e^{\sqrt{V_2}} - 1)}{e^{\sqrt{V_2}}(t_a - t)},$$

with $2U_1 + 2U_2 = \ -E_1 - E_2 - \phi_2(z_1 + z_2) - M(z_1 + z_2)$, where $\phi_2(t, \cdot)$ is the arbitrary time term given by: $\phi_2(t, z_1 + z_2) = \frac{\gamma_2(e^{(z_1+z_2)}-1)}{e^{(z_1+z_2)}(t_a-t)}$ if $t \in [t_0, t_a)$, else $\phi_2(t, z_1 + z_2) = 0$ for $t \geq t_a$, $\gamma_2 > 1$ and $M > 0$.

For all $t \geq t_a$, the designed controls U_1 and U_2 will maintain the dynamics (7.6) at the origin, hence, $\dot{V} = 0$ for all $t \geq t_a$. Now, let $w_i = \sqrt{V_i}$, $i = 1, 2$, then $\dot{w}_i = \frac{\dot{V}_i}{2\sqrt{V_i}} \leq \frac{-\gamma_i(e^{w_i}-1)}{e^{w_i}(t_a-t)}$. Thus, the comparison system constructed over $t \in [t_0, t_a)$ is $\dot{w}_i = \frac{-\gamma_i(e^{w_i}-1)}{e^{w_i}(t_a-t)}$. For all $t \geq t_a$, $\dot{w}_i = 0$, for $i = 1, 2$. The comparison system is quasi-monotone non-decreasing and arbitrary time stable in time t_a with $\gamma_1 > 2$ and $\gamma_2 > 2$ as $p = 2$. Hence, it follows that system states (7.4) stabilize at the origin in the set arbitrary time t_a respectively. In a similar way, results can be extended to solve regulation or tracking problems.

7.3.1 Simulation Results

We provide the simulation results to validate the above-derived results in Fig. 7.9 along with control inputs. We consider simulation data: $\gamma_1 = \gamma_2 = 2.5, k_{p1} = k_{p2} = 1, M = 1$ and $t_a = 5, 10$ sec. From this figure, it is clear that system states are stabilized within the set arbitrary time t_a.

Comparative analysis: It is to be noted that for many years, linear controllers such as PID and LQR controllers have been frequently used in most of the practical systems such as the Helicopter model and robotic system. However, these controllers work well for a linearized or approximated model of a nonlinear system providing local stability. In fact, a nonlinear controller in a nonlinear system brings simultaneous performance benefits than a linear scheme does not. Thus, the nonlinear control design scheme and its implementation in this chapter is appreciable in comparison to existing linear controllers. On the other hand, it is better than other existing rated convergence control algorithms such as finite-time and fixed-time cases in the essence that, it provides convergence of the system states in the arbitrary time set by the user, which is independent from initial conditions.

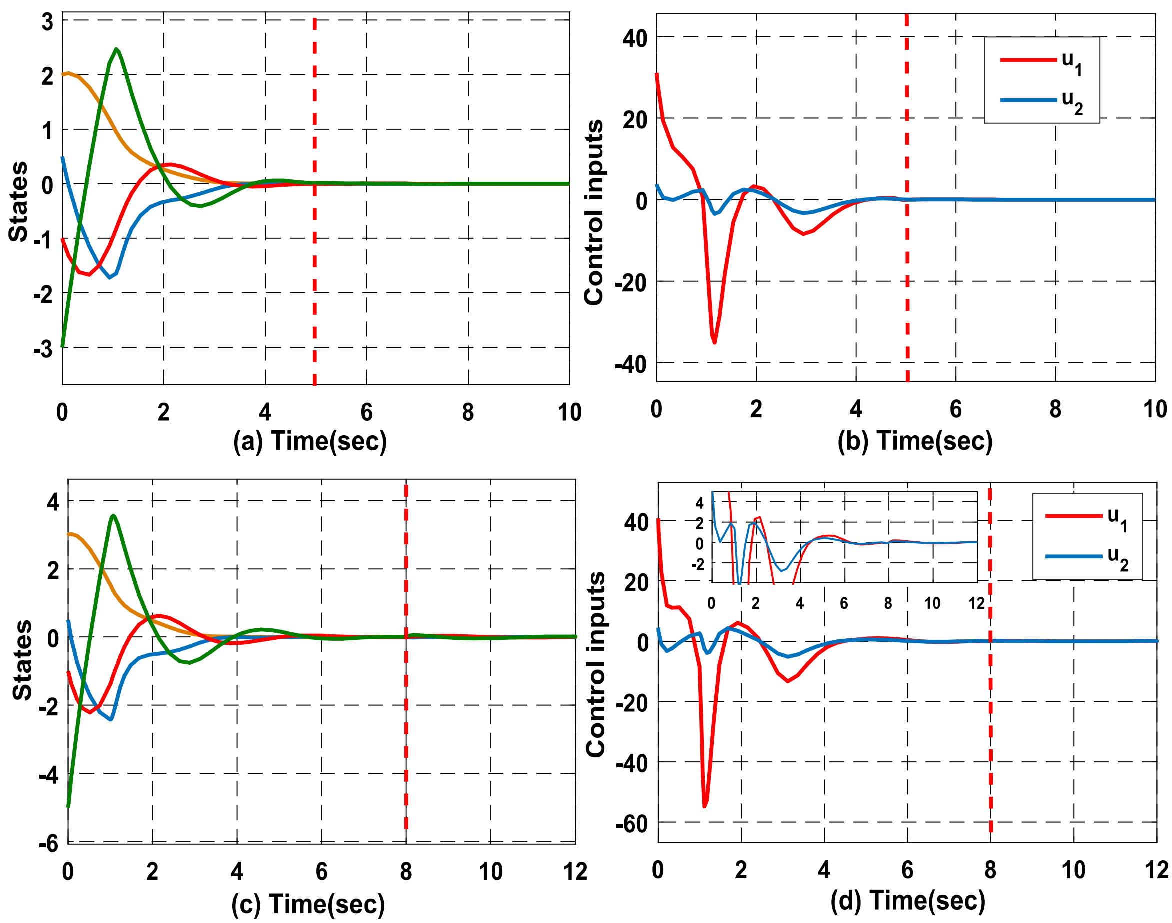

Figure 7.9: Evolution of system (7.4) states and control inputs with set arbitrary time $t_a = 5$ sec (a, b) and $t_a = 8$ sec (c, d)

7.4 Conclusion

In this chapter, we exploited vector framework based stability analysis tool, in particular, vector Lyapunov functions, to design control and implemented it experimentally on a Helicopter system to achieve arbitrary time convergence. Further, simulation results are provided for a two-link manipulator. Simulation and experimental results show that the proposed results are easy, efficient and straightforward to apply on practical and experimental systems.