

STUDY OF SOME PROBLEMS ON NEURAL
NETWORKS BASED ON THE LYAPUNOV AND
MATRIX MEASURE METHODS



*The thesis submitted in partial fulfilment
for the Award of Degree
DOCTOR OF PHILOSOPHY*

by

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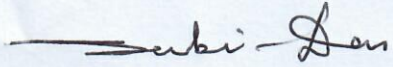
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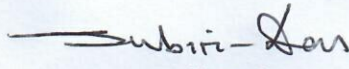
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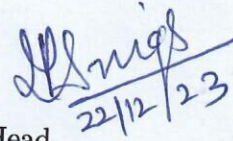
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Dedicated

to

my beloved *Family*

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Contents

List of Figures	xi
List of Tables	xv
Preface	xvii
1 Introduction	1
1.1 Background	1
1.2 Biological neurons	2
1.2.1 Biological model	6
1.3 Artificial Neural Network	11
1.3.1 Activation function	15
1.3.2 Major Architectures of Artificial Neural Networks.	19
1.3.3 Architecture of a Single-Layer Feedforward Network	20
1.4 Architectures of Multi-Layer Feedforward Networks	20
1.4.1 Architecture of Recurrent or Feedback Networks	22
1.4.2 Mesh Architectures	23
1.4.3 Basics of electrical circuit	25
1.4.4 Additive model	27
1.4.5 Hopfield neural network	29
1.4.6 Cohen-Grossberg neural network	33
1.5 Glimpse of Some Useful Mathematical Concepts	35
1.5.1 Stability Theory	35
1.5.2 Lyapunov Stability	37
1.5.3 Finite and Fixed Time Stability	38
1.5.4 Delay Differential Equations	40
1.5.5 Matrix measure theory	44
1.6 Synchronization	48
1.7 Global dissipativity	51
1.8 Inertial Neural Networks	51
1.9 Complex Valued Neural networks	53

1.10	Quaternion valued Neural networks	55
1.10.1	Background	55
1.10.2	Quaternion definition	57
2	Fixed time synchronization of quaternion-valued neural networks with mixed time-varying delay	61
2.1	Introduction	61
2.2	Model Description and Preliminaries	62
2.3	Main results	70
2.4	Numerical Example	76
2.5	Conclusion	79
3	Quasi Projective-synchronization of complex-valued recurrent neural networks with proportional delay and mismatched parameters via matrix measure approach	81
3.1	Introduction	81
3.2	Model Description and Preliminaries	83
3.3	Main results	88
3.4	Numerical Example	96
3.5	Conclusion	104
4	Global exponential stability of inertial Cohen-Grossberg neural networks with time-varying delays	107
4.1	Introduction	107
4.2	Model Description and Preliminaries	108
4.3	Global exponential stability	110
4.4	Numerical Example	122
4.5	Conclusion	126
5	Anti-synchronization of inertial neural networks with quaternion-valued and unbounded delays: Non-reduction and non-separation approach	127
5.1	Introduction	127
5.2	Model Description and Preliminaries	129
5.3	Main results	131
5.4	Numerical Examples	145
5.5	Conclusion	149
6	Global dissipativity analysis for a class of quaternion-valued neural networks with inertial term and mixed time delays: Non-separation approach	161
6.1	Introduction	161
6.2	Model Description and Preliminaries	163

6.3	Main Results	165
6.4	Numerical Examples	180
6.5	Conclusion	193
7	Conclusion and Future Works	195
	Bibliography	203
	List of Publications	225

List of Figures

1.1	Schematic diagram of a single neuron.	3
1.2	Diagram illustrating a neuron and a network: Neuron v_i has a potential $x_i(t)$ relative to equilibrium and sends a signal S_{ij} along its axon to a target neuron v_j . This signal impacts the target neuron with a coupling strength Z_{ij} , which can either be excitatory ($Z_{ij} > 0$) or inhibitory ($Z_{ij} < 0$). Furthermore, there could be additional inputs I_i to the neuron, representing external stimuli in the neuron model.	7
1.3	Fundamental diagram of an artificial neuron of the network.	13
1.4	(a) Sigmoidal function, (b) tanh function, (c) Piecewise activation function, (d) Threshold function, (e) RELU function	19
1.5	Single-layer feedforward network	21
1.6	Multi-layer feedforward network	22
1.7	Recurrent feedforward network	23
1.8	Structure of a Mesh network	23
1.9	RC-circuit with a source of voltage.	25
1.10	Additive model of a neuron.	27
1.11	An architecture of Hopfield neural network consisting of $n = 4$ neurons.	30
1.12	Encoding-Decoding illustration between the space of fundamental memories ξ_μ and the space of stored vectors x_μ	30
1.13	Phase portrait for stable and unstable equilibrium point	39
2.1	Plots of trajectories of $z_1^R(t)$ and $s_1^R(t)$, $z_1^I(t)$ and $s_1^I(t)$, $z_1^J(t)$ and $s_1^J(t)$, $z_1^K(t)$ and $s_1^K(t)$ for the drive-response systems (2.3) and (2.4) without controllers.	77
2.2	Plots of the trajectories of $e_1^R(t)$, $e_1^I(t)$, $e_1^J(t)$, $e_1^K(t)$ and $e_2^R(t)$, $e_2^I(t)$, $e_2^J(t)$, $e_2^K(t)$ error systems (2.5).	78
3.1	The state trajectories of the system (3.19) in three-dimensional space for (a) $\alpha_1(t)$, $\beta_1(t)$, and (b) $\alpha_2(t)$ and $\beta_2(t)$	100
3.2	The state trajectories of the systems (3.19) and (3.20) for (a) $\alpha_1(t)$, $\tilde{\alpha}_1(t)$, and (b) $\alpha_2(t)$, $\tilde{\alpha}_2(t)$ with time t for the case I.	101
3.3	The state trajectories of the systems (3.19) and (3.20) for (a) $\beta_1(t)$, $\tilde{\beta}_1(t)$, and (b) $\beta_2(t)$, $\tilde{\beta}_2(t)$ with time t for the case I.	101

3.4	Plots of the error system (3.7) for (a) without controllers and (b) with controllers for case I.	102
3.5	The state trajectories of the systems (3.19) and (3.20) for (a) $\alpha_1(t)$, $\tilde{\alpha}_1(t)$, and (b) $\alpha_2(t)$, $\tilde{\alpha}_2(t)$ with time t for the case II.	102
3.6	The state trajectories of the systems (3.19) and (3.20) for (a) $\beta_1(t)$, $\tilde{\beta}_1(t)$, and (b) $\beta_2(t)$, $\tilde{\beta}_2(t)$ with time t for the case II.	103
3.7	Plots of the system (3.7) for (a) without controllers and (b) with controllers for case II.	103
3.8	Plots of the system (3.7) for (a) without controllers and (b) with controllers for case III.	104
4.1	Phase trajectory in the absence of any control schemes of the system (4.28)	124
4.2	Stabilization of considered model (4.28) by feedback controller	124
4.3	Stabilization of considered model (4.28) by adaptive controller	125
4.4	Evolution of adaptive control gains	125
5.1	Time behaviour of state variables q_1^R, q_2^R and q_1^I, q_2^I of the system (5.27)	151
5.2	Time behaviour of state variables q_1^J, q_2^J and q_1^K, q_2^K of the system (5.27)	151
5.3	Anti-synchronization of q_1^R, v_1^R and q_2^R, v_2^R with control (5.3)	152
5.4	Anti-synchronization of q_1^R, v_1^R and q_2^R, v_2^R with control (5.3)	152
5.5	Anti-synchronization of q_1^I, v_1^I and q_2^I, v_2^I with control (5.3)	153
5.6	Anti-synchronization of q_1^J, v_1^J and q_2^J, v_2^J with control (5.3)	153
5.7	Error $\varepsilon_1^R, \varepsilon_2^R$ and $\varepsilon_1^I, \varepsilon_2^I$ with the control (5.3)	154
5.8	Error $\varepsilon_1^J, \varepsilon_2^J$ and $\varepsilon_1^K, \varepsilon_2^K$ with the control (5.3)	154
5.9	Anti-synchronization of q_1^R, v_1^R and q_2^R, v_2^R with control (5.17)	155
5.10	Anti-synchronization of q_1^I, v_1^I and q_2^I, v_2^I with control (5.17)	155
5.11	Anti-synchronization of q_1^J, v_1^J and q_2^J, v_2^J with control (5.17)	156
5.12	Anti-synchronization of q_1^K, v_1^K and q_2^K, v_2^K with control (5.17)	156
5.13	Error $\varepsilon_1^R, \varepsilon_2^R$ and $\varepsilon_1^I, \varepsilon_2^I$ with the control (5.17)	157
5.14	Error $\varepsilon_1^J, \varepsilon_2^J$ and $\varepsilon_1^K, \varepsilon_2^K$ with the control (5.17)	157
5.15	Time response of control gains $\alpha_1, \alpha_2, \beta_1, \beta_2$ (5.17)	158
5.16	Original color image of pattern "S"	158
5.17	Simulation of retrieving image "S" with random initial values of time t .	159
6.1	(a) Phase plots of state variables r_1^R, r_2^R and (b) r_1^I, r_2^I of the system (6.12).	186
6.2	(c) Phase plots of state variables r_1^J, r_2^J and (d) r_1^K, r_2^K of the system (6.12).	187
6.3	(a), (b) Time response of state variables r_1^R, r_2^R and r_1^I, r_2^I of the system (6.12) respectively.	187
6.4	(a), (b) Time response of state variables r_1^J, r_2^J and r_1^K, r_2^K of the system (6.12) respectively.	188

6.5	(a), (b) Time response of state variables r_1^R, r_2^R and r_1^I, r_2^I of the system (6.12) with u_1^R, u_2^R, u_1^I and $u_2^I = 0$.	188
6.6	(a), (b) Time response of state variables r_1^J, r_2^J and r_1^K, r_2^K of the system (6.12) with u_1^J, u_2^J, u_1^K and $u_2^K = 0$.	189
6.7	(a) Phase plots of state variables r_1^R, r_2^R and (b) r_1^I, r_2^I of the system (6.14).	189
6.8	(c) Phase plots of state variables r_1^J, r_2^J and (d) r_1^K, r_2^K of the system (6.14).	190
6.9	(a), (b) Time response of state variables r_1^R, r_2^R and r_1^I, r_2^I of the system (6.14) respectively.	190
6.10	(c), (d) Time response of state variables r_1^J, r_2^J and r_1^K, r_2^K of the system (6.14) respectively.	191
6.11	(a), (b) Time response of state variables r_1^R, r_2^R and r_1^I, r_2^I of the system (6.12) with u_1^R, u_2^R, u_1^I and $u_2^I = 0$.	191
6.12	(c), (d) Time response of state variables r_1^J, r_2^J and r_1^K, r_2^K of the system (6.12) with u_1^J, u_2^J, u_1^K and $u_2^K = 0$.	192
6.13	Original color image of pattern "T."	192
6.14	Simulation of retrieving image "T" with random initial values of time t .	193

List of Tables

1.1 Comparison between artificial and biological neural network.	12
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