Chapter 1

Introduction

1.1 Motivation

Domain decomposition methods and operational matrices provide a powerful technique for approximating solutions of partial differential equations (PDEs), partial integro-differential equations (PIDEs) and fractional partial differential equations (FPDEs). The basic idea behind domain decomposition method is to split the computational domain into smaller sub-domains with or without overlap and then apply numerical techniques such as wavelet operational matrices for approximately solving the underlying proposed problems on the sub-domains independently. The operational matrices also reduces existing singularities in PDEs, PIDEs and FPDEs and converts it into algebraic equations.

So for, there is a great deal of literature devoted to differential equations, integral equations and integro-differential equations, but there are only a few papers (see for instant [?], [2], [3], [4]) available for solving PDEs, PIDEs and FPDEs, based on two dimensional operational matrix methods. In this thesis, we discuss two dimensional operational matrix methods for PDEs, PIDEs and FPDEs.

In chapter 2, we have used two dimensional operational matrices based on two dimensional polynomials approximation.

In chapter 3-5, we have used domain decomposition method and two dimensional operational matrices based on two dimensional wavelets approximation.

The thesis has contained three types of wavelets (Legendre, Chebyshev, and Bernoulli) and all these wavelets may be deals with domain decomposition methods for solving each proposed problems based on operational matrices. Moreover, we have analyzed convergence analysis of approximations and have derived error estimations.

1.2 Introduction

The main objective of this thesis is to analyze domain decomposition method for partial differential equations, fractional partial differential equations and partial integro-differential equations based on operational matrices. At the beginning, we would like to mention that the thesis may be decomposed into three parts. The part I (chapters 2 and 3) deals fractional partial differential equation via two-dimensional shifted Legendre polynomial, Legendre wavelet and Chebyshev wavelet based on operational matrices. In part II (chapter 4), we have deal partial integro-differential equation via two-dimensional Legendre and Chebyshev wavelets based on operational matrices and finally in part III (chapter 5), we have deal complex partial differential equation via two-dimensional Legendre and Bernoulli wavelets based on operational matrices.

1.3 Review of Wavelets

As we know that mathematics is subdivided in several specialized areas, most of areas have attained such a technical sophistication that they become rather inaccessible for the non-specialist. Sometimes, but not so often, a novel concept breaks through which reverts this tendency, because it uses techniques from , and is relevant for many different areas of mathematics and has, moreover, many applications of very different nature. An example of such a concept is given by wavelets. Truly, the range of pure and applied fields touched by wavelets are extremely wide areas. Wavelets are widely used in approximation theory, functional analysis, signal analysis, operator theory, image processing, computer science, electrical engineering and physics, etc.

Nowadays, “Wavelets” has been a very popular topic of conversations in many areas of scientific and engineering aspects. Some view wavelets as a new basis for representing functions, some consider it as a technique for time-frequency analysis, and others think of it as a new mathematical subject. Of course, all of them are right, since "wavelets" is a versatile tool with very rich mathematical content and great potential for applications. However, as this subject is still in the midst of rapid development, it is definitely too early to give a unified presentation.

The first use of wavelets was by Haar (1909) [5]. He was interested in finding a basis on a functional space similar to Fourier's basis in frequency space. In physics, wavelets were used in the characterization of Brownian motion. This work led to some of the ideas used to construct wavelet bases. Wavelets were also used for analysis of coherent states of a particular quantum system and in the signal processing field, S. Mallat ([6]) discovered that filter banks have important connections with wavelet basis functions.

Wavelets attracted attention in the 1980-90's through the work of several researchers form various disciplines Stromberg [7], Morlet ([8], [9]), Grossmann [10]), Meyer ([11], [12], [13], [14], [15]), Battle([16], [17], [18], [19]), Lemarie([20], [21], [22]), Daubechies([23], [24], [25], [26], [27]), Mallat ([28], [29], [30], [31]), Coifman([32]), Chui([33], [34], [35]) to name a few.

There are several reasons for their present success. On the other hand, the concept of wavelets can be viewed as a synthesis of ideas which originated during the last forty or fifty years in engineering (like sub band coding), physics (like: coherent states and re-normalization group), pure and applied mathematics.

Wavelets are a fairly simple mathematical tool with a great variety of possible applications. The wavelet analysis is useful for problems in many applied discipline as well as within mathematics itself tells us that there is something special about it: Wavelet analysis provides a systematic new way to represent and analyze multi-scale structures. The prevalence of multi-scale structure in nature and in engineering is one reason that wavelets are broadly valuable. Wavelet analysis is also a far-reaching generalization of orthogonal bases of functions whose particular new contribution is a systematic way to represent functions on unbounded domains by linear combinations of orthogonal basis functions that are compactly supported and overlapped.

A wavelet is a wave-like oscillation with an amplitude that begins at zero, increases, and then decreases back to zero. It can typically be visualized as a "brief oscillation" like one might see recorded by a seismograph or heart monitor. Generally, wavelets are purposefully crafted to have specific properties that make them useful for signal processing. Wavelets can be combined, using a "reverse, shift, multiply and integrate" technique called convolution, with portions of a known signal to extract information from the unknown signal.

As a mathematical tool, wavelets can be used to extract information from many different kinds of data, including but certainly not limited to audio signals and images. Also, a wavelet is a mathematical function useful in digital signal processing and image compression. In signal processing, wavelets make it possible to recover weak signals from noise. This has proven useful especially in the processing of X-ray and magnetic-resonance images in medical applications. Images processed in this way can be "cleaned up" without blurring or muddling the details.

The subject of wavelets has taken a place in the heart of science, engineering, and mathematics. Wavelet analysis is an exciting new method for solving difficult problems in mathematics, physics, and engineering, with modern applications as diverse as wave propagation, data compression, signal processing, image processing, pattern recognition, computer graphics, the detection of aircraft and submarines and other medical image technology. Wavelets allow complex information such as music, speech, images and patterns to be decomposed into elementary forms at different positions and scales and subsequently reconstructed with high precision. Signal transmission is based on transmission of a series of numbers. The series representation of a function is important in all types of signal transmission. The wavelet representation of a function is a new technique. Wavelet transform of a function is the improved version of Fourier transform. Fourier transform is a powerful tool for analyzing the components of a stationary signal. But it is failed for analyzing the non stationary signal where as wavelet transform allows the components of a non-stationary signal to be analyzed.

The family of wavelets have been considered due to their useful properties. As the contribution of wavelets by Legendre, Chebyshev and Bernoulli wavelet based solution of partial differential equations, fractional partial differential equations and partial integro-differential equations gained momentum in attractive way. Advantages of Wavelets bases over operational matrix method have led to tremendous application in science and engineering. In the present work two dimensional wavelets have been applied to solve partial differential equations, fractional partial differential equations, and Integral equations. Solution obtained numerically has been compared with exact solution. The good agreement of mathematical results, with the exact solution proves the accuracy and efficiency of wavelets operational matrix methods. Also, the wavelets operational matrix method is simple, efficient and produces very accurate numerical results in considerably small number of basis functions and hence reduces computational effort. Moreover, the technique is easy to apply for multidimensional problems.

1.4 Preliminaries

1.4.1 Definition of wavelets

An oscillatory function $\Phi(t) \in L^2(\mathbf{R})$ is a **wavelet** if it has the desirable properties:

- **Smoothness** $\Phi(t)$ is an n times differentiable and that their derivatives are continuous.

- **Localization** $\Phi(t)$ is well localized both in time and frequency domains, i.e., $\Phi(t)$ and its derivatives must decay very rapidly. For frequency localization $\widehat{\Phi}(\omega)$ must decay sufficiently fast as $|\omega| \to \infty$ and that $\widehat{\Phi}(\omega)$ becomes flat in the neighborhood of $\omega = 0$. The flatness is associated with number of vanishing moments of $\Phi(t)$, i.e.,

$$\int_{-\infty}^{\infty} x^k \Phi(t) dk = 0,$$

or
$$\frac{\partial^k}{\partial \omega^k} \widehat{\Phi}(\omega) = 0, \qquad k = 0, 1, ..., n,$$

in this sense that larger the number of vanishing moments more is the flatness when ω is small.

- **Admissibility condition**

$$\int_{-\infty}^{\infty} \frac{|\widehat{\Phi}(\omega)|^2}{\omega} d\omega < \infty$$

suggests that $|\widehat{\Phi}(\omega)|^2$ decay at least as $|\omega|^{-1}$ or $|x|^{\epsilon-1}$ for $\epsilon > 0$. and also $\widehat{\Phi}$ is a continuous function and it implies $\widehat{\Phi}(0) = 0$ or equivalently,

$$\int_{-\infty}^{\infty} \Phi(t) dt = 0.$$

1.4.2 Basis of wavelets

Wavelets basis constitute a family of functions constructed from dilation and translation of single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets:

$$\Phi_{a,b}(t) = |a|^{-\frac{1}{2}} \Phi\left(\frac{t-b}{a}\right), a, b \in \mathbf{R}, a \neq 0. \tag{1.1}$$

If the parameter a and b are restricted to the discrete values as $a = a_0^{-k,}, b = nb_0 a_0^{-k}, a_0 > 1, b_0 > 0$, n and k are positive integers, from the $Eq.(1.1)$, we have the following family of discrete wavelets :

$$\Phi_{k,n}(t) = |a_0|^{\frac{k}{2}} \Phi(a_0^k t - nb_0), \tag{1.2}$$

where, In Eq.(1.2), $\Phi_{k,n}(t)$ form a wavelet basis for $L^2(\mathbf{R})$.In particular, when $a_0 = 2$ and $b_0 = 1$ then $\Phi_{k,n}(t)$ form an orthonormal basis (see [36]).

1.4.3 Orthonormality of wavelets

A function $\Phi \in L^2(\mathbf{R})$ is called an **orthogonal wavelet (or orthonormal wavelet)**, if the family $\{\Phi_{j,k}\} = 2^{j/2}\Phi(2^j x - k), \quad j,k \in \mathbf{Z}$, is an orthonormal basis of $L^2(\mathbf{R})$; that is

$$\langle \Phi_{j,k}, \Phi_{l,m} \rangle = \delta_{j,l}\delta_{k,m}, \qquad j,k,l,m \in \mathbf{Z},$$

and every $f \in L^2(\mathbf{R})$ can be written as

$$f(t) = \sum_{j,k=-\infty}^{\infty} c_{j,k}\Phi_{j,k}(t), \tag{1.3}$$

where the **convergence of the series** in (1.3) is in $L^2(\mathbf{R})$, namely:

$$\lim_{M_1,N_1,M_2,N_2 \to \infty} \|f - \sum_{j=-M_2}^{N_2} \sum_{k=-M_1}^{N_1} c_{j,k}\Phi_{j,k}\|_2 = 0$$

The simplest **example** of an orthogonal wavelet is the **Haar function** Φ_H defined by

$$\Phi_H^{(1)}(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2}, \\ -1, & \frac{1}{2} \le t < 1, \\ 0, & \text{otherwise.} \end{cases}$$

The series representation of f in (1.3) is called a **wavelet series**. Analogous to the notion of wavelet coefficients $c_{j,k}$ are given by

$$c_{j,k} =< f, \Phi_{j,k} > .$$

1.4.4 Legendre Polynomials

The well-known Legendre polynomials are defined on the interval [-1,1], which are orthogonal with respect to the weight function $w(y) = 1$ on the interval $[-1, 1]$, and can be determined with the aid of the following recurrence formulae:

$$(m+1)L_m(y) = (2m+1)yL_m(y) - mL_{m-1}(y), \quad m \in N,$$

where,

$$L_0(y) = 1, L_1(y) = y,$$

$$L_{m+1}(y) = \frac{2m+1}{m+1}yL_m - \frac{m}{m+1}L_{m-1}(y), \quad m = 1, 2, ..., \quad -1 \leq y \leq 1.$$

In order to use Legendre polynomials on the interval $[0, 1]$, we define the so-called shifted Legendre polynomials by introducing the change of variable $y = 2x - 1$. Let the shifted Legendre polynomials $L_m(2x-1)$ be denoted by $P_m(x)$. Then $P_m(x)$ can be obtained as follows:

$$(m+1)P_{m+1}(x) = (2m+1)(2x-1)P_m(x) - mP_{m-1}(x), m = 1, 2, 3, ...,$$

where, $P_0(x) = 1$ and $P_1(x) = 2x - 1$. The shifted Legendre polynomial $P_m(x)$ has the following analytic form

$$P_m(x) = \sum_{k=0}^{m} (-1)^{m+k} \frac{(m+k)!x^k}{(m-k)!(k!)^2},$$

and the orthogonality condition is

$$\int_0^1 P_m(x)P_n(x)dx = \begin{cases} \frac{1}{2m+1}, & m = n, \\ 0, & m \neq n. \end{cases} \tag{1.4}$$

1.4.5 Two-dimensional shifted Legendre polynomials

2D shifted Legendre polynomials (TDSLPs) are defined on $([0,1]\times[0,1])=\Omega$ as follows:

$$\phi_{nm}(x,y)=P_n(x)P_m(y), \qquad n,m=0,1,2,... \tag{1.5}$$

here P_n and P_m are the well-known Legendre function of order n and m, which are defined on the interval [0,1], $P_n(x)$ TDSLPs are orthogonal with respect to weight function $\omega(t,x)$ such that

$$\int_0^1\int_0^1 \omega(t,x)\phi_{nm}(x,y)\phi_{ij}(x,y)dxdy=\begin{cases}\frac{1}{(2n+1)(2m+1)}, & i=n, j=m\\ 0, & \text{otherwise.}\end{cases} \tag{1.6}$$

1.4.6 One dimensional Legendre wavelets

The Legendre wavelets is constructed from Legendre polynomial. The Legendre functions satisfy the Legendre differential equation. Legendre wavelet $\Phi^C_{n,m}(t)=\Phi^C(k,\hat{n},m,t)$ have four arguments; $\hat{n}=2n-1$, $k\in N$, $n=1,2,...,2^{k-1}$, and m is the degree of the Legendre polynomial and t is the normalized time. One dimension Legendre wavelets $\Phi^L_{n,m}(x)$ over $[0,1]$ defined as follows [36]:

$$\Phi^L_{n,m}(x)=\begin{cases}\sqrt{m+\frac{1}{2}}\,2^{\frac{k}{2}}P_m(2^{k-1}x-\hat{n}) \ , & \frac{\hat{n}-1}{2^k}\le x<\frac{\hat{n}+1}{2^k},\\ 0, & \text{otherwise,}\end{cases} \tag{1.7}$$

where, $m=0,1,2,...,M-1, n=1,2,3,...,2^{k-1}$.

Since $L^2(\Omega)$ is the inner product space so the inner product of this space is defined as follows :

$$<\Phi^L_{n,m}(x),\Phi^L_{n',m'}(x)>=\int_0^1 \omega(x)\Phi^L_{n,m}(x)\Phi^L_{n',m'}(x)dx.$$

1.4.7 Two dimensional Legendre wavelets

Two dimensional Legendre wavelets in L^2 over Ω in terms of one dimensional Legendre wavelets can be express as follows :

$$\Phi^L_{n,m,n',m'}(x,y) = \begin{cases} \Phi^L_{n,m}(x)\Phi^L_{n',m'}(y) \quad , & \text{for } \frac{\hat{n}-1}{2^k} \le x < \frac{\hat{n}+1}{2^k}, \\ & \frac{\hat{n'}-1}{2^{k'}} \le y < \frac{\hat{n'}+1}{2^{k'}}, \\ 0, & \text{otherwise,} \end{cases} \tag{1.8}$$

where,

$$\begin{aligned} \Phi^L_{n,m}(x) &= \left(m+\frac{1}{2}\right)^{\frac{1}{2}} 2^{\frac{k}{2}} P_m(2^k x - \hat{n}), \\ \Phi^L_{n',m'}(y) &= \left(m'+\frac{1}{2}\right)^{\frac{1}{2}} 2^{\frac{k'}{2}} P_{m'}(2^{k'} y - \hat{n'}), \end{aligned} \tag{1.9}$$

and $m = 0,1,2,...,M-1, m' = 0,1,2,...,M'-1, n = 1,2,3,...,2^{k-1}, n' = 1,2,3,...,2^{k'-1}$, here, P_m and $P_{m'}$ are Legendre functions of order m and m' defined over the interval $[0,1]$ and also two-dimensions Legendre wavelets are orthonormal set over Ω. Since $L^2(\Omega)$ is the inner product space so the inner product of this space is defined as follows :

$$< \Phi^L_{n,m}(x,y), \Phi^L_{n',m'}(x,y) >= \int_0^1 \int_0^1 \omega(x,y)\Phi^L_{n,m}(x,y)\Phi^L_{n',m'}(x,y)dx.$$

1.4.8 One-dimensional Chebyshev wavelet

Chebyshev wavelet $\Phi^C_{n,m}(t) = \Phi^C(k,\hat{n},m,t)$ have four arguments; $\hat{n} = 2n-1$, $k \in N$, $n = 1,2,...,2^{k-1}$ and m is the degree of the Chebyshev polynomial of the fist kind and t is the normalized time i.e. $t \in [0,1]$. They are defined on the interval $[0,1]$ as follows [37]:

$$\Phi^C_{n,m}(t) = \Phi^C(k,\hat{n},m,t) = \begin{cases} 2^{k/2}\widehat{T}_m(2^k t - \hat{n}), & \text{if } \frac{\hat{n}-1}{2^k} \le t < \frac{\hat{n}+1}{2^k}; \\ 0, & \text{otherwise,} \end{cases} \tag{1.10}$$

where,

$$\widehat{T}_m(t) = \begin{cases} \frac{1}{\sqrt{\pi}}, & \text{if } m = 0; \\ \sqrt{\frac{2}{\pi}} T_m(t), & \text{if m>0}, \end{cases} \tag{1.11}$$

$m = 0, 1, ..., M-1$, and M is a fixed positive integer. Here, $\{T_m(t), m \in N \cup \{0\}\}$ is the set of well-known Chebyshev polynomials of degree m which are orthogonal with respect to the weight function $w(t) = \frac{1}{\sqrt{1-t^2}}$ on the interval $[-1, 1]$ and satisfy the following recursive formula:

$$T_0(t) = 1,\ T_1(t) = t,\ T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t), \quad m = 1, 2, ...$$

Remark 1 *A recurrence relation in the Chebyshev polynomial which is used for convergence analysis as follows:*

$$T_m(t) = \frac{1}{2}\left(\frac{T'_{m+1}(t)}{m+1} - \frac{T'_{m-1}(t)}{m-1}\right).$$

We should note that in dealing with Chebyshev polynomials the weight function $\widehat{w}(t) = w(2t-1)$ have to be dilated and translated as $w_n(t) = w(2^k t - 2n + 1)$ to get orthogonal wavelets.

1.4.9 Two-dimensional Chebyshev wavelet

Two-dimensional Chebyshev wavelets can be written in the product of one-dimensional Chebyshev wavelets as follows:

$$\Phi^C_{n,m,n',m'}(x,y) = \begin{cases} \Phi^C_{n,m}(x)\Phi^C_{n',m'}(y) \quad , & \text{for } \frac{\hat{n}-1}{2^k} \le x < \frac{\hat{n}+1}{2^k}, \\ & \frac{\hat{n}'-1}{2^{k'}} \le y < \frac{\hat{n}'+1}{2^{k'}}, \\ 0, & \text{otherwise}, \end{cases} \tag{1.12}$$

where,

$$\begin{aligned}\Phi^C_{n,m}(x) &= 2^{\frac{k}{2}} T_m(2^k x - \hat{n}),\\ \Phi^C_{n',m'}(y) &= 2^{\frac{k'}{2}} T_{m'}(2^{k'} y - \hat{n'}),\end{aligned} \tag{1.13}$$

and $m = 0, 1, 2, ..., M-1, m' = 0, 1, 2, ..., M'-1, n = 1, 2, 3, ..., 2^{k-1}, n' = 1, 2, 3, ..., 2^{k'-1},$, here, T_m and $T_{m'}$ are Chebyshev polynomials of first kind of order m and m' respectively defined over the interval $[0, 1]$ and also two-dimensions Chebyshev wavelets are orthonormal set over Ω.

1.4.10 One-dimensional Bernoulli wavelet

Bernoulli wavelet $\Phi^B_{n,m}(t) = \Phi^B(k, \hat{n}, m, t)$ have four arguments; $\hat{n} = n-1$, $k \in N$, $n = 1, 2, ..., 2^{k-1}$, and m is the degree of the Bernoulli polynomial of the fist kind and t is the normalized time i.e. $t \in [0, 1)$. They are defined on the interval $[0, 1)$ as (see [38]):

$$\Phi^B_{n,m}(t) = \Phi^B(k, \hat{n}, m, t) = \begin{cases} 2^{(k-1)/2}\widehat{B}_m(2^{k-1}t - \hat{n}), & , \frac{\hat{n}-1}{2^k} \leq t < \frac{\hat{n}+1}{2^k}; \\ 0, & \text{otherwise} \end{cases}, \tag{1.14}$$

with

$$\widehat{B}_m(t) = \begin{cases} 1, & \text{if } m = 0; \\ \sqrt{\frac{2m!}{(-1)^{m-1}(m!)^2\alpha_{2m}}} B_m(t), & \text{if m>0.} \end{cases} \tag{1.15}$$

where, $m = 0, 1, ..., M-1$, and $n = 1, 2, ..2^{k-1}$. The coefficient $\sqrt{\frac{2m!}{(-1)^{m-1}(m!)^2\alpha_{2m}}}$ is for normality, the dilation parameter is $g = 2^{-(k-1)}$ and the translation parameter $h = \widehat{n}2^{-(k-1)}$. Here, $B_m(t)$ are the well-known Bernoulli polynomial of of order m which can be defined as follows:

$$B_m(t) = \sum_{i=0}^{m} {}^mC_i \alpha_{m-1} t^i,$$

where, $\alpha_i,\ i = 0, 1, ..., m$ are Bernoulli numbers. These numbers are a sequence of signed rational numbers which arise in the series expansion of trigonometric function and can be defined by

$$\frac{t}{e^t-1}=\sum_{i=0}^{\infty}\alpha_i\frac{t^i}{i!}.$$

The first few Bernoulli polynomials are $B_0(t)=1$, $B_1(t)=t-\frac{-1}{2}$, $B_2(t)=t^2-t+\frac{1}{6}$, $B_3(t)=t^3-\frac{3}{2}t^2+\frac{1}{2}, ...$ These polynomials satisfy the following relation:

$$\int_0^1 B_n(t)B_m(t)=(-1)^{n-1}\frac{m!n!}{(m+n)!}\alpha_{n+m},\quad m,n\geq 1.$$

According to [39], Bernoulli polynomial form a complete basis over the interval $[0,1]$, Bernoulli polynomials can be defined in many ways such as the following form:

(i) $B_n'(t)=nB_{n-1}(t)\ \forall n\geq 1.$

(ii) $\int_0^1 B_n(t)dt=0\ \forall n\geq 1.$

(iii) $B_0(t)=1.$

1.4.11 Two-dimensional Bernoulli wavelet

Two-dimensional Bernoulli wavelet can be written in the product of one-dimensional Bernoulli wavelet as follows:

$$\Phi^B_{n,m,n',m'}(a,b)=\Phi^B_{n,m}(a)\Phi^B_{n',m'}(b), \tag{1.16}$$

where, $\Phi^B_{n,m}(a)$ and $\Phi^B_{n',m'}(b)$ are defined as $Eq.$(1.10), $n'=1,2,...,2^{k'-1}$ and $m'=0,1,2,...,M'-1$.

1.4.12 Kronecker multiplication

We recall the definition of Kronecker multiplication of matrices and also an important property which is related to Kronecker multiplications defined in ([107], [108]).

Remark 2 *Suppose that U and V are two dimensions $m\times n$ and $p\times q$, respectively, then the Kronecker multiplication of U and V is denoted by $U\otimes V=kron(U,V)$ and is defined in the following form :*

$$W = U_{m\times n} \otimes V_{p\times q} = \begin{pmatrix} u_{11}V & u_{12}V & \dots & u_{1n}V \\ u_{21}V & u_{22}V & \dots & u_{2n}V \\ \vdots & \vdots & \ddots & \vdots \\ u_{m1}V & a_{m2}V & \dots & u_{mn}V \end{pmatrix},$$

where, W is $mp \times nq$ matrix.

Remark 3 *If matrices U_1, U_2 and V_1, V_2 with appropriate dimensions then following interesting property is satisfied :*

$$(U_1U_2) \otimes (V_1V_2) = (U_1 \otimes V_1)(U_2 \otimes V_2).$$

1.5 Review of fractional partial differential equations

Fractional calculus is a branch of mathematics which investigates the properties of derivatives and integrals of non-integer orders (called fractional derivatives and integrals). In particular, this discipline involves the notion and methods of solving of differential equations involving fractional derivatives of the unknown function (called fractional differential equations). The history of fractional calculus started almost at the same time when classical calculus was established. It was first mentioned in Leibniz's letter to L'Hospital in 1695, where the idea of semiderivative was suggested. During time fractional calculus was built on formal foundations by many famous mathematicians, e.g. Liouville, Grunwald, Riemann, Euler, Lagrange, Heaviside, Fourier and Abel etc. A lot of them proposed original approaches, which can be found chronologically in [10]. The theory of fractional calculus includes even complex orders of integro-differential and left and right integro-differential (analogously to left and right derivatives).

The applications of fractional differential and integral operators in mathematical models have become increasingly widespread in recent years. Several forms of fractional differential equations have been proposed in standard models, and there has been significant interest in developing numerical schemes for their solution. These methods include Laplace transforms [40], Fourier transforms [41], Adomian decomposition method (ADM)

([42],[43]), eigenvector expansion [44], Fractional Differential Transform Method (FDTM) ([45],[46]), Variational Iteration Method (VIM) ([47], [48]), Fractional Difference Method (FDM) [49], and Power Series Method [50]. But, few papers reported application of wavelet to solve the fractional order differential equations ([51], [52]).

Many definitions and studies of fractional calculus have been proposed in the last two centuries. These definitions included Riemann-Liouville, Caputo, Weyl, Reize, Compos and Nashimoto fractional derivative operators (see [66]). Fractional derivative operators have application in physics, mathematics, engineering and applied science as physical sciences phenomena in area like Damping law , Diffusion process, Electrochemistry, Arterial sciences, the theory of Ultra-slow processes, etc. Now a days, we can find applications and models involving fractional derivative in Probability, Astrophysics, Anomalous diffusion, Chemical physics, Finance, Robust control, Electromagnetism, Optic and signal processing, Seismic analysis, Viscoelasticity, Acoustics, Biology,etc.

In view of successful application of wavelet operational matrix in system analysis ([53], [54]), system identification ([55], [56]), optimal control ([57], [58]) and numerical solution of integral and differential equations ([59], [60], [61], [62], [63], [64], [77]), together with the characteristic of wavelet functions, we hold that they should be applicable to solve the fractional order systems. In this thesis we consider only the most common definitions named Riemann Liouville which will be introduced below.

The fractional derivative in the Riemann-Liouville sense

Using partial Riemann-Liouville fractional derivative operators (PRLFDOs) to find the easy numerical solvability of our proposed fractional partial differential equations (FPDEs), we converted the FPDEs into fractional partial integro-differential equations (FPIDEs) by using PRLFDOs [66] which are defined as follows:

$$({}_0D_t^{\alpha}u)(t,x) = \frac{1}{\Gamma(1-\{\alpha\})}(\frac{\partial}{\partial t})^{[\alpha]+1}\int_0^t \frac{u(t,x)}{(t-s)^{\{\alpha\}}}ds \quad \forall t>0, x>0, \alpha>0, \qquad (1.17)$$

where, $[\alpha]$ and $\{\alpha\}$ being the integral and fractional parts of α respectively.

1.6 Review of operational matrices

Approximation by orthogonal and orthonormal family of functions have played a key role in the mathematical analysis in particular since long time and also in additional, they have important role in physical sciences, engineering and technology in general. In the last four or five decades, orthogonal and orthonormal family have been playing an important role in the evaluation of new techniques to solve problem such as identification, analysis and optimal control. The main aim of these techniques has been to obtain effective algorithms that are suitable for the digital computers. Their major effort has been concentrated on the methods of the orthogonal as well as orthonormal functions.

The motivation and philosophy behind the operational matrix approach have some important characteristic. These are enlisted:

- It is reduce singularities from the proposed mathematical problems in easy way.
- It is not only simplifies the proposed problem but also speedup the computation.
- It is transform the PDEs, PIDEs and FPIDEs into algebraic system.
- The method is computer oriented, thus solving higher order partial differential equations (PDEs), partial integro-differential equations (PIDEs), fractional partial integro-differential equations (FPIDEs) becomes a matter of dimension increasing.
- The solution is convergent, even though the size of increment may be large.
- Wavelets operational matrices are have sparsity i.e. maximum element of these matrices are zero which are very useful for computation and simplification in easy way.

The basic idea of operational matrix technique is as follows:

- The unknown function or its derivatives with respect to time(or space) in the given PDEs/PIDEs/FPIDEs are approximated by linear combinations of the orthonormal basis functions and truncating them upto optimal levels.

- After operational matrix approximation the proposed problem converted into simple algebraic equations whose solutions can be obtained using collocation method with Sylevester's approach; this gives approximate solutions for PDEs/PIDEs/ FPIDEs.

The key idea of operational matrix technique depends on the following properties of the basis vector $\Phi(t)$

- Operational matrix of integration with respect to t as

$$\int_a^t \dots \int_a^t \Phi(s)(ds)^k = P_{m+1}^{(k)}\Phi(t),$$

- Operational matrix of differentiation with respect to t as

$$\frac{\partial^k \Phi(t)}{\partial t^k} = D_{m+1}^{(k)}\Phi(t),$$

where, $\Phi(t) = [\Phi_0(t), \Phi_1(t), ..., \Phi_t]^T$ is a basis function which is orthogonal or orthonormal on a certain interval [a,b], P_{m+1} and D_{m+1} are the operational matrices for the integration and differentiation of $\Phi(t)$ respectively. Here, It should be noticed that P_{m+1} and D_{m+1} are constant matrices of order $(m+1)$.

Much work has been done on the operational matrix of integration and differentiation technique for several types of orthogonal and orthonormal basis functions, such as the Walsh function (Chen and Hsiao ([67], [68])), block-pulse function (Chen et al.[69]), Sannuti([70])), Lagurerre series (King and Paraskevopoulos [71], Hwang and Shih[72], [73]), Chebshev polynomials [74], Legendre polynomials [75]. Later Gu and Jiang [76] derived the Haar wavelets operational matrix of integration followed by Razzaghi and Yousefi [77], who gave the Legendre wavelets operational matrix of integration.

Also, the operational matrix of differentiation has been determined for several types of orthogonal and orthonormal basis functions, such as Legendre polynomial (Saadatmandi and M. Dehghan [78]), Legendre wavelet (Mohammadi and Hosseini [79]) Chebshev wavelet (Hosseini and Mohammadi [80]), Jacobi operational matrix

(Doha et al. [81]).

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