Chapter 5

Stability Criteria based on First and second order polynomials

5.1 Introduction

Consider a Time-delay system as:

$$\dot{x}(t) = Ax(t) + A_{\tau}x(t - \tau_t), \tag{5.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $A, A_{\tau} \in \mathbb{R}^{n \times n}$ are the constant system matrices with continuously differentiable initial condition. The time-varying delay τ_t is otherwise represented as $\tau(t)$ that satisfies the following properties:

$$0 \leq \tau_t \leq \hbar, \quad \mu_0 \leq \dot{\tau}_t \leq \mu_1 \tag{5.2}$$

where $\hbar, \mu_0$ and μ_1 are constants. The problem of stability analysis for time-delay system (1) has been widely investigated over the years and consistent improvement have been attained using the Lyapunov-Krasovskii (LK) approach [99–102]. Performance of such stability criteria is assessed by the maximum permissible upper bound (MPUB) of τ_t. The key steps involved in obtaining improved results are by constructing appropriate LK Functional (LKF) involving augmentation of several states including state integrals and delayed states, and to find precise bound of the integral function in the derivative of LKF. It was shown in [77, 82, 103, 104] that LKF must include the states involved in the integral function such that interaction among various states are established in the resulting criterion. Therefore, construction of LKFs depends on the integral function to be used in the derivation of the stability criterion.

On the other hand, regarding integral inequalities, Jensen and Wirtinger inequalities [1, 77] have been widely used to obtain bound of the integral function. Due to this reason, stability criteria are obtained as a first-order polynomials $f(\tau_t) = a_1\tau_t + a_0$, where $a_i(i = 0, 1) \in \mathbb{R}$ and independent of τ_t. To make $f(\tau_t) < 0$ for all $\tau_t \in [0, \hbar]$, two inequalities are to be satisfied as $f(0) < 0$ and $f(\hbar) < 0$.

Recent developments involve the use of (i) double integral states to construct the LKF and (ii) higher order inequalities, such as auxiliary-polynomial based inequality [78] and Bessel Legendre function based inequality (BLBI) [94] to obtain bound of the integral function. The stability criteria are then obtained in quadratic form as: $f(\tau_t) = a_2\tau_t^2 + a_1\tau_t + a_0$ with $a_i(i = 0, 1, 2) \in \mathbb{R}$. In [105], the above is approximated as $f(0) < 0$ and $f(\hbar) < 0$ to obtain $f(\tau_t) < 0$, but it applies when $a_2 \geq 0$. The requirement $a_2 \geq 0$ has been relaxed in [85] by introducing an additional condition $-\hbar^2 a_2 + f(\hbar) < 0$. Similar works have been reported in [106–109], where some new set of conditions for negativity of a quadratic function have been developed. In [107] and [108], tuning parameters are introduced that lead to solving inequalities that are dependent on a tuning parameter to obtain less conservative results. In [106], the entire delay interval $[0, \hbar]$ is divided uniformly in multiple subintervals and for each of the sub-intervals, two end points are considered. It has been illustrates that less conservative results can be obtained by increasing the number of subintervals. Recently, two lemmas have been introduced in [109] to ensure the negativity of quadratic polynomial. The first inequality uses the property of cross-point between two tangent lines of the end points whereas the other inequality exploits the condition of finite interval. It should be noted that such quadratic inequalities also introduce conservativeness, since one has to approximate the polynomial. Then question arises "Can the appearance of quadratic form of τ_t be avoided while still using higher-order integral inequalities?". An investigation on the appearance of τ_t^2 dependent terms in the derivative of LKFs reveals that it is due to the combination of product of delay interval and interval-normalized states. To this end, a further inspection leads to the fact that these terms can be avoided by treating integral and its interval-normalized states as separate individual states. Moreover, not only this, out of the several possible LKFs, one requires to choose appropriate LKF that leads to less conservative result. This motivates the present work, which incorporates the following.

(i) An augmented-type delay-product LKF has been constructed by including single and double integral states along with their interval-normalized forms to incorporate more τ_t^2 dependent terms in the resulting stability criterion.

(ii) Based on this LKF and negative-determination lemma (NDL) introduced in [85] to obtain negative-definiteness of the resulting criterion, a stability criteria is derived leading to quadratic function based LMI conditions.

(iii) Another new stability criterion is derived by avoiding quadratic function based LMI conditions by treating integral states and their normalized forms as individual states in the derivative of the LKF.

Finally, three examples are considered to illustrate the less conservativeness of the proposed stability criterion that does not involve the NDL. It is demonstrated that not using the NDL leads to considerably less conservative results.

5.2 Useful Lemmas

In this section, the following Lemmas are recalled that will be used to derive the main result. These are useful to deal with inversely weighted positive convexity parameters, integral function and second order polynomial conditions.

Lemma 9 *[110] For a real scalar $\alpha \in (0,1)$, symmetric matrices $R_1 \geq 0, R_2 \geq 0$ and any matrices U_1 and U_2, the following inequality holds:*

$$\begin{bmatrix} \frac{1}{\alpha}R_1 & 0 \\ 0 & \frac{1}{1-\alpha}R_2 \end{bmatrix} \geq \begin{bmatrix} R_1 + (1-\alpha)Z_1 & (1-\alpha)U_1 + \alpha U_2 \\ * & R_2 + \alpha Z_2 \end{bmatrix} \tag{5.3}$$

where $Z_1 = R_1 - U_2 R_2^{-1} U_2^T$ and $Z_2 = R_2 - U_1^T R_1^{-1} U_1$

The above is known as the Reciprocity Lemma and it is used to obtain a bound of quadratic terms involving reciprocal parameters. Next, we introduce the second-order Bessel-legendre inequality, which supersede the inequalities proposed in [78] and [81], respectively.

Lemma 10 *[94] For any constant matrix $R \geq 0$, the following inequality holds for all continuously differentiable function $w \in [a,b] \to \mathbb{R}^n$*

$$(b-a)\int_a^b \dot{w}^T(s) R \dot{w}(s) ds \geq \vartheta_1^T R \vartheta_1 + 3\vartheta_2^T R \vartheta_2 + 5\vartheta_3^T R \vartheta_3 \tag{5.4}$$

where

$\vartheta_1 = w(b) - w(a),$

$\vartheta_2 = w(b) + w(a) - \frac{2}{(b-a)}\int_a^b w(s)ds,$

$\vartheta_3 = \vartheta_1 - \frac{6}{(b-a)}\int_a^b w(s)ds + \frac{12}{(b-a)^2}\int_a^b\int_a^s w(r)drds$

The next Lemma is known as the NDL and it guarantees the negative definiteness of a quadratic function in the interval $[0, \hbar]$ irrespective of its concave or convex nature.

Lemma 11 *[85] The given quadratic function $z(u) = a_2u^2 + a_1u + a_0$, where $a_0, a_1, a_2 \in \mathbb{R}$, satisfy $z(u) < 0$ for all $u \in [0, \hbar]$ if the following three inequalities hold:*

$$(a) z(0) < 0 \qquad (b) z(\hbar) < 0 \qquad (c) - \hbar^2 a_2 + z(0) < 0$$

To simplify matrix and vector representations, the following notations are subsequently used:

$$\begin{aligned}
&1 - \dot{\tau}(t) = \tilde{\tau}_t \\
&w_0(t) = \left[x^T(t), x^T(t-\tau_t)\right]^T \\
&w_1(t) = \left[\int_{t-\tau_t}^{t} x^T(s)ds,\ \frac{1}{\tau_t}\int_{t-\tau_t}^{t}\int_{t-\tau_t}^{s} x^T(r)drds\right]^T \\
&w_2(t) = \left[\int_{t-\hbar}^{t-\tau_t} x^T(s)ds,\ \frac{1}{\hbar-\tau_t}\int_{t-\hbar}^{t-\tau(t)}\int_{t-\hbar}^{s} x^T(r)drds\right]^T \\
&w_3(t) = \left[x^T(t-\hbar),\ \dot{x}(t-\tau_t)\right]^T \\
&\psi_0(t) = \left[w_0^T(t),\ w_1^T(t),\ w_2^T(t)\right]^T \\
&\psi_1(t) = \left[w_0^T(t),\ \frac{1}{\tau_t}w_1^T(t),\ w_1^T(t)\right]^T \\
&\psi_2(t) = \left[w_0^T(t),\ \frac{1}{\hbar-\tau_t}w_2^T(t),\ w_2^T(t)\right]^T \\
&\psi_3(s) = \left[x^T(s), \dot{x}^T(s), \int_{t-\tau_t}^{s} \dot{x}^T(s)ds\right]^T
\end{aligned}$$

Note that the vectors $w_i, (i = 1, 2)$ contain the integral states for the delay intervals $[t - \tau_t, t]$ and $[t - \hbar, t - \tau_t]$. Their interval normalized forms are obtained by multiplying reciprocal of the delay interval with the integral states, for example, $\frac{1}{\tau_t}w_1(t)$ and $\frac{1}{\hbar-\tau_t}w_2(t)$. In this chapter, both the integral and their interval normalized states are utilized to define the LKF.

5.3 Main Results

In this section, two stability criteria are derived for system (5.1). One leads to a criterion involving the second-order polynomial and the other one does not. The construction of LKF is discussed next.

5.3.1 Lyapunov-Krasovskii functional

In [88], a new type of LKF has been introduced in which single integral states have been used as pivot elements of augmented vectors to construct delay-coefficient based quadratic terms. The time-derivative of this LKF introduces terms involving both the delay and its derivative, so that these terms contribute to reduce conservativeness in the stability criterion. By extending this idea, in [90] and [91], new delay-product based LKF have been constructed by using double integral states. On the basis of these works, a new LKF is constructed by involving both the integral states and its interval-normalized forms. Further, appropriate zero equalities are used to exploit the time-dependency of the states. The following LKF is used in this work.

$$V(t) = V_0(t) + V_1(t) + V_2(t) \tag{5.5}$$

where

$$\begin{aligned}
V_0(t) &= \psi_0^T(t)P\psi_0(t) + \tau_t\psi_1^T(t)P_1\psi_1(t) \\
&\quad + (\hbar - \tau_t)\psi_2^T(t)P_2\psi_2(t) \\
V_1(t) &= \int_{t-\tau_t}^{t} \psi_3^T(s)Q_1\psi_3(s)ds \\
&\quad + \int_{t-\hbar}^{t-\tau_t} x^T(s)Q_2x(s)ds \\
V_2(t) &= \int_{-\tau_t}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R_1\dot{x}(s)dsd\theta \\
&\quad + \int_{-\hbar}^{-\tau_t}\int_{t+\theta}^{t} \dot{x}^T(s)R_2\dot{x}(s)dsd\theta
\end{aligned}$$

and ψ_i, $i = 0, 1, 2, 3$ are defined in section II. P, P_1, P_2, Q_1, Q_2, R_1 and R_2 are positive definite matrices.

Remark 7 *In the functional $V_0(t)$, additional integral states $w_1(t)$ and $w_2(t)$ are introduced in the augmented vectors of delay-coefficient based quadratic terms corresponding* *to the delay intervals $[t-\tau_t, t]$ and $[t-\hbar, t-\tau_t]$, such that more τ_t and $\dot{\tau}_t$ dependent terms appear in the derived conditions. Note that these terms have not been used in LKFs defined earlier in [89, 111, 112]. Also, using these states, delay-dependent zero-equalities are formulated to exploit the time-relation of the delayed states.*

Remark 8 *A new form of single integral functionals have been proposed in [88], in which the cross terms $x(s)$, $\dot{x}(s)$ in the interval $[t-\tau_t, t]$ and $x(s)$ in $[t-\hbar, t-\tau_t]$ have been introduced. By extending this idea, $V_1(t)$ incorporates cross-terms $\int_{t-\tau_t}^{s}\dot{x}(s)ds$, $\dot{x}(s)$, $x(s)$ in one interval whereas $x(s)$ in the other one to avoid the inclusion of $x(t-\hbar)$ in the delay coefficient based terms and in the Lyapunov-matrix related term. This also avoids the appearance of additional state in the form of derivative of $x(t-\hbar)$ in the derivative of the LKF. By doing so the number of LMI variables and maximum order of LMI decreases which in turn reduces the computational burden. In addition, the work in [95] demonstrated the advantage of using $V_2(t)$ that leads to delay-derivative dependent integral terms.*

5.3.2 Zero-equalities

Zero-equalities are often used to exploit the time relation of the states [113, 114]. In [115], the integral states and its interval-normalized forms are utilized to construct new zero-equalities. The following two zero-equalities with slack variable matrices $N_i, i = 1, 2, 3, 4$, of appropriate dimensions are used in this work.

$$(I).\quad 2\zeta^T(t)N_1\left[\hat{E}_1 w_1(t) - \tau_t\hat{E}_1\left(\frac{1}{\tau_t}w_1(t)\right)\right] + 2\zeta^T(t)N_2 \times\left[\hat{E}_1 w_2(t) - (\hbar-\tau_t)\hat{E}_1\left(\frac{1}{\hbar-\tau_t}w_2(t)\right)\right] = 0 \tag{5.6}$$

$$(II),\quad 2\zeta^T(t)N_3\left[\hat{E}_2 w_1(t) - \tau_t\hat{E}_2\left(\frac{1}{\tau_t}w_1(t)\right)\right] + 2\zeta^T(t)N_4 \times\left[\hat{E}_2 w_2(t) - (\hbar-\tau_t)\hat{E}_2\left(\frac{1}{\hbar-\tau_t}w_2(t)\right)\right] = 0 \tag{5.7}$$

where $\hat{E}_1 = \begin{bmatrix} I & 0 \end{bmatrix}$, $\hat{E}_2 = \begin{bmatrix} 0 & I \end{bmatrix}$ and $\zeta(t)$ is a column vector to be appropriately chosen. Note that the above zero-equalities (5.6) and (5.7) are quadratic forms of interval-normalized states and involves zero-functions of both $w_1(t)$ and $w_2(t)$ in the sense that the bracketed terms are zero by virtue of their time-dependency. These inequalities will be augmented in the derivative of the quadratic LKF so that the time relation among the integrals can be exploited. The slack variables introduce some freedom in the interplay of the constraints imposed on other variables arising from the LKF.

5.3.3 Stability Analysis

This section presents two stability criteria for system (5.1) incorporating LKF (5.5). The first method leads to terms involving τ_t^2 and thereby leads to a criterion as a second-order polynomial function of τ_t. Whereas the second one involves only a first-order polynomial that does not require further approximation for a polytopic representation. The first result is presented next.

Theorem 6 *System (1) is asymptotically stable if there exist positive definite matrices* $P \in \mathbb{R}^{6n\times 6n}, P_1, P_2 \in \mathbb{R}^{6n\times 6n}, Q_1 \in \mathbb{R}^{3n\times 3n}, Q_2 \in \mathbb{R}^{n\times n}, R_1, R_2 \in \mathbb{R}^{n\times n}$, *and matrices* $U_1, U_2 \in \mathbb{R}^{3n\times 3n}$, *satisfying the following LMIs.*

$$\begin{bmatrix} \Upsilon(0,\mu_i) & E_1^T U_2 \\ * & -\hbar\bar{\mathcal{J}}(\mu_i) \end{bmatrix} < 0, \tag{5.8}$$

$$\begin{bmatrix} \Upsilon(\hbar,\mu_i) & E_2^T U_1^T \\ * & -\hbar\bar{\mathcal{J}}(\mu_i) \end{bmatrix} < 0 \tag{5.9}$$

$$\begin{bmatrix} -\hbar^2\mathcal{G}_0(\mu_i) + \Upsilon(0,\mu_i) & E_1^T U_2 \\ * & -\hbar\bar{\mathcal{J}}(\mu_i) \end{bmatrix} < 0 \tag{5.10}$$

with $i = 0, 1$ *and*

$$\bar{\mathcal{J}}(\mu_i) > 0, \tag{5.11}$$

where

$$\Upsilon(\tau_t,\dot{\tau}_t) = \Phi_0(\tau_t,\dot{\tau}_t) + \Phi_1(\tau_t,\dot{\tau}_t) + \Phi_2(\tau_t,\dot{\tau}_t) \tag{5.12}$$

$$\mathcal{G}_0(\dot{\tau}_t) = \dot{\tau}_t(\Lambda_{21}^T P_1\Lambda_{21} - \Lambda_{41}^T P_2\Lambda_{41}) + Sym\{(\Lambda_{21}^T P_1\Lambda_{31} + \Lambda_{41}^T P_2\Lambda_{51})\} \tag{5.13}$$

$$\begin{aligned} \Phi_0(\tau_t,\dot{\tau}_t) &= Sym\{\Lambda_0^T P\Lambda_1\} + \dot{\tau}_t(\Lambda_{20} + \tau_t\Lambda_{21})^T P_1(\Lambda_{20} + \tau_t\Lambda_{21}) \\ &+ Sym\{(\Lambda_{20} + \tau_t\Lambda_{21})^T P_1(\Lambda_{30} + \tau_t\Lambda_{31})\} - \dot{\tau}_t(\Lambda_{40} + (\hbar - \tau_t)\Lambda_{41})^T P_1(\Lambda_{40} + (\hbar - \tau_t)\Lambda_{41}) \\ &+ Sym\{(\Lambda_{40} + (\hbar - \tau_t)\Lambda_{41})^T \times P_2(\Lambda_{50} + (\hbar - \tau_t)\Lambda_{51})\} \end{aligned} \tag{5.14}$$

$$\Phi_1(\tau_t,\dot{\tau}_t) = \Lambda_6^T Q_1\Lambda_6 - \tilde{\tau}_t\Lambda_7^T Q_1\Lambda_7 + Sym\{\Lambda_8^T Q_1\Lambda_9\} + \tilde{\tau}_t e_2^T Q_2 e_2 - e_7^T Q_2 e_7 \tag{5.15}$$

$$\Phi_2(\tau_t,\dot{\tau}_t) = e_s^T[\tau_t R_1 + (\hbar - \tau_t)R_2]e_s - \frac{1}{\hbar}[E_1^T(2-\alpha)\tilde{\mathcal{J}}(\dot{\tau}_t)E_1 + E_2^T(1+\alpha)\tilde{R}_2 U_2 \\ - 2U_1^T(\alpha U_1 + (1-\alpha)U_2)E_2] \tag{5.16}$$

The Λ_i's in the above comprises of vectors as the following:

$$\begin{aligned}
\Lambda_0 =& [e_1^T, e_2^T, \tau_t e_3^T, \tau_t e_4^T, (\hbar-\tau_t)e_5^T, (\hbar-\tau_t)e_6^T]^T \\
\Lambda_1 =& [e_s^T, \tilde{\tau}_t e_8^T, e_1^T - \tilde{\tau}_t e_2^T, e_3^T - \tilde{\tau}_t e_2^T - \dot{\tau}_t e_4^T, \tilde{\tau}_t e_2^T - e_7^T, \tilde{\tau}_t e_5^T - e_7^T + \dot{\tau}_t e_6^T]^T \\
\Lambda_{20} =& [e_1^T, e_2^T, e_3^T, e_4^T, e_0^T, e_0^T]^, \ \Lambda_{21} = [e_0^T, e_0^T, e_0^T, e_0^T, e_3^T, e_4^T]^T \\
\Lambda_{30} =& [e_0^T, e_0^T, e_1 - \tilde{\tau}_t e_2^T - \dot{\tau}_t e_3^T, e_3^T - \tilde{\tau}_t e_2^T - 2\dot{\tau}_t e_4^T, e_0^T, e_0^T]^T \\
\Lambda_{31} =& [e_s^T, \tilde{\tau}_t e_8^T, e_0^T, e_0^T, e_1^T - \tilde{\tau}_t e_2^T, e_3^T - \tilde{\tau}_t e_2^T - \dot{\tau}_t e_4^T]^T \\
\Lambda_{40} =& [e_1^T, e_2^T, e_5^T, e_6^T, e_0^T, e_0^T]^T, \ \Lambda_{41} = [e_0^T, e_0^T, e_0^T, e_0^T, e_5^T, e_6^T]^T \\
\Lambda_{50} =& [e_0^T, e_0^T, \tilde{\tau}_t e_2 - e_7^T + \dot{\tau}_t e_5^T, \tilde{\tau}_t e_5^T - e_7^T + 2\dot{\tau}_t e_6^T, e_0^T, e_0^T]^T \\
\Lambda_{51} =& [e_s^T, \tilde{\tau}_t e_8^T, e_0^T, e_0^T, \tilde{\tau}_t e_2^T - e_7^T, \tilde{\tau}_t e_5^T - e_7^T + \dot{\tau}_t e_6^T]^T \\
\Lambda_6 =& [e_1^T, e_s^T, e_1^T - e_2^T]^T, \ \Lambda_7 = [e_2^T, e_7^T, e_0^T]^T \\
\Lambda_8 =& [\tau_t e_3^T, e_1^T - e_2^T, \tau_t e_3^T - \tau_t e_2^T]^T. \ \Lambda_9 = [e_0^T, e_0^T, -\tilde{\tau}_t e_8^T]^T
\end{aligned}$$

The above involves block vectors defined as:

$$\begin{aligned}
e_i =& [0_{n\times(i-1)}, I_n, 0_{n\times(8-i)}], i = 1, 2, ..., 8, \\
e_s =& Ae_1 + A_\tau e_2, \ e_0 = [0]_{n\times 8n}
\end{aligned}$$

Proof: By taking the time-derivative of the individual terms in the LKF (5.5) and defining $\xi_1(t) = [w_0^T(t), \frac{1}{\tau_t}w_1^T(t), \frac{1}{\hbar-\tau_t}w_2^T(t), w_3^T(t)]^T$, one gets

$$\dot{V}_0(t) = \xi_1^T(t)\Phi_0(\tau_t, \dot{\tau}_t)\xi_1(t) \tag{5.17}$$

$$\dot{V}_1(t) = \xi_1^T(t)\Phi_1(\tau_t, \dot{\tau}_t)\xi_1(t) \tag{5.18}$$

$$\dot{V}_2(t) = \xi_1^T(t)e_s^T[\tau_t R_1 + (\hbar-\tau_t)R_2]e_s\xi_1(t) - \mathcal{I}(t) \tag{5.19}$$

where $\Phi_0(\tau_t, \dot{\tau}_t)$ and $\Phi_1(\tau_t, \dot{\tau}_t)$ are defined in (5.12) and (5.13) respectively, and

$$\begin{aligned}
\mathcal{I}(t) =& \int_{t-\tau_t}^{t} \dot{x}^T(s)\mathcal{J}(\dot{\tau}_t)\dot{x}(s)ds + \int_{t-\hbar}^{t-\tau_t} \dot{x}^T(s)R_2\dot{x}(s)ds \\
\mathcal{J}(\dot{\tau}_t) =& \tilde{\tau}_t R_1 + \dot{\tau}_t R_2
\end{aligned}$$

Next, the bound for the integral term $\mathcal{I}(t)$ is obtained. Using Lemma 2, one can write

$$\mathcal{I}(t) \geq \frac{\xi_1^T(t)}{\hbar}\left(\frac{\hbar}{\tau_t}E_1^T\tilde{\mathcal{J}}(\dot{\tau}_t)E_1 + \frac{\hbar}{(\hbar-\tau_t)}E_2^T\tilde{R}_2E_2\right)\xi_1(t) \tag{5.20}$$

where

$$E_1 =[e_1^T - e_2^T, e_1^T + e_2^T - 2e_3^T, e_1^T - e_2^T - 6e_3^T + 12e_4^T]^T$$
$$E_2 =[e_2^T - e_7^T, e_2^T + e_7^T - 2e_5^T, e_2^T - e_7^T - 6e_5^T + 12e_6^T]^T$$
$$\tilde{\mathcal{J}} =diag\{\mathcal{J}(\dot{\tau}_t), 3\mathcal{J}(\dot{\tau}_t), 5\mathcal{J}(\dot{\tau}_t)\}$$
$$\tilde{R}_2 =diag\{R_2, 3R_2, 5R_2\}$$

The RHS of (5.20) involves inverse of τ_t. It is taken care of by using Lemma 9. This step yields

$$\mathcal{I}(t) \geq \frac{1}{\hbar}\xi_1(t)^T[(2-\alpha)E_1^T\tilde{\mathcal{J}}(\dot{\tau}_t)E_1 + (1+\alpha)E_2^T\tilde{R}_2E_2 + 2E_1^T\{\alpha U_1 + (1-\alpha)U_2\}E_2 - \Delta_1]\xi_1(t) \tag{5.21}$$

where

$$\Delta_1 = (1-\alpha)E_1^TU_2\tilde{R}_2^{-1}U_2^TE_1 + \alpha E_2^TU_1^T\tilde{\mathcal{J}}^{-1}(\dot{\tau}_t)U_1E_2$$

By replacing (5.21) into (5.19), one obtains

$$\dot{V}_2(t) \leq \xi_1^T(t)[\Phi_2(\tau_t, \dot{\tau}_t) + \frac{\Delta_1}{\hbar}]\xi_1(t) \tag{5.22}$$

where $\Phi_2(\tau_t, \dot{\tau}_t)$ is defined in (4.11). Now, one obtains $\dot{V}(t)$ by collecting the derivatives of individual terms in the LKFs (5.17), (5.18) and (5.22) as

$$\dot{V}(t) \leq \xi_1^T(t)(\Upsilon(\tau_t, \dot{\tau}_t) + \frac{\Delta_1}{\hbar})\xi_1(t) \tag{5.23}$$

where $\Upsilon(\tau_t, \dot{\tau}_t)$ is defined in (13).

The matrix $\bar{\Phi}_0(\dot{\tau}_t, \tau_t)$ of $\Upsilon(\dot{\tau}_t, \tau_t)$ contains τ_t^2 terms, so $\Upsilon(\dot{\tau}_t, \tau_t) + \frac{\Delta_1}{\hbar}$ can be expressed in quadratic form of τ_t as

$$\Upsilon(\dot{\tau}_t, \tau_t) + \Delta_1 = \tau_t^2\mathcal{G}_0(\dot{\tau}_t) + \tau_t\mathcal{G}_1(\dot{\tau}_t) + \mathcal{G}_2(\dot{\tau}_t) \tag{5.24}$$

where $\mathcal{G}_0(\dot{\tau}_t)$ is defined in (13), $\mathcal{G}_1(\dot{\tau}_t)$ and $\mathcal{G}_2(\dot{\tau}_t)$ are symmetric matrices and all are independent of τ_t. Therefore, using Lemma 11, the matrix $\Upsilon(\dot{\tau}_t, \tau_t) + \Delta_1 < 0$ holds for $\tau_t \in [0, \hbar]$ and $\dot{\tau}_t \in [\mu_0, \mu_1]$ if the following inequalities, for $i = 0, 1$, hold:

$$\Upsilon(0, \mu_i) + \Delta_1(0, \mu_i) < 0$$
$$\Upsilon(\hbar, \mu_i) + \Delta_1(\hbar, \mu_i) < 0$$
$$-\hbar^2\mathcal{G}_0(\mu_i) + \Upsilon(0, \mu_i) + \Delta_1(0, \mu_i) < 0$$

Using Schur complement, one can transform the above inequalities into LMIs (5.8)-(5.10). Hence, if LMIs (5.8)-(5.11) are satisfied, then $\dot{V}(t) < 0$, which ensures the asymptotic stability of system (5.1). This completes the proof. □ In Theorem 6, $\dot{V}_0(t)$ yields τ_t^2 terms because the integral states $w_1(t)$ and $w_2(t)$ are considered as product of delay-interval and their interval-normalized form. Hence, the stability criterion is in the form of quadratic function of τ_t, and this enforces using Lemma 11. However, Lemma 3 has inherent conservatism that make the stability result conservative.

To deal with this issue, an improved criteria is proposed in the next theorem by treating $w_1(t)$ and $w_2(t)$ and their interval-normalized forms $\frac{1}{\tau_t}w_1(t)$ and $\frac{1}{\hbar-\tau_t}w_2(t)$ are considered as separate individual states. This leads to no τ_t^2 term in the derivative of the LKF and thereby not introducing conservativeness invited by the use of Lemma 11. Note that, this step invokes more decision variables in the resulting criterion, which is a trade-off with the reduction in conservativeness.

Theorem 7 *System (5.1) is asymptotically stable if there exist positive-definite matrices, $P \in \mathbb{R}^{6n\times 6n}, P_1, P_2 \in \mathbb{R}^{6n\times 6n}, Q_1 \in \mathbb{R}^{3n\times 3n}, Q_2 \in \mathbb{R}^{n\times n}, R_1, R_2 \in \mathbb{R}^{n\times n}$, any matrices $U_1, U_2 \in \mathbb{R}^{3n\times 3n}$ and N_1, N_2 of appropriate dimensions, satisfying the following LMI conditions.*

$$\begin{bmatrix} \bar{\Upsilon}(0,\mu_i) & \bar{E}_1^T U_2 \\ * & -\hbar\bar{\mathcal{J}}(\mu_i) \end{bmatrix} < 0 \tag{5.25}$$

$$\begin{bmatrix} \bar{\Upsilon}(\hbar,\mu_i) & \bar{E}_2^T U_1^T \\ * & -\hbar\bar{\mathcal{J}}(\mu_i) \end{bmatrix} < 0 \tag{5.26}$$

with

$$\mathcal{J}(\mu_i) \geq 0, for\ i = 0, 1 \tag{5.27}$$

where

$$\bar{\Upsilon}(\tau_t,\dot{\tau}_t) = \bar{\Phi}_0(\tau_t,\dot{\tau}_t) + \bar{\Phi}_1(\tau_t,\dot{\tau}_t) + \bar{\Phi}_2(\tau_t,\dot{\tau}_t) \tag{5.28}$$

$$\bar{\Phi}_0(\tau_t,\dot{\tau}_t) = Sym\{\bar{\Lambda}_0^T P\bar{\Lambda}_1\} + \dot{\tau}_t\bar{\Lambda}_2^T P_1\bar{\Lambda}_2 + Sym\{\bar{\Lambda}_2^T P_1\bar{\Lambda}_3\} - \dot{\tau}_t\bar{\Lambda}_4^T P_2\bar{\Lambda}_4 + Sym\{\bar{\Lambda}_4^T P_2\bar{\Lambda}_5\} \tag{5.29}$$

$$\begin{aligned}\bar{\Phi}_1(\tau_t,\dot{\tau}_t) = {} & \bar{\Lambda}_6^T Q_1\bar{\Lambda}_6 - \tilde{\tau}_t\bar{\Lambda}_7^T Q_1\bar{\Lambda}_7 + Sym\{\bar{\Lambda}_8^T Q_1\bar{\Lambda}_9\} + \tilde{\tau}_t\bar{e}_2^T Q_2\bar{e}_2 - \bar{e}_7^T Q_2\bar{e}_7 \\ & + Sym\{N_1(\bar{e}_9 - \tau_t\bar{e}_3) + N_2(\bar{e}_{10} - (\hbar-\tau_t)\bar{e}_5)\}\end{aligned}$$

$$+ Sym\{N_3(\bar{e}_{11} - \tau_t\bar{e}_4) + N_4(\bar{e}_{12} - (\hbar - \tau_t)\bar{e}_6)\} \tag{5.30}$$

$$\bar{\Phi}_2(\tau_t, \dot{\tau}_t) = \bar{e}_s^T[(\tau_t)R_1 + (\hbar - \tau_t)R_2]\bar{e}_s - \frac{1}{\hbar}[\bar{E}_1^T(2-\alpha)\tilde{\mathcal{J}}(\dot{\tau}_t)\bar{E}_1 + \bar{E}_2^T(1+\alpha)\tilde{R}_2\bar{E}_2$$

$$- 2\bar{E}_1^T(\alpha S_1 + (1-\alpha)S_2)\bar{E}_2] \tag{5.31}$$

The $\bar{\Lambda}$ related quadratic term in the above consists of the following vectors:

$$\bar{\Lambda}_0 =[\bar{e}_1^T, \bar{e}_2^T, \bar{e}_9^T, \bar{e}_{10}^T, \bar{e}_{11}^T, \bar{e}_{12}^T]^T$$

$$\bar{\Lambda}_1 =[\bar{e}_s^T, \tilde{\tau}_t\bar{e}_8^T, \bar{e}_1^T - \tilde{\tau}_t\bar{e}_2^T, \bar{e}_3^T - \tilde{\tau}_t\bar{e}_2^T - \dot{\tau}_t\bar{e}_4^T, \tilde{\tau}_t\bar{e}_2^T - \bar{e}_7^T, \tilde{\tau}_t\bar{e}_5^T - \bar{e}_7^T + \dot{\tau}_t\bar{e}_6^T]^T$$

$$\bar{\Lambda}_2 =[\bar{e}_1^T, \bar{e}_2^T, \bar{e}_3^T, \bar{e}_4^T, \bar{e}_9^T, \bar{e}_{10}^T]^T, \ \bar{\Lambda}_4 = [\bar{e}_1^T, \bar{e}_2^T, \bar{e}_5^T, \bar{e}_6^T, \bar{e}_{11}^T, \bar{e}_{12}^T]^T$$

$$\bar{\Lambda}_3 =[\tau_t\bar{e}_s^T, \tilde{\tau}_t\tau_t\bar{e}_8^T, \bar{e}_1 - \tilde{\tau}_t\bar{e}_2^T - \dot{\tau}_t\bar{e}_3^T, \bar{e}_3^T - \tilde{\tau}_t\bar{e}_2^T - 2\dot{\tau}_t\bar{e}_4^T, \tau_t(\bar{e}_1 - \tilde{\tau}_t\bar{e}_2)^T,$$

$$\tau_t(\bar{e}_3 - \tilde{\tau}_t\bar{e}_2 - \dot{\tau}_t\bar{e}_4)^T]^T$$

$$\bar{\Lambda}_5 =[(\hbar - \tau_t)\bar{e}_s^T, (\hbar - \tau_t)\tilde{\tau}_t\bar{e}_8^T, \tilde{\tau}_t\bar{e}_2^T - \bar{e}_7^T + \dot{\tau}_t\bar{e}_5^T, \tilde{\tau}_t\bar{e}_5^T - \bar{e}_7^T + 2\dot{\tau}_t\bar{e}_6^T, (\hbar - \tau_t)(\tilde{\tau}_t\bar{e}_2 - \bar{e}_7)^T,$$

$$(\hbar - \tau_t)(\tilde{\tau}_t\bar{e}_5 - \bar{e}_7 + \dot{\tau}_t\bar{e}_6)^T]^T$$

$$\bar{\Lambda}_6 =[\bar{e}_1^T, \bar{e}_s^T, \bar{e}_1^T - \bar{e}_2^T]^T, \ \bar{\Lambda}_7 = [\bar{e}_2^T, \bar{e}_8^T, \bar{e}_0^T]^T$$

$$\bar{\Lambda}_8 =[\bar{e}_9^T, \bar{e}_1^T - \bar{e}_2^T, \bar{e}_9^T - \tau_t\bar{e}_2^T]^T, \ \bar{\Lambda}_9 = [\bar{e}_0^T, \bar{e}_0^T, -\tilde{\tau}_t\bar{e}_8^T]^T$$

The above expression involves block vectors $\bar{e}_0, ..., \bar{e}_{12}$ defined as:

$$\bar{e}_i =[0_{n\times(i-1)}, I_n, 0_{n\times(12-i)}], i = 1, 2, ..., 12,$$

$$\bar{e}_s =A\bar{e}_1 + A_\tau\bar{e}_2, \ \bar{e}_0 = [0]_{n\times 12n}$$

Proof : Similar to Theorem 6, consider the LKF (5.5). Next, by taking the derivative of $V_0(t)$ and by treating the integral and its normalized version as individual states to reformulate second order polynomial into first order, one obtains

$$\dot{V}_0(t) = \xi_2^T(t)\Phi_0(\tau_t, \dot{\tau}_t)\xi_2(t) \tag{5.32}$$

where $\Phi_0(\tau_t, \dot{\tau}_t)$ is defined in (29) and $\xi_2(t) = [w_0^T(t), \frac{1}{\tau_t}w_1^T(t), \frac{1}{\hbar-\tau_t}w_2^T(t), w_3^T(t), w_1^T(t), w_2^T(t)]^T$. In the zero equalities of (5.6) and (5.7), the integral states $w_1(t)$ and $w_{2(t)}$ and their normalized version are considered as separate individual states then one obtains

$$2\xi_2^T(t)[N_1(\bar{e}_9 - \tau_t\bar{e}_3) + N_2(\bar{e}_{10} - (\hbar - \tau_t)\bar{e}_5)]\xi_2(t) = 0 \tag{5.33}$$

$$2\xi_2^T(t)[N_3(\bar{e}_{11} - \tau_t\bar{e}_4) + N_4(\bar{e}_{12} - (\hbar - \tau_t)\bar{e}_6)]\xi_2(t) = 0 \tag{5.34}$$

Next, by including (5.33) and (5.34) in the derivatives of $V_1(t)$, we have

$$\dot{V}_1(t) = \xi_2^T(t)\Phi_1(\tau_t, \dot{\tau}_t)\xi_2(t) \tag{5.35}$$

where $\Phi_1(\tau_t, \dot{\tau}_t)$ is defined in (5.10). Similarly, the derivative of $V_2(t)$ can be expressed as:

$$\dot{V}_2(t) = \xi_2^T(t)\bar{e}_s^T[\tau_t R_1 + (\hbar - \tau_t)R_2]\bar{e}_s\xi_2(t) - \mathcal{I}(t) \tag{5.36}$$

where

$$\begin{aligned}\mathcal{I}(t) &= \int_{t-\tau_t}^{t} \dot{x}^T(s)\mathcal{J}(\dot{\tau}_t)\dot{x}(s)ds + \int_{t-\hbar}^{t-\tau_t} \dot{x}^T(s)R_2\dot{x}(s)ds \\ \mathcal{J}(\dot{\tau}_t) &= [\tilde{\tau}_t R_1 + \dot{\tau}_t R_2]\end{aligned} \tag{5.37}$$

By bounding the integral function $\mathcal{I}(t)$ using Lemma 9 and 2, and invoking (5.34), one obtains

$$\dot{V}_2(t) \leq \xi_2^T(t)[\Phi_2(\tau_t, \dot{\tau}_t) + \frac{\mathit{\Pi}_1}{\hbar}]\xi_2(t), \tag{5.38}$$

where $\Phi_2(\tau_t, \dot{\tau}_t)$ is defined in (5.31) and

$$\mathit{\Pi}_1 = (1-\alpha)\bar{E}_1^T S_2 \tilde{R}_2^{-1} S_2^T \bar{E}_1 + \alpha \bar{E}_2^T S_1^T \tilde{\mathcal{J}}^{-1}(\tau_t) S_1 \bar{E}_2$$

where

$$\begin{aligned}\bar{E}_1 =& [\bar{e}_1^T - \bar{e}_2^T, \bar{e}_1^T + \bar{e}_2^T - 2\bar{e}_3^T, \bar{e}_1^T - \bar{e}_2^T - 6\bar{e}_3^T + 12\bar{e}_4^T]^T \\ \bar{E}_2 =& [\bar{e}_2^T - \bar{e}_7^T, \bar{e}_2^T + \bar{e}_7^T - 2\bar{e}_5^T, \bar{e}_2^T - \bar{e}_7^T - 6\bar{e}_5^T + 12\bar{e}_6^T]^T \\ \tilde{\mathcal{J}} =& diag\{\mathcal{J}(\dot{\tau}_t), 3\mathcal{J}(\dot{\tau}_t), 5\mathcal{J}(\dot{\tau}_t)\} \\ \tilde{R}_2 =& diag\{R_2, 3R_2, 5R_2\}\end{aligned}$$

Finally, one can write $\dot{V}(t)$ using (5.32)-(5.34) as

$$\dot{V}(t) \leq \xi_2^T(t)(\mathit{\Upsilon}(\tau_t, \dot{\tau}_t) + \frac{\mathit{\Pi}_1}{\hbar})\xi_2(t) \tag{5.39}$$

where $\mathit{\Upsilon}(\tau_t, \dot{\tau}_t)$ is defined in (5.28).

If the matrix $\mathit{\Upsilon}(\tau_t, \dot{\tau}_t) + \frac{\mathit{\Pi}_1}{\hbar}$ is negative definite for all $\tau_t \in [0, \hbar]$ and $\dot{\tau}_t \in [\mu_0, \mu_1]$, then $\dot{V}(t) < 0$. Now, using Schur complement one can transform $\mathit{\Upsilon}(\tau_t, \dot{\tau}_t) + \frac{\mathit{\Pi}_1}{\hbar} < 0$ into LMIs (5.25) and (5.26). This completes the proof. □

Remark 9 *In Theorem 7, the integral and their interval-normalized forms are treated as separate individual states, to avoid the use of Lemma 11. Therefore, in Theorem 6, additional states $w_1(t)$ and $w_2(t)$ are considered as compared to Theorem 6. Two zero equalities are also modified to take these states into account. In doing so, the maximum order of the LMI criterion and number of LMI variables are increased. This results in computational burden and convergence time. In general, the potential to yield better result using Theorem 7 in comparison to Theorem 6 is at the price of more computational burden and convergence time. This yet again provides the trade-off between complexity and conservativeness.*

5.4 Numerical Examples

In order to demonstrate the less conservativeness of Theorem 7, following three examples are considered and comparisons are constructed in terms of MPUB of $\hbar$ and number of LMI variables (NLVs).

5.4.1 System Parameters

Consider system (5.1) with the following three sets of system matrices:

(1) Example 1:

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_\tau = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \tag{5.40}$$

(2) Example 2:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad A_\tau = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \tag{5.41}$$

(3) Example 3:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad A_\tau = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \tag{5.42}$$

5.4.2 Comparative results

(a) Delay upper-bound ($\hbar$): The MPUB of delay for various values of μ obtained using different approaches are tabulated in Tables 5.1, 5.2 and 5.3 for example 1, 2 and 3,

Table 5.1: For Example 1 the LAUB of delay $\hbar$ for various values of μ

Methods	$\mu = -\mu_0 = \mu_1$				**NLVs**
	0.1	**0.2**	**0.5**	**0.8**	
Theorem 1 (C3) [106]	4.939	-	3.298	2.869	$104n^2 + 15n$
Theorem 2(C1) [95]	4.940	4.262	3.304	2.877	$69n^2 + 12n$
Theorem 3 [116]	4.944	4.274	3.305	2.850	$221.5n^2 + 12.5n$
Theorem 1 [117]	4.942	-	3.309	2.882	$108n^2 + 12n$
Theorem 8(N=4) [79]	5.01	4.29	3.19	2.70	$146.5n^2 + 9.5n$
Theorem 2 [108]	5.041	-	3.431	2.980	$162.5n^2 + 16.5n$
Theorem 3 [109]	5.026	-	3.428	2.977	$252.5n^2 + 0.5n$
Corollary 2 [118]	5.044	-	3.443	2.983	$235n^2 + 34n$
Theorem 6	4.9438	4.2673	3.3104	2.8880	$76n^2 + 11n$
Theorem 7	5.0588	4.4317	3.4606	2.9981	$124n^2 + 11n$

respectively. It can be seen that the MPUBs obtained using Theorem 7 is larger than Theorem 6. This shows that use of Lemma 11 due to the involvement of τ_t^2 terms introduces considerably conservativeness. This can be avoided if the states are appropriately augmented.

For a comparison among the proposed results and the existing ones in literature, one can observe that Theorem 6 in example 1 is more conservative than all the approaches except Theorem 6 and 2 of [106] and [95], respectively. However, Theorem 7 yields better results as compared to all other approaches consistently. In case of example 2, Theorem 6 mostly provides conservative result except Proposition of [115] and Theorem 2(N=2) of [97]. Further, for example 3, Theorem 6 yields better results among all that are listed in table 3 except Theorem 7 of [95].

(b) Complexity computation: The computational complexity depends on the maximum order of LMIs (MOL) involved in the stability criteria and also on the number of scalar LMI variables (NLV) used. Due to the use of extra states and zero equalities in the stability analysis of Theorem 7, the MOL is increased by four as compared to Theorem 6. Also the NLVs used in Theorem 7 are larger than that of Theorem 6. So, the computational burden and convergence time is more in Theorem 7 as compared to Theorem 6. Further,

Table 5.2: For Example 2 the LAUB of delay $\hbar$ for various values of μ

Methods	$\mu = -\mu_0 = \mu_1$				NLVs
	0.1	**0.2**	**0.5**	**0.8**	
Proposition 1 [115]	7.230	4.556	2.509	1.940	$54.5n^2 + 6.5n$
Theorem 2(N=2) [97]	7.263	4.591	2.575	2.011	$65n^2 + 8n$
Theorem 1 [117]	7.400	4.795	2.717	2.089	$108n^2 + 12n$
Theorem 1 (C3) [106]	7.401	4.765	2.709	2.094	$104n^2 + 15n$
Theorem 3 [116]	7.550	4.902	2.714	2.054	$221.5n^2 + 12.5n$
Theorem 2 [108]	7.616	4.949	2.798	2.142	$162.5n^2 + 16.5n$
Theorem 3 [109]	7.651	4.936	2.764	2.114	$252.5n^2 + 0.5n$
Theorem 6	7.3083	4.6622	2.6401	2.0585	$76n^2 + 11n$
Theorem 7	7.6657	4.9675	2.7948	2.1335	$124n^2 + 11n$

for comparison with the existing methods, the NLVs are listed in all the Tables. One can observe that the NLVs required in proposed Theorem 7 are less as compared to the other approaches except in Theorem 1 (C3) of [106], Theorem 2(C1) of [95] and Theorem 1 of [117]. Similarly, for example 2, the NLVs used in proposed Theorem 7 are more in comparison to the other methods listed in Table 5.2 except Theorem 3 of [116] and Theorem 2 of [108]. Further solving in example 3 requires large number of LMI variables in comparison to all the criterion listed in Table 5.3. Therefore, on the basis of the above comparisons, it can observed that Theorem 7 requires more convergence time and computational burden than the some of the existing approaches.

5.4.3 Simulation verification

From Table 5.1, one can observed that proposed Theorem 7 guarantees the stability of system (5.1) for example 1 until $\hbar = 5.0588$ for $\mu = 0.1$. Similarly from Table 5.2, for example 2 until $\hbar = 7.6657$ for $\mu = 0.1$. Further from Table 5.3 for example 3 until $\hbar = 2.6692$ for $\mu = 0.05$.

For verification of proposed results, we consider the following parameters.

Example 1: $x(0) = [1, -1]^T$ and $\tau_t = 4.9588 + 0.1 sint$

Table 5.3: For Example 3 the LAUB of delay $\hbar$ for various values of μ

Methods	$\mu = -\mu_0 = \mu_1$			**NLVs**
	0.05	**0.1**	**0.5**	
Theorem 3 [92]	2.5903	2.4382	2.0260	$70n^2 + 12n$
Proposition 1 [115]	2.6370	2.4742	2.0424	$54.5n^2 + 6.5n$
Theorem 2(N=2) [97]	2.6387	2.4770	2.0447	$65n^2 + 8n$
Theorem 1 [95]	2.646	2.498	2.113	$100.5n^2 + 8.5n$
Theorem 6	2.6435	2.4905	2.0962	$76n^2 + 11n$
Theorem 7	2.6692	2.5300	2.1370	$124n^2 + 11n$

Example 2: $x(0) = [0.5, -1.5]^T$ and $\tau_t = 7.5657 + 0.1sint$

Example 3: $x(0) = [1, -1.5]^T$ and $\tau_t = 2.6192 + 0.05sint$

The state responses of all the three examples are shown in Figs. 5.1-5.3. In all the cases, the systems are seen to be asymptotically stable that corroborates the obtained results.

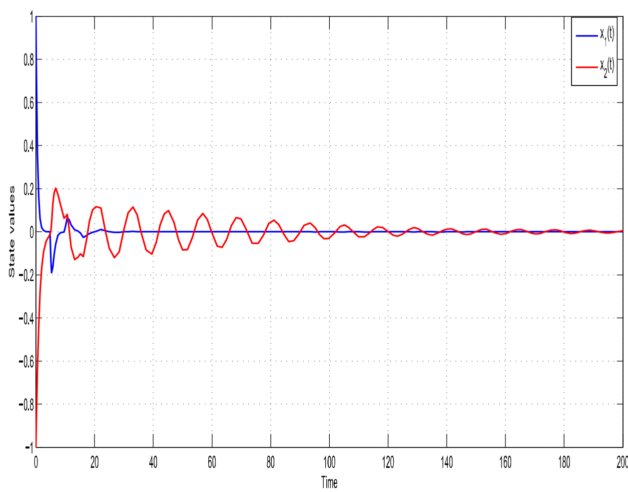

Figure 5.1: State responses of Example 1

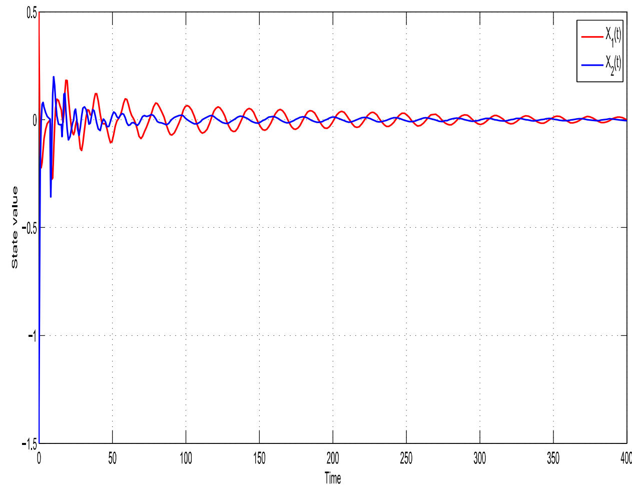

Figure 5.2: State responses of Example 2

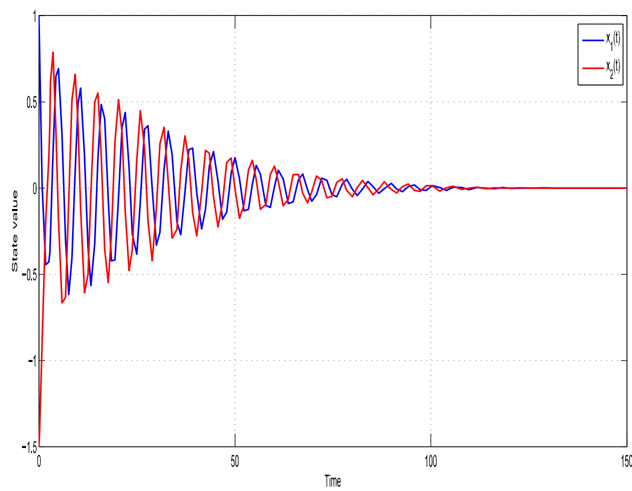

Figure 5.3: State responses of Example 3

5.5 Summary

Two new criteria in the form of second and first-order polynomial functions of τ_t have been proposed. Both the criteria are obtained using same LKF for the stability analysis of systems with time-varying delay. Using comparative studies, it is observed that the conservativeness in the stability criterion due to use of NDL is considerable and it can be removed by avoiding appearence of the quadratic term of τ_t. However, zero-equalities are used with additional states which leads to more number of LMI variables and complexities as well. The effectiveness of the proposed LMI conditions are demonstrated by considering three examples, which shows that the proposed Theorem 7 yields improved results.