

Chapter 5

Characterizations of the Inversion Formula of the Continuous Bessel Wavelet Transform of Distributions in $H'_{\mu}(\mathbb{R}^+)$

5.1 Introduction

Hankel transform had considerable effects on the problems of the theory of distributions. Exploiting the theory of distributions, Zemanian [60–63], Koh [29, 30], Lee [31], Dube et al. [17] and Arteaga and Marrero [1] gave significant contributions in this area. Using the Hankel transform approach, Betancor et al. [2–12] and Pathak et al. [40–45] discussed many properties of functional spaces and Hankel convolution. The Hankel convolution made an adequate foundation for the formation of the

Bessel wavelet transform. Considering the Zemanian theory of the Hankel transform, Upadhyay et al. [51–56] discussed the continuous Bessel wavelet transform and studied its properties.

Motivated by the aforesaid results, author discussed the properties of the continuous Bessel wavelet transform in Besov and Triebel-Lizorkin type spaces in chapter 4 by taking Hankel transform tool. Our main objective in this chapter is to investigate the inversion formula of the Bessel wavelet transform of distributions in $H'_\mu(\mathbb{R}^+)$ by considering the Hankel transform theory.

The present chapter is organized in the following way:

Section 5.1 is introductory, which gives a brief history regarding the development of the inversion formula of the continuous Bessel wavelet transform of distributions in $H'_\mu(\mathbb{R}^+)$. Section 5.2 provides the basic properties of Bessel wavelet and Bessel wavelet transform of distributions in form of Lemmas and Theorem. Section 5.3 deals the structure formula of distributions in $H'_\mu(\mathbb{R}^+)$ and some important results of the continuous Bessel wavelet transform in the classical sense which are useful to find the inversion formula of the continuous Bessel wavelet transform in $H'_\mu(\mathbb{R}^+)$. Section 5.4 is devoted to find some results related to the inversion formula of the continuous Bessel wavelet transform in $H'_\mu(\mathbb{R}^+)$ and finally the inversion formula of the continuous Bessel wavelet transform of distributions is given. In Section 5.5, some applications of the inversion formula of the continuous Bessel wavelet transform of distributions in $H'_\mu(\mathbb{R}^+)$ in the form of the Calderón reproducing formula and heat equation in $H'_\mu(\mathbb{R}^+)$ are given by author.

5.2 Properties of the Bessel wavelet and Bessel wavelet transform of distributions

In this section, properties of the Bessel wavelet and the Bessel wavelet transform of distributions are found in $H'_\mu(\mathbb{R}^+)$ -space.

Lemma 5.2.1. *Let $\psi \in H_\mu(\mathbb{R}^+)$, $\mu \geq -\frac{1}{2}$. Then, ψ is a basic Bessel wavelet if and only if $\int_0^\infty x^{\mu+\frac{1}{2}}\psi(x)dx = 0$.*

Proof. Let $\psi \in H_\mu(\mathbb{R}^+)$ be the basic Bessel wavelet which satisfies the admissibility condition (1.5.1).

Since $\omega^{-\mu-\frac{1}{2}}(h_\mu\psi)(\omega)$ continuous on \mathbb{R}^+ , therefore

$$\begin{aligned}\omega^{-\mu-\frac{1}{2}}(h_\mu\psi)(\omega) &= \omega^{-\mu-\frac{1}{2}} \int_0^\infty (x\omega)^{\frac{1}{2}} J_\mu(x\omega)\psi(x)dx \\ &= \int_0^\infty (x\omega)^{-\mu} J_\mu(x\omega)x^{\mu+\frac{1}{2}}\psi(x)dx.\end{aligned}$$

If we take $\omega \rightarrow 0$, then $(x\omega)^{-\mu} J_\mu(x\omega) \rightarrow \frac{1}{2^\mu\Gamma\mu+1}$ and from (1.5.1) it is clear that $\omega^{-\mu-\frac{1}{2}}(h_\mu\psi)(\omega)$ must go to zero. Hence from the last expression, we have

$$\frac{1}{2^\mu\Gamma\mu+1} \int_0^\infty x^{\mu+\frac{1}{2}}\psi(x)dx = 0,$$

so that

$$\int_0^\infty x^{\mu+\frac{1}{2}}\psi(x)dx = 0. \quad (5.2.1)$$

Conversely, for $\psi \in H_\mu(\mathbb{R}^+) \subset L^2(\mathbb{R}^+)$, we can write

$$\int_0^\infty \frac{|(h_\mu\psi)(\omega)|^2}{\omega^{2\mu+2}}d\omega = \int_0^1 \frac{|(h_\mu\psi)(\omega)|^2}{\omega^{2\mu+2}}d\omega + \int_1^\infty \frac{|(h_\mu\psi)(\omega)|^2}{\omega^{2\mu+2}}d\omega. \quad (5.2.2)$$

The second integral of the R.H.S. of (5.2.2) can be estimated as

$$\begin{aligned} \int_1^\infty \frac{|(h_\mu\psi)(\omega)|^2}{\omega^{2\mu+2}} d\omega &\leq \int_1^\infty |(h_\mu\psi)(\omega)|^2 d\omega \\ &\leq \int_0^\infty |(h_\mu\psi)(\omega)|^2 d\omega \\ &= \int_0^\infty |\psi(x)|^2 dx. \end{aligned}$$

Therefore,

$$\int_1^\infty \frac{|(h_\mu\psi)(\omega)|^2}{\omega^{2\mu+2}} d\omega \leq \|\psi(x)\|_2^2. \quad (5.2.3)$$

Next, we can show the bound of the first integral of the R.H.S. of (5.2.2). Since $\omega^{-\mu-\frac{1}{2}}(h_\mu\psi)(\omega) \rightarrow 0$ as $\omega \rightarrow 0$. Therefore,

$$\begin{aligned} \left| \omega^{-\mu-\frac{1}{2}}(h_\mu\psi)(\omega) \right| &= \left| \lim_{\epsilon \rightarrow 0} \int_\epsilon^\omega \frac{\partial}{\partial y} y^{-\mu-\frac{1}{2}}(h_\mu\psi)(y) dy \right| \\ &\leq \lim_{\epsilon \rightarrow 0} \int_\epsilon^\omega \left| yy^{-1} \frac{\partial}{\partial y} y^{-\mu-\frac{1}{2}}(h_\mu\psi)(y) \right| dy \\ &\leq \lim_{\epsilon \rightarrow 0} \int_\epsilon^\omega y^{-\frac{1}{2}} \left| (1+y^2)y^{-1} \frac{\partial}{\partial y} y^{-\mu-\frac{1}{2}}(h_\mu\psi)(y) \right| dy \\ &\leq \lim_{\epsilon \rightarrow 0} \sup_{x \in \mathbb{R}^+} \left| (1+y^2)y^{-1} \frac{\partial}{\partial y} y^{-\mu-\frac{1}{2}}(h_\mu\psi)(y) \right| \int_\epsilon^\omega y^{-\frac{1}{2}} dy. \end{aligned}$$

Using (1.6.1), we find

$$\begin{aligned} \left| \omega^{-\mu-\frac{1}{2}}(h_\mu\psi)(\omega) \right| &\leq \gamma_{1,1}^\mu(h_\mu\psi) \lim_{\epsilon \rightarrow 0} \int_\epsilon^\omega y^{-\frac{1}{2}} dy \\ &\leq \gamma_{1,1}^\mu(h_\mu\psi) 2 \omega^{\frac{1}{2}}, \end{aligned}$$

so that

$$\left| \omega^{-\mu-1}(h_\mu\psi)(\omega) \right| \leq 2\gamma_{1,1}^\mu(h_\mu\psi).$$

Therefore,

$$\int_0^1 \left| \omega^{-\mu-1} (h_\mu \psi)(\omega) \right|^2 d\omega \leq 4 \left\{ \gamma_{1,1}^\mu (h_\mu \psi) \right\}^2 \int_0^1 1^2 d\omega.$$

This implies that

$$\int_0^1 \frac{|(h_\mu \psi)(\omega)|^2}{\omega^{2\mu+2}} d\omega \leq 4 \left\{ \gamma_{1,1}^\mu (h_\mu \psi) \right\}^2, \quad (5.2.4)$$

for $\psi \in H_\mu(\mathbb{R}^+)$ and $(h_\mu \psi)(y) \in H_\mu(\mathbb{R}^+)$.

This shows that

$$\int_0^\infty \frac{|(h_\mu \psi)(\omega)|^2}{\omega^{2\mu+2}} d\omega < \infty. \quad (5.2.5)$$

□

Definition 5.2.2. Let $\psi \in H_\mu(\mathbb{R}^+)$ be the Bessel wavelet and $f \in H'_\mu(\mathbb{R}^+)$, then the Bessel wavelet transform of f is defined by

$$(B_\psi f)(a, b) = \left\langle f(x), a^{\mu-\frac{1}{2}} \psi\left(\frac{x}{a}, \frac{b}{a}\right) \right\rangle,$$

where $a > 0, b > 0$ and $\mu > -\frac{1}{2}$.

Lemma 5.2.3. For $\mu \geq -\frac{1}{2}$, $Cx^{\mu+\frac{1}{2}}$ belongs to $H'_\mu(\mathbb{R}^+)$.

Proof. We have $f(x) = Cx^{\mu+\frac{1}{2}}$ and $\phi \in H_\mu(\mathbb{R}^+)$ then

$$\begin{aligned} \langle f, \phi \rangle &= \int_0^\infty f(x)\phi(x)dx \\ &= \int_0^\infty Cx^{\mu+\frac{1}{2}}\phi(x)dx \\ &= C \int_0^\infty x^{2\mu+1}x^{-\mu-\frac{1}{2}}\phi(x)dx. \end{aligned}$$

Therefore,

$$\begin{aligned} |\langle f, \phi \rangle| &= \left| C \int_0^\infty \frac{x^{2\mu+1}}{(1+x^2)^m} (1+x^2)^m x^{-\mu-\frac{1}{2}} \phi(x) dx \right| \\ &\leq C \int_0^\infty \frac{x^{2\mu+1}}{(1+x^2)^m} \left| (1+x^2)^m x^{-\mu-\frac{1}{2}} \phi(x) \right| dx \\ &\leq \sup_{x \in \mathbb{R}^+} \left| (1+x^2)^m x^{-\mu-\frac{1}{2}} \phi(x) \right| C \int_0^\infty \frac{x^{2\mu+1}}{(1+x^2)^m} dx. \end{aligned}$$

By the definition of (1.6.1), we get

$$|\langle f, \phi \rangle| \leq C \gamma_{m,0}^\mu(\phi) \int_0^\infty \frac{x^{2\mu+1}}{(1+x^2)^m} dx.$$

Putting $(1+x^2) = u$, $2xdx = du$, then

$$\begin{aligned} |\langle f, \phi \rangle| &\leq C \gamma_{m,0}^\mu(\phi) \int_1^\infty \frac{(u-1)^{2\mu+1}}{2u^m(u-1)^{\frac{1}{2}}} du \\ &\leq C \gamma_{m,0}^\mu(\phi) \int_1^\infty \frac{(u-1)^{2\mu+1}}{2u^m} du. \end{aligned}$$

Now, taking $u = \frac{1}{v}$, $du = -\frac{1}{v^2} dv$, then the above expression yields

$$\begin{aligned} |\langle f, \phi \rangle| &\leq \frac{C}{2} \gamma_{m,0}^\mu(\phi) \int_0^1 v^m \left(\frac{1}{v} - 1 \right)^\mu \frac{dv}{v^2} \\ &\leq \frac{C}{2} \gamma_{m,0}^\mu(\phi) \int_0^1 (1-v)^\mu v^{m-\mu-2} dv. \end{aligned}$$

By definition of β -function, we get

$$\begin{aligned} |\langle f, \phi \rangle| &\leq \frac{C}{2} \gamma_{m,0}^\mu(\phi) \frac{\Gamma(\mu+1)\Gamma(m-\mu-1)}{\Gamma(\mu+1+m-\mu-1)} \\ &\leq \frac{C}{2} \gamma_{m,0}^\mu(\phi) \frac{\Gamma(\mu+1)\Gamma(m-\mu-1)}{\Gamma m}, \end{aligned}$$

for $m - \mu - 1 \geq 0$.

□

Theorem 5.2.4. *Let $\psi \in H_\mu(\mathbb{R}^+)$ be the Bessel wavelet, then the Bessel wavelet transform of distributions $Cx^{\mu+\frac{1}{2}}$ is zero, where C is constant.*

Proof. From Definition 5.2.2, we have the Bessel wavelet transform of the distribution $Cx^{\mu+\frac{1}{2}}$ as

$$\begin{aligned} (B_\psi Cx^{\mu+\frac{1}{2}})(a, b) &= \left\langle Cx^{\mu+\frac{1}{2}}, a^{\mu-\frac{1}{2}}\psi\left(\frac{x}{a}, \frac{b}{a}\right) \right\rangle \\ &= \int_0^\infty Cx^{\mu+\frac{1}{2}}a^{\mu-\frac{1}{2}}\psi\left(\frac{x}{a}, \frac{b}{a}\right)dx \\ &= \int_0^\infty Cx^{\mu+\frac{1}{2}}a^{\mu-\frac{1}{2}} \int_0^\infty \psi(z)D_\mu\left(\frac{x}{a}, \frac{b}{a}, z\right)dzdx \\ &= \int_0^\infty \int_0^\infty Cx^{\mu+\frac{1}{2}}a^{\mu-\frac{1}{2}}\psi(z)D_\mu\left(\frac{x}{a}, \frac{b}{a}, z\right)dx dz. \end{aligned}$$

Putting $\frac{x}{a} = t$ and using (1.4.8), we find

$$\begin{aligned} (B_\psi Cx^{\mu+\frac{1}{2}})(a, b) &= \int_0^\infty Ca^{2\mu+1}\psi(z) \int_0^\infty t^{\mu+\frac{1}{2}}D_\mu\left(t, \frac{b}{a}, z\right)dt dz \\ &= \int_0^\infty Ca^{2\mu+1}\psi(z) \frac{\left(\frac{bz}{a}\right)^{\mu+\frac{1}{2}}}{2^\mu\Gamma(\mu+1)} dz \\ &= C \frac{(ab)^{\mu+\frac{1}{2}}}{2^\mu\Gamma(\mu+1)} \int_0^\infty z^{\mu+\frac{1}{2}}\psi(z)dz \\ &= 0. \end{aligned}$$

So, Bessel wavelet transform of distributions $Cx^{\mu+\frac{1}{2}}$ will be zero. \square

Example 5.1. *Let $\psi(x) \in H_\mu(\mathbb{R}^+)$ such that*

$$\psi(x) = S_{\mu,x}\{x^{\mu+\frac{1}{2}}e^{-x^2}\},$$

where $S_{\mu,x} = x^{-\mu-\frac{1}{2}}D_x x^{2\mu+1}D_x x^{-\mu-\frac{1}{2}}$ is Bessel operator. Then ψ satisfies the admissibility condition (1.5.1) and represents a Bessel wavelet in $H_\mu(\mathbb{R}^+)$ whose all the moments weighted by $x^{\mu+\frac{1}{2}}$ are non-zero.

Solution of Example 5.1 is given below stepwise:

Step I: For $\psi(x) = S_{\mu,x}\{x^{\mu+\frac{1}{2}}e^{-x^2}\}$, using (1.7.8) and the formula [19, p.30], we get

$$(h_\mu\psi)(y) = h_\mu\{S_{\mu,x}x^{\mu+\frac{1}{2}}e^{-x^2}\}(y) = -y^2h_\mu\{x^{\mu+\frac{1}{2}}e^{-x^2}\}(y) = -y^2\frac{y^{\mu+\frac{1}{2}}}{2^{\mu+1}}e^{-\frac{y^2}{4}}.$$

Therefore,

$$\begin{aligned} \int_0^\infty \frac{|(h_\mu\psi)(y)|^2}{y^{2\mu+2}} dy &= \int_0^\infty \frac{y^{2\mu+5}e^{-\frac{y^2}{2}}}{2^{2\mu+2}y^{2\mu+2}} dy = \frac{1}{2^{2\mu+2}} \int_0^\infty y^3e^{-\frac{y^2}{2}} dy \\ &= \frac{2}{2^{2\mu+2}} \int_0^\infty te^{-t} dt, \quad \left(\text{taking } \frac{y^2}{2} = t\right) \\ &= \frac{1}{2^{2\mu+1}} \Gamma(2) = \frac{1}{2^{2\mu+1}} < \infty. \end{aligned}$$

This implies that $\psi(x) \in H_\mu(\mathbb{R}^+) \subset L^2(\mathbb{R}^+)$ satisfies the admissibility condition (1.5.1).

Step II: Also, for $\psi(x) = S_{\mu,x}\{x^{\mu+\frac{1}{2}}e^{-x^2}\}$, we have

$$\begin{aligned} \int_0^\infty x^{\mu+\frac{1}{2}}\psi(x)dx &= \int_0^\infty x^{\mu+\frac{1}{2}}S_{\mu,x}\{x^{\mu+\frac{1}{2}}e^{-x^2}\}dx \\ &= \int_0^\infty x^{\mu+\frac{1}{2}}x^{-\mu-\frac{1}{2}}D_x x^{2\mu+1}D_x x^{-\mu-\frac{1}{2}}x^{\mu+\frac{1}{2}}e^{-x^2}dx \\ &= \int_0^\infty D_x\{x^{2\mu+1}(-2x)e^{-x^2}\}dx \\ &= \left\{x^{2\mu+1}(-2x)e^{-x^2}\right\}\Big|_{x=0}^\infty = 0. \end{aligned}$$

By Lemma 5.2.1, we conclude that $\psi(x) \in H_\mu(\mathbb{R}^+)$ is a Bessel wavelet.

Step III: Now, the moment of $\psi(x)$ of order $m \in \mathbb{N}$, weighted by $x^{\mu+\frac{1}{2}}$ be

$$\begin{aligned}
\int_0^\infty x^m x^{\mu+\frac{1}{2}} \psi(x) dx &= \int_0^\infty x^m x^{\mu+\frac{1}{2}} S_{\mu,x} x^{\mu+\frac{1}{2}} e^{-x^2} dx \\
&= \int_0^\infty x^m x^{\mu+\frac{1}{2}} x^{-\mu-\frac{1}{2}} D_x x^{2\mu+1} D_x x^{-\mu-\frac{1}{2}} x^{\mu+\frac{1}{2}} e^{-x^2} dx \\
&= \int_0^\infty x^m D_x x^{2\mu+1} D_x e^{-x^2} dx \\
&= \left\{ x^m x^{2\mu+1} D_x e^{-x^2} \right\} \Big|_{x=0}^\infty - \int_0^\infty m x^{m-1} x^{2\mu+1} D_x e^{-x^2} dx \\
&= -m \int_1^2 x^{m-1} x^{2\mu+1} (-2x) e^{-x^2} dx \\
&= 2m \int_1^2 x^{m+2\mu+1} e^{-x^2} dx \neq 0.
\end{aligned}$$

This implies that moments of all orders weighted by $x^{\mu+\frac{1}{2}}$ of $\psi(x)$ are non-zero.

From the aforesaid Lemma 5.2.3 and Theorem 5.2.4, it is clear that the continuous Bessel wavelet transform of the distribution of type $Cx^{\mu+\frac{1}{2}}$ will be zero. Therefore, the inversion formula of the continuous Bessel wavelet transform in $H'_\mu(\mathbb{R}^+)$ can not be defined. So the inversion formula of the continuous Bessel wavelet transform will be valid for modulo a $Cx^{\mu+\frac{1}{2}}$ distributions.

For investigating the inversion formula of the continuous Bessel wavelet transform of distributions, our observations will be based on the following points:

(A) Inversion formula of the continuous Bessel wavelet transform of distribution will be found on $H'_{\mu,F}(\mathbb{R}^+)$ which can be obtained by filtering

- (i) all the non-zero $Cx^{\mu+\frac{1}{2}}$ distributions from $H'_\mu(\mathbb{R}^+)$.
- (ii) all non-zero $Cx^{\mu+\frac{1}{2}}$ that appear with a distribution as a union. For example in the distribution $\frac{x^{\mu+\frac{5}{2}}}{1+x^2} = x^{\mu+\frac{1}{2}} - \frac{x^{\mu+\frac{1}{2}}}{1+x^2}$, we would omit $x^{\mu+\frac{1}{2}}$ and retain only $-\frac{x^{\mu+\frac{1}{2}}}{1+x^2}$.

- (B) The Bessel wavelet kernel under consideration for determining the continuous Bessel wavelet transform are those Bessel wavelets whose all the moments weighted by $x^{\mu+\frac{1}{2}}$ are non-zero.

5.3 Properties of the continuous Bessel wavelet transform in $L^2(\mathbb{R}^+)$ and structure formula in $H'_\mu(\mathbb{R}^+)$

In this section, the various properties of the continuous Bessel wavelet transform in $L^2(\mathbb{R}^+)$ and the structure formula in $H'_\mu(\mathbb{R}^+)$ are discussed by taking the theory of Hankel transform.

Note: If $f \in L^1(\mathbb{R}^+)$, the integral in 1 – dimension can be expressed as

$$\int_0^\infty f(x)dx = \int_0^N f(x)dx + \int_N^\infty f(x)dx, \quad N \in (0, \infty), \quad (5.3.1)$$

provided the integral is convergent. Then we have,

$$\int_0^\infty f(x)dx = \int_0^N f(x)dx + R(N), \quad (5.3.2)$$

where $R(N)$ is the remainder term. If f is integrable then $R(N) \rightarrow 0$ as $N \rightarrow \infty$.

We use this technique in the following theorem.

Lemma 5.3.1. *Let $\psi \in L^2(\mathbb{R}^+)$, $a > 0$ and $b > 0$ with $\mu \geq -\frac{1}{2}$. Then $\psi\left(\frac{x}{a}, \frac{b}{a}\right)$ belongs to $L^2_b(\mathbb{R}^+)$ as well as in $L^2_x(\mathbb{R}^+)$ and its Hankel transform with respect to x and b will be*

$$h_\mu^x \psi\left(\frac{x}{a}, \frac{b}{a}\right)(y) = a^{-\mu+\frac{1}{2}} y^{-\mu-\frac{1}{2}} (by)^{\frac{1}{2}} J_\mu(by) (h_\mu \psi)(ay), \quad (5.3.3)$$

and

$$h_\mu^b \psi \left(\frac{x}{a}, \frac{b}{a} \right) (y) = a^{-\mu+\frac{1}{2}} y^{-\mu-\frac{1}{2}} (xy)^{\frac{1}{2}} J_\mu(xy) (h_\mu \psi)(ay). \quad (5.3.4)$$

Proof. From Upadhyay et al. [56], we have

$$\psi \left(\frac{x}{a}, \frac{b}{a} \right) = a^{-\mu+\frac{1}{2}} \int_0^\infty y^{-\mu-\frac{1}{2}} (xy)^{\frac{1}{2}} J_\mu(xy) (by)^{\frac{1}{2}} J_\mu(by) (h_\mu \psi)(ay) dy. \quad (5.3.5)$$

Since $t^{-\mu} J_\mu(t)$ uniformly bounded on \mathbb{R}^+ and $h_\mu \psi \in H_\mu(\mathbb{R}^+) \subset L^2(\mathbb{R}^+)$, $y^{-\mu-\frac{1}{2}} (xy)^{\frac{1}{2}} J_\mu(xy) (h_\mu \psi)(ay)$ and $y^{-\mu-\frac{1}{2}} (by)^{\frac{1}{2}} J_\mu(by) (h_\mu \psi)(ay) \in L^2(\mathbb{R}^+)$. Hence, (5.3.3) and (5.3.4) both exist. \square

Theorem 5.3.2. Let $\psi \in H_\mu(\mathbb{R}^+)$ be the Bessel wavelet, $f \in L^2(\mathbb{R}^+)$ and $(B_\psi f)(a, b)$ be the continuous Bessel wavelet transform of f . Then the Hankel transform of $(B_\psi f)(a, b)$ with respect to variable b can be written as

$$h_\mu^b \{(B_\psi f)(a, b)\}(y) = a^{\mu-\frac{1}{2}} \left\langle f(x), h_\mu^b \psi \left(\frac{x}{a}, \frac{b}{a} \right) \right\rangle. \quad (5.3.6)$$

Proof. First we prove that $(B_\psi f)(a, b) \in L_b^2(\mathbb{R}^+)$. The Bessel wavelet transform of function $f \in L^2(\mathbb{R}^+)$ with respect to the Bessel wavelet $\psi \in H_\mu(\mathbb{R}^+)$ be

$$(B_\psi f)(a, b) = a^{\mu-\frac{1}{2}} \left\langle f(x), \psi \left(\frac{x}{a}, \frac{b}{a} \right) \right\rangle.$$

Since $f \in L^2(\mathbb{R}^+)$ and $\psi \left(\frac{x}{a}, \frac{b}{a} \right) \in L_x^2(\mathbb{R}^+)$, then by the Parseval formula and Lemma 5.3.1

$$\begin{aligned} (B_\psi f)(a, b) &= a^{\mu-\frac{1}{2}} \left\langle (h_\mu f)(y), h_\mu^x \psi \left(\frac{x}{a}, \frac{b}{a} \right) (y) \right\rangle \\ &= a^{\mu-\frac{1}{2}} \int_0^\infty (h_\mu f)(y) a^{-\mu+\frac{1}{2}} y^{-\mu-\frac{1}{2}} (by)^{\frac{1}{2}} J_\mu(by) (h_\mu \psi)(ay) dy \\ &= \int_0^\infty (by)^{\frac{1}{2}} J_\mu(by) (h_\mu f)(y) y^{-\mu-\frac{1}{2}} (h_\mu \psi)(ay) dy. \end{aligned}$$

Since $\psi \in H_\mu(\mathbb{R}^+)$, so $y^{-\mu-\frac{1}{2}}(h_\mu\psi)(ay)$ uniformly bounded on \mathbb{R}^+ for fix $a > 0$ and $h_\mu f \in L^2(\mathbb{R}^+)$. Therefore $y^{-\mu-\frac{1}{2}}(h_\mu\psi)(ay)(h_\mu f)(y) \in L^2_y(\mathbb{R}^+)$ and by the inverse Hankel transformation, $(B_\psi f)(a, b)$ belongs to $L^2_y(\mathbb{R}^+)$. Now, let

$$\begin{aligned} I &= \int_0^\infty f(x)\psi\left(\frac{x}{a}, \frac{b}{a}\right)dx \\ &= \int_0^N f(x)\psi\left(\frac{x}{a}, \frac{b}{a}\right)dx + \int_N^\infty f(x)\psi\left(\frac{x}{a}, \frac{b}{a}\right)dx \\ &= I(N, b) + R(N, b). \end{aligned}$$

So,

$$\begin{aligned} h_\mu^b(I) &= h_\mu^b(I(N, b)) + h_\mu^b(R(N, b)) \\ &= \int_0^N f(x)h_\mu^b\psi\left(\frac{x}{a}, \frac{b}{a}\right)dx + \int_N^\infty f(x)h_\mu^b\psi\left(\frac{x}{a}, \frac{b}{a}\right)dx, \end{aligned}$$

where $R(N, b)$ as a function of b goes to zero in $L^2(\mathbb{R}^+)$ as $N \rightarrow \infty$, therefore $h_\mu^b R(N, b) \rightarrow 0$ in $L^2(\mathbb{R}^+)$ as a function of y . Hence letting $N \rightarrow \infty$, we get the result (5.3.6). Since integrals

$$\int_0^\infty \int_0^\infty (by)^{\frac{1}{2}} J_\mu(by) f(x)\psi\left(\frac{x}{a}, \frac{b}{a}\right) dx db,$$

and

$$\int_0^\infty \int_0^\infty (by)^{\frac{1}{2}} J_\mu(by) f(x)\psi\left(\frac{x}{a}, \frac{b}{a}\right) db dx,$$

are not absolutely convergent and no way of proving that the above integrals exist.

So we are unable to switch the order of integration directly, that is, we can not apply the Fubini's theorem. \square

Theorem 5.3.3. Let $f \in L^2(\mathbb{R}^+)$ be continuous at a point $x = x_0$, then

$$\lim_{\epsilon \rightarrow 0} \int_0^{\infty} f(x) \phi_{\epsilon}(x) dx = f(x_0), \quad (5.3.7)$$

where the function $\phi_{\epsilon}(x) \in D(\mathbb{R}^+)$ and is defined by

$$\phi_{\epsilon}(x) = \begin{cases} C_{\epsilon} \exp\left(-\frac{\epsilon}{x-x_0} - \frac{\epsilon}{x_0+\epsilon-x}\right), & x_0 < x < x_0 + \epsilon \\ 0, & x \leq x_0 \text{ and } x \geq x_0 + \epsilon \end{cases} \quad (5.3.8)$$

the constant C_{ϵ} chosen so that

$$\int_0^{\infty} \phi_{\epsilon}(x) dx = 1.$$

Proof. Since the integral of $\phi_{\epsilon}(x)$ over \mathbb{R}^+ is 1, therefore we choose C_{ϵ} in such a way that

$$\int_{x_0}^{x_0+\epsilon} C_{\epsilon} \exp\left(-\frac{\epsilon}{x-x_0} - \frac{\epsilon}{x_0+\epsilon-x}\right) dx = 1. \quad (5.3.9)$$

Putting $\frac{x-x_0}{\epsilon} = t$ in (5.3.9), we get

$$\epsilon C_{\epsilon} \int_0^1 \exp\left(-\frac{1}{t(1-t)}\right) dt = 1.$$

Now,

$$\begin{aligned} \int_0^{\infty} f(x) \phi_{\epsilon}(x) dx &= \epsilon C_{\epsilon} \int_0^1 f(x_0 + \epsilon t) \exp\left(-\frac{1}{t(1-t)}\right) dt \\ &= \epsilon C_{\epsilon} \int_0^1 [f(x_0 + \epsilon t) - f(x_0)] \exp\left(-\frac{1}{t(1-t)}\right) dt \\ &\quad + f(x_0) \epsilon C_{\epsilon} \int_0^1 \exp\left(-\frac{1}{t(1-t)}\right) dt. \end{aligned}$$

This implies,

$$\int_0^\infty f(x)\phi_\epsilon(x)dx = \epsilon C_\epsilon \int_0^1 [f(x_0 + \epsilon t) - f(x_0)] \exp\left(-\frac{1}{t(1-t)}\right) dt + f(x_0). \quad (5.3.10)$$

Since f is continuous at $x = x_0$ and $0 < t < 1$, we can choose sufficiently small $\epsilon > 0$ corresponding to δ , so that $|f(x_0 + \epsilon t) - f(x_0)| < \delta$, where $\delta > 0$ is arbitrarily chosen small positive number.

Therefore, using continuity property of f , we get

$$\begin{aligned} & \left| \epsilon C_\epsilon \int_0^1 [f(x_0 + \epsilon t) - f(x_0)] \exp\left(-\frac{1}{t(1-t)}\right) dt \right| \\ & \leq \epsilon C_\epsilon \int_0^1 |f(x_0 + \epsilon t) - f(x_0)| \exp\left(-\frac{1}{t(1-t)}\right) dt \\ & < \delta \epsilon C_\epsilon \int_0^1 \exp\left(-\frac{1}{t(1-t)}\right) dt = \delta. \end{aligned}$$

Since δ is an arbitrarily chosen small positive number, we have

$$\lim_{\epsilon \rightarrow 0} \epsilon C_\epsilon \int_0^1 [f(x_0 + \epsilon t) - f(x_0)] \exp\left(-\frac{1}{t(1-t)}\right) dt = 0. \quad (5.3.11)$$

From (5.3.10) and (5.3.11), we get

$$\lim_{\epsilon \rightarrow 0} \int_0^\infty f(x)\phi_\epsilon(x)dx = f(x_0). \quad (5.3.12)$$

□

Theorem 5.3.4. *Let $f \in L^2(\mathbb{R}^+)$ be continuous function at $x = x_0$ and $h_\mu f$ be its Hankel transform. Let $\phi_\epsilon(x)$ be the function which is defined in Theorem 5.3.3; then*

we prove the following

$$(i) \quad \lim_{\epsilon \rightarrow 0} \int_0^\infty \int_0^\infty (xy)^{\frac{1}{2}} J_\mu(xy) (h_\mu f)(y) \phi_\epsilon(x) dy dx \\ = \int_0^\infty (yx_0)^{\frac{1}{2}} J_\mu(yx_0) (h_\mu f)(y) dy. \quad (5.3.13)$$

$$(ii) \quad \int_0^\infty (yx_0)^{\frac{1}{2}} J_\mu(yx_0) (h_\mu f)(y) dy = f(x_0), \quad (5.3.14)$$

converges in a pointwise sense. This is called the inverse Hankel transform in the sense of pointwise convergence.

Proof. From Theorem 5.3.3, we have

$$\phi_\epsilon(x) = \begin{cases} C_\epsilon \exp\left(-\frac{\epsilon}{x-x_0} - \frac{\epsilon}{x_0+\epsilon-x}\right), & x_0 < x < x_0 + \epsilon \\ 0, & x \leq x_0 \text{ and } x \geq x_0 + \epsilon \end{cases}. \quad (5.3.15)$$

It is easy to see that $\phi_\epsilon^2(x) \leq \phi_\epsilon(x)$. Then, we can find

$$\|\phi_\epsilon\|_2^2 = \int_0^\infty \phi_\epsilon^2(x) dx \leq \int_0^\infty \phi_\epsilon(x) dx = 1.$$

We have

$$\left\langle \int_0^\infty (xy)^{\frac{1}{2}} J_\mu(xy) (h_\mu f)(y) dy, \phi_\epsilon(x) \right\rangle = \left\langle \int_0^N (xy)^{\frac{1}{2}} J_\mu(xy) (h_\mu f)(y) dy, \phi_\epsilon(x) \right\rangle \\ + \left\langle \int_N^\infty (xy)^{\frac{1}{2}} J_\mu(xy) (h_\mu f)(y) dy, \phi_\epsilon(x) \right\rangle. \quad (5.3.16)$$

Since $\int_0^N (xy)^{\frac{1}{2}} J_\mu(xy) (h_\mu f)(y) dy$ is a continuous function of x , for all $x \in \mathbb{R}^+$ therefore using Theorem 5.3.3, we find

$$\lim_{\epsilon \rightarrow 0} \left\langle \int_0^N (xy)^{\frac{1}{2}} J_\mu(xy) (h_\mu f)(y) dy, \phi_\epsilon(x) \right\rangle = \int_0^N (yx_0)^{\frac{1}{2}} J_\mu(yx_0) (h_\mu f)(y) dy. \quad (5.3.17)$$

This implies that after taking the certain limit

$$\lim_{N \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \left\langle \int_0^N (xy)^{\frac{1}{2}} J_\mu(xy) (h_\mu f)(y) dy, \phi_\epsilon(x) \right\rangle = \int_0^\infty (yx_0)^{\frac{1}{2}} J_\mu(yx_0) (h_\mu f)(y) dy.$$

In view of the Holder's inequality, the 2nd term on the R.H.S. of (5.3.16) can be obtained as

$$\begin{aligned} \left| \left\langle \int_N^\infty (xy)^{\frac{1}{2}} J_\mu(xy) (h_\mu f)(y) dy, \phi_\epsilon(x) \right\rangle \right| &\leq \left\| \int_N^\infty (xy)^{\frac{1}{2}} J_\mu(xy) (h_\mu f)(y) dy \right\|_2 \|\phi_\epsilon(x)\|_2 \\ &\leq \left\| \int_N^\infty (xy)^{\frac{1}{2}} J_\mu(xy) (h_\mu f)(y) dy \right\|_2. \end{aligned}$$

Since, $\left\| \int_N^\infty (xy)^{\frac{1}{2}} J_\mu(xy) (h_\mu f)(y) dy \right\|_2 \rightarrow 0$ as $N \rightarrow \infty$. Therefore,

$$\lim_{N \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \left\langle \int_N^\infty (xy)^{\frac{1}{2}} J_\mu(xy) (h_\mu f)(y) dy, \phi_\epsilon(x) \right\rangle = 0. \quad (5.3.18)$$

Thus,

$$\lim_{\epsilon \rightarrow 0} \left\langle \int_0^\infty (xy)^{\frac{1}{2}} J_\mu(xy) (h_\mu f)(y) dy, \phi_\epsilon(x) \right\rangle = \int_0^\infty (yx_0)^{\frac{1}{2}} J_\mu(yx_0) (h_\mu f)(y) dy. \quad (5.3.19)$$

Next, we prove (ii), the pointwise convergence of the inverse Hankel transformation by using the Parseval's formula

$$\langle h_\mu f, h_\mu \phi_\epsilon \rangle = \langle f, \phi_\epsilon \rangle,$$

for $f \in L^2(\mathbb{R}^+)$ and $\phi_\epsilon \in D(\mathbb{R}^+) \subset L^2(\mathbb{R}^+)$.

Letting $\epsilon \rightarrow 0$,

$$\lim_{\epsilon \rightarrow 0} \langle h_\mu f, h_\mu \phi_\epsilon \rangle = \lim_{\epsilon \rightarrow 0} \langle f, \phi_\epsilon \rangle. \quad (5.3.20)$$

From (5.3.19), (5.3.20) and Theorem 5.3.3, we get

$$\int_0^\infty (yx_0)^{\frac{1}{2}} J_\mu(yx_0) (h_\mu f)(y) dy = f(x_0), \quad (5.3.21)$$

for pointwise convergence. \square

Theorem 5.3.5. *Let $\phi \in H_\mu(\mathbb{R}^+)$ be a Bessel wavelet and $f \in L^2(\mathbb{R}^+)$. Then the following classical inversion formula holds:*

$$f(x) = \frac{1}{C_{\mu,\psi}} \int_0^\infty \int_0^\infty (B_\psi f)(a, b) a^{\mu-\frac{1}{2}} \psi\left(\frac{x}{a}, \frac{b}{a}\right) \frac{dbda}{a^{2\mu+2}},$$

where $(B_\psi f)(a, b)$ is defined to be the integral Bessel wavelet transform of $f \in L^2(\mathbb{R}^+)$ by

$$(B_\psi f)(a, b) = \int_0^\infty f(x) a^{\mu-\frac{1}{2}} \psi\left(\frac{x}{a}, \frac{b}{a}\right) dx,$$

in duality notation, we write it in the form

$$(B_\psi f)(a, b) = \left\langle f(x), a^{\mu-\frac{1}{2}} \psi\left(\frac{x}{a}, \frac{b}{a}\right) \right\rangle,$$

where

$$C_{\mu,\psi} = \int_0^\infty \frac{|(h_\mu \psi)(t)|^2}{t^{2\mu+2}} dt < \infty.$$

Proof. From Lemma 5.3.1 and Theorem 5.3.2, we have

$$h_\mu^b \psi\left(\frac{x}{a}, \frac{b}{a}\right)(y) = a^{-\mu+\frac{1}{2}} y^{-\mu-\frac{1}{2}} (xy)^{\frac{1}{2}} J_\mu(xy) (h_\mu \psi)(ay),$$

and

$$h_\mu^b\{(B_\psi f)(a, b)\}(y) = a^{\mu-\frac{1}{2}} \left\langle f(x), h_\mu^b \psi\left(\frac{x}{a}, \frac{b}{a}\right) \right\rangle.$$

Therefore,

$$\begin{aligned} h_\mu^b\{(B_\psi f)(a, b)\}(y) &= \int_0^\infty f(x) y^{-\mu-\frac{1}{2}} (xy)^{\frac{1}{2}} J_\mu(xy) (h_\mu \psi)(ay) dx \\ &= y^{-\mu-\frac{1}{2}} (h_\mu \psi)(ay) \int_0^\infty (xy)^{\frac{1}{2}} J_\mu(xy) f(x) dx \\ &= y^{-\mu-\frac{1}{2}} (h_\mu \psi)(ay) (h_\mu f)(y). \end{aligned}$$

Let us consider that

$$F(x) = \frac{1}{C_{\mu, \psi}} \int_0^\infty \int_0^\infty (B_\psi f)(a, b) a^{\mu-\frac{1}{2}} \psi\left(\frac{x}{a}, \frac{b}{a}\right) \frac{db da}{a^{2\mu+2}}.$$

Now using the Parseval's formula of the Hankel transform with respect to variable b , we get

$$\begin{aligned} F(x) &= \frac{1}{C_{\mu, \psi}} \int_0^\infty a^{\mu-\frac{1}{2}} \left[\int_0^\infty h_\mu^b\{(B_\psi f)(a, b)\}(y) h_\mu^b\left\{\psi\left(\frac{x}{a}, \frac{b}{a}\right)\right\}(y) dy \right] \frac{da}{a^{2\mu+2}} \\ &= \frac{1}{C_{\mu, \psi}} \int_0^\infty a^{\mu-\frac{1}{2}} \left[\int_0^\infty y^{-\mu-\frac{1}{2}} (h_\mu \psi)(ay) (h_\mu f)(y) \right. \\ &\quad \left. \times a^{-\mu+\frac{1}{2}} y^{-\mu-\frac{1}{2}} (xy)^{\frac{1}{2}} J_\mu(xy) (h_\mu \psi)(ay) dy \right] \frac{da}{a^{2\mu+2}} \\ &= \frac{1}{C_{\mu, \psi}} \int_0^\infty (xy)^{\frac{1}{2}} J_\mu(xy) (h_\mu f)(y) \left[\int_0^\infty \frac{|(h_\mu \psi)(ay)|^2 da}{(ay)^{2\mu+1} a} \right] dy \\ &= \frac{C_{\mu, \psi}}{C_{\mu, \psi}} \int_0^\infty (xy)^{\frac{1}{2}} J_\mu(xy) (h_\mu f)(y) dy \\ &= \int_0^\infty (xy)^{\frac{1}{2}} J_\mu(xy) (h_\mu f)(y) dy \\ &= f(x), \end{aligned}$$

where putting $ay = t$ in $\int_0^\infty \frac{|(h_\mu \psi)(ay)|^2 da}{(ay)^{2\mu+1} a}$, it will be $\int_0^\infty \frac{|(h_\mu \psi)(t)|^2 dt}{t^{2\mu+2}} = C_{\mu, \psi}$. Here convergence is in $L^2(\mathbb{R}^+)$ sense. \square

Theorem 5.3.6. Let $\psi \in H_\mu(\mathbb{R}^+)$ be a Bessel wavelet and $f \in L^2(\mathbb{R}^+)$ be continuous at $x = x_0$, then

$$\frac{1}{C_{\mu,\psi}} \int_0^\infty \int_0^\infty \int_0^\infty f(t) \psi\left(\frac{t}{a}, \frac{b}{a}\right) \psi\left(\frac{x_0}{a}, \frac{b}{a}\right) \frac{dt db da}{a^3} = f(x_0),$$

in the pointwise sense.

Proof. Applying the same technique, as in the proof of Theorem 5.3.5, at $x = x_0$

$$\frac{1}{C_{\mu,\psi}} \int_0^\infty \int_0^\infty \int_0^\infty f(t) \psi\left(\frac{t}{a}, \frac{b}{a}\right) \psi\left(\frac{x_0}{a}, \frac{b}{a}\right) \frac{dt db da}{a^3} = \int_0^\infty (yx_0)^{\frac{1}{2}} J_\mu(yx_0) (h_\mu f)(y) dy. \quad (5.3.22)$$

In view of Theorem 5.3.4 (ii), at the point of continuity $x = x_0$ of f , we have

$$\int_0^\infty (yx_0)^{\frac{1}{2}} J_\mu(yx_0) (h_\mu f)(y) dy = f(x_0), \quad (5.3.23)$$

in the sense of pointwise convergence. From (5.3.22) and (5.3.23), we get the required result. \square

Theorem 5.3.7. Let $\mu \geq -\frac{1}{2}$. A functional $f \in H'_\mu(\mathbb{R}^+)$, then there exists $r \in \mathbb{N}_0$ and $f_k \in L^q(\mathbb{R}^+)$, ($1 < q \leq \infty$) $k = 0, 1, 2, \dots, r$; such that

$$f = \sum_{k=0}^r S_{\mu,x}^k (1+x^2)^r x^{-\mu-\frac{1}{2}} f_k, \quad (5.3.24)$$

where $S_{\mu,x} = x^{-\mu-\frac{1}{2}} D_x x^{2\mu+1} D_x x^{-\mu-\frac{1}{2}}$ is a Bessel operator.

Proof. From Marrero and Arteaga [1, p. 8], we have

$$f = \sum_{k=0}^r S_{\mu,x}^k (1+x^2)^r x^{-\mu-\frac{1}{2}} f_k, \quad (5.3.25)$$

Therefore,

$$\begin{aligned}\langle f, \phi \rangle &= \left\langle \sum_{k=0}^r S_{\mu,x}^k (1+x^2)^r x^{-\mu-\frac{1}{2}} f_k(x), \phi(x) \right\rangle \\ &= \sum_{k=0}^r \left\langle S_{\mu,x}^k (1+x^2)^r x^{-\mu-\frac{1}{2}} f_k(x), \phi(x) \right\rangle.\end{aligned}$$

Using (1.7.9) and distributional differentiation, we find

$$\begin{aligned}\langle f, \phi \rangle &= \sum_{k=0}^r \left\langle f_k(x), (1+x^2)^r x^{-\mu-\frac{1}{2}} S_{\mu,x}^k \phi(x) \right\rangle \\ &= \sum_{k=0}^r \left\langle f_k(x), (1+x^2)^r x^{-\mu-\frac{1}{2}} \sum_{j=0}^k b_j x^{2j+\mu+\frac{1}{2}} \left(x^{-1} \frac{\partial}{\partial x}\right)^{k+j} x^{-\mu-\frac{1}{2}} \phi(x) \right\rangle \\ &= \sum_{k=0}^r \sum_{j=0}^k \left\langle f_k(x), (1+x^2)^r x^{-\mu-\frac{1}{2}} b_j x^{2j+\mu+\frac{1}{2}} \left(x^{-1} \frac{\partial}{\partial x}\right)^{k+j} x^{-\mu-\frac{1}{2}} \phi(x) \right\rangle.\end{aligned}$$

So that the above expression can be written as

$$\begin{aligned}&|\langle f, \phi \rangle| \\ &\leq \sum_{k=0}^r \sum_{j=0}^k \left| \left\langle f_k(x), (1+x^2)^r x^{-\mu-\frac{1}{2}} b_j x^{2j+\mu+\frac{1}{2}} \left(x^{-1} \frac{\partial}{\partial x}\right)^{k+j} x^{-\mu-\frac{1}{2}} \phi(x) \right\rangle \right| \\ &\leq \sum_{k=0}^r \sum_{j=0}^k \int_0^\infty \left| f_k(x) (1+x^2)^r x^{-\mu-\frac{1}{2}} b_j x^{2j+\mu+\frac{1}{2}} \left(x^{-1} \frac{\partial}{\partial x}\right)^{k+j} x^{-\mu-\frac{1}{2}} \phi(x) \right| dx \\ &\leq \sum_{k=0}^r \sum_{j=0}^k \int_0^\infty \left| f_k(x) (1+x^2)^r b_j x^{2j} \left(x^{-1} \frac{\partial}{\partial x}\right)^{k+j} x^{-\mu-\frac{1}{2}} \phi(x) \right| dx \\ &\leq \sum_{k=0}^r \sum_{j=0}^k \int_0^\infty \left| f_k(x) (1+x^2)^{r+j} b_j \left(x^{-1} \frac{\partial}{\partial x}\right)^{k+j} x^{-\mu-\frac{1}{2}} \phi(x) \right| dx \\ &\leq \sum_{k=0}^r \sum_{j=0}^k b_j \gamma_{r+j+1, k+j}^\mu(\phi) \int_0^\infty \frac{|f_k(x)|}{1+x^2} dx.\end{aligned}$$

Let q be a conjugate exponent of p ($1 \leq p \leq \infty$) such that $\frac{1}{p} + \frac{1}{q} = 1$; then using the Holder's inequality we get

$$\begin{aligned} |\langle f, \phi \rangle| &\leq \sum_{k=0}^r \sum_{j=0}^k b_j \gamma_{r+j+1, k+j}^\mu(\phi) \left(\int_0^\infty |f_k(x)|^q dx \right)^{\frac{1}{q}} \left(\int_0^\infty \frac{1}{(1+x^2)^p} dx \right)^{\frac{1}{p}} \\ &\leq \sum_{k=0}^r \sum_{j=0}^k C_p b_j \gamma_{r+j+1, k+j}^\mu(\phi) \|f_k\|_q. \end{aligned}$$

□

5.4 Inversion formula of the continuous Bessel wavelet transform of the distributions

In this section, using structure formula and the theory of Hankel transformation, inversion formula of the continuous Bessel wavelet transform of distributions and its properties in $H'_\mu(\mathbb{R}^+)$ are obtained.

Lemma 5.4.1. *Let $\psi \in H_\mu(\mathbb{R}^+)$ be a Bessel wavelet and $\phi \in H_\mu(\mathbb{R}^+)$, then*

$$\frac{1}{C_{\mu, \psi}} \int_0^\infty \int_0^\infty \int_0^\infty S_{\mu, t}^k \phi(t) \psi\left(\frac{t}{a}, \frac{b}{a}\right) \psi\left(\frac{x_0}{a}, \frac{b}{a}\right) \frac{dt db da}{a^3} = S_{\mu, x}^k \phi(x) \Big|_{x=x_0},$$

in the pointwise sense.

Proof. Since $\phi \in H_\mu(\mathbb{R}^+)$, from Theorem 5.3.6 we have

$$\frac{1}{C_{\mu, \psi}} \int_0^\infty \int_0^\infty \int_0^\infty S_{\mu, t}^k \phi(t) \psi\left(\frac{t}{a}, \frac{b}{a}\right) \psi\left(\frac{x_0}{a}, \frac{b}{a}\right) \frac{dt db da}{a^3} = S_{\mu, x}^k \phi(x) \Big|_{x=x_0},$$

where $S_{\mu, x}^k$ is Bessel operator of order $k \in \mathbb{N}_0$.

□

Lemma 5.4.2. Let $\psi \in H_\mu(\mathbb{R}^+)$ be a Bessel wavelet and $\phi \in H_\mu(\mathbb{R}^+)$, then

$$\frac{1}{C_{\mu,\psi}} \int_0^\infty \int_0^\infty \int_0^\infty S_{\mu,t}^k \phi(t) \psi\left(\frac{t}{a}, \frac{b}{a}\right) \psi\left(\frac{x}{a}, \frac{b}{a}\right) \frac{dt db da}{a^3},$$

converges uniformly to $S_{\mu,x}^k \phi(x)$ for every $x \in \mathbb{R}^+$.

Proof. From Theorem 5.3.5, convergence in $L^2(\mathbb{R}^+)$ sense, we have

$$\begin{aligned} \frac{1}{C_{\mu,\psi}} \int_0^\infty \int_0^\infty \int_0^\infty S_{\mu,t}^k \phi(t) \psi\left(\frac{t}{a}, \frac{b}{a}\right) \psi\left(\frac{x}{a}, \frac{b}{a}\right) \frac{dt db da}{a^3} \\ = \int_0^\infty (yx)^{\frac{1}{2}} J_\mu(yx) (h_\mu S_{\mu,x}^k \phi(x))(y) dy = S_{\mu,x}^k \phi(x). \end{aligned}$$

By Weierstrass M -test, we have

$$\left| \int_0^\infty (yx)^{\frac{1}{2}} J_\mu(yx) (h_\mu S_{\mu,x}^k \phi(x))(y) dy \right| \leq \int_0^\infty \left| (h_\mu S_{\mu,x}^k \phi(x))(y) \right| dy < \infty,$$

and $S_{\mu,x}^k \phi(x) \in H_\mu(\mathbb{R}^+)$. Therefore,

$$\frac{1}{C_{\mu,\psi}} \int_0^\infty \int_0^\infty \int_0^\infty S_{\mu,t}^k \phi(t) \psi\left(\frac{t}{a}, \frac{b}{a}\right) \psi\left(\frac{x}{a}, \frac{b}{a}\right) \frac{dt db da}{a^3}, \quad (5.4.1)$$

converges uniformly to $S_{\mu,x}^k \phi(x)$. \square

Lemma 5.4.3. Let $\mu \geq -\frac{1}{2}$ and $\psi\left(\frac{x}{a}, \frac{b}{a}\right)$ be Hankel translation and dilation of $\psi \in H_\mu(\mathbb{R}^+)$ then

$$S_{\mu,x} \psi\left(\frac{x}{a}, \frac{b}{a}\right) = S_{\mu,b} \psi\left(\frac{x}{a}, \frac{b}{a}\right).$$

Proof. From (1.4.4), we have

$$S_{\mu,x} \psi\left(\frac{x}{a}, \frac{b}{a}\right) = S_{\mu,x} \int_0^\infty \psi(z) D_\mu\left(\frac{x}{a}, \frac{b}{a}, z\right) dz$$

$$\begin{aligned}
&= S_{\mu,x} \int_0^\infty \phi(z) \int_0^\infty t^{-\mu-\frac{1}{2}} \left(\frac{xt}{a}\right)^{\frac{1}{2}} J_\mu\left(\frac{xt}{a}\right) \left(\frac{bt}{a}\right)^{\frac{1}{2}} J_\mu\left(\frac{bt}{a}\right) (zt)^{\frac{1}{2}} J_\mu(zt) dt dz \\
&= \int_0^\infty \phi(z) \int_0^\infty t^{-\mu-\frac{1}{2}} S_{\mu,x} \left\{ \left(\frac{xt}{a}\right)^{\frac{1}{2}} J_\mu\left(\frac{xt}{a}\right) \right\} \left(\frac{bt}{a}\right)^{\frac{1}{2}} J_\mu\left(\frac{bt}{a}\right) (zt)^{\frac{1}{2}} J_\mu(zt) dt dz.
\end{aligned}$$

On using differential operators from (1.7.6) and (1.7.7), we find that

$$\begin{aligned}
S_{\mu,x} x^{\frac{1}{2}} J_\mu\left(\frac{xt}{a}\right) &= x^{-\mu-\frac{1}{2}} D_x x^{2\mu+1} D_x x^{-\mu} J_\mu\left(\frac{xt}{a}\right) \\
&= x^{-\mu-\frac{1}{2}} D_x x^{2\mu+1} \left(-\frac{t}{a}\right) x^{-\mu} J_{\mu+1}\left(\frac{xt}{a}\right) \\
&= \left(-\frac{t}{a}\right) x^{-\mu-\frac{1}{2}} D_x x^{\mu+1} J_{\mu+1}\left(\frac{xt}{a}\right) \\
&= -\left(\frac{t}{a}\right)^2 x^{-\mu-\frac{1}{2}} x^{\mu+1} J_\mu\left(\frac{xt}{a}\right) \\
&= -\left(\frac{t}{a}\right)^2 x^{\frac{1}{2}} J_\mu\left(\frac{xt}{a}\right).
\end{aligned}$$

Similarly,

$$S_{\mu,b} b^{\frac{1}{2}} J_\mu\left(\frac{bt}{a}\right) = -\left(\frac{t}{a}\right)^2 b^{\frac{1}{2}} J_\mu\left(\frac{bt}{a}\right).$$

So,

$$\begin{aligned}
&S_{\mu,x} \psi\left(\frac{x}{a}, \frac{b}{a}\right) \\
&= \int_0^\infty \phi(z) \int_0^\infty t^{-\mu-\frac{1}{2}} (-1) \left(\frac{t}{a}\right)^2 \left(\frac{xt}{a}\right)^{\frac{1}{2}} J_\mu\left(\frac{xt}{a}\right) \left(\frac{bt}{a}\right)^{\frac{1}{2}} J_\mu\left(\frac{bt}{a}\right) (zt)^{\frac{1}{2}} J_\mu(zt) dt dz \\
&= \int_0^\infty \phi(z) \int_0^\infty t^{-\mu-\frac{1}{2}} \left(\frac{xt}{a}\right)^{\frac{1}{2}} J_\mu\left(\frac{xt}{a}\right) S_{\mu,b} \left\{ \left(\frac{bt}{a}\right)^{\frac{1}{2}} J_\mu\left(\frac{bt}{a}\right) \right\} (zt)^{\frac{1}{2}} J_\mu(zt) dt dz \\
&= S_{\mu,b} \int_0^\infty \psi(z) D_\mu\left(\frac{x}{a}, \frac{b}{a}, z\right) dz \\
&= S_{\mu,b} \psi\left(\frac{x}{a}, \frac{b}{a}\right).
\end{aligned}$$

□

Theorem 5.4.4. Let $f \in H'_{\mu,F}(\mathbb{R}^+)$ and $\psi \in H_\mu(\mathbb{R}^+)$ be a Bessel wavelet. Define the continuous Bessel wavelet transform of f with respect to Bessel wavelet ψ by

$$(B_\psi f)(a, b) = \left\langle f(t), a^{\mu-\frac{1}{2}} \psi\left(\frac{t}{a}, \frac{b}{a}\right) \right\rangle, \quad (5.4.2)$$

where \langle, \rangle is functional notation.

Then the inversion formula of the continuous Bessel wavelet transform is given by,

$$\lim_{\substack{A, B \rightarrow \infty \\ \epsilon \rightarrow 0^+}} \left\langle \frac{1}{C_{\mu, \psi}} \int_\epsilon^A \int_0^B (B_\psi f)(a, b) a^{\mu-\frac{1}{2}} \psi\left(\frac{x}{a}, \frac{b}{a}\right) \frac{db da}{a^{2\mu+2}}, \phi \right\rangle = \langle f, \phi \rangle, \quad (5.4.3)$$

for all $\phi \in H_\mu(\mathbb{R}^+)$.

Proof. From Definition 5.2.2, for $\psi \in H_\mu(\mathbb{R}^+)$, the Bessel wavelet and $f \in H'_{\mu,F}(\mathbb{R}^+)$

$$(B_\psi f)(a, b) = \left\langle f(t), a^{\mu-\frac{1}{2}} \psi\left(\frac{t}{a}, \frac{b}{a}\right) \right\rangle.$$

From Theorem 5.3.7, the above expression can be written as

$$\begin{aligned} (B_\psi f)(a, b) &= \left\langle \sum_{k=0}^r S_{\mu, t}^k (1+t^2)^r t^{-\mu-\frac{1}{2}} f_k(t), a^{\mu-\frac{1}{2}} \psi\left(\frac{t}{a}, \frac{b}{a}\right) \right\rangle \\ &= \sum_{k=0}^r \left\langle S_{\mu, t}^k (1+t^2)^r t^{-\mu-\frac{1}{2}} f_k(t), a^{\mu-\frac{1}{2}} \psi\left(\frac{t}{a}, \frac{b}{a}\right) \right\rangle. \end{aligned}$$

Applying distributional differentiation, we find

$$\begin{aligned} (B_\psi f)(a, b) &= \sum_{k=0}^r \left\langle f_k(t), (1+t^2)^r t^{-\mu-\frac{1}{2}} S_{\mu, t}^k a^{\mu-\frac{1}{2}} \psi\left(\frac{t}{a}, \frac{b}{a}\right) \right\rangle \\ &= \sum_{k=0}^r \left\langle f_k(t), (1+t^2)^r t^{-\mu-\frac{1}{2}} a^{\mu-\frac{1}{2}} S_{\mu, t}^k \psi\left(\frac{t}{a}, \frac{b}{a}\right) \right\rangle. \end{aligned} \quad (5.4.4)$$

Let us consider

$$F(x) = \frac{1}{C_{\mu,\psi}} \int_{\epsilon}^A \int_0^B (B_{\psi}f)(a,b) a^{\mu-\frac{1}{2}} \psi\left(\frac{x}{a}, \frac{b}{a}\right) \frac{dbda}{a^{2\mu+2}}. \quad (5.4.5)$$

Therefore,

$$\begin{aligned} & \lim_{\substack{A,B \rightarrow \infty \\ \epsilon \rightarrow 0+}} \langle F(x), \phi(x) \rangle \\ &= \lim_{\substack{A,B \rightarrow \infty \\ \epsilon \rightarrow 0+}} \left\langle \frac{1}{C_{\mu,\psi}} \int_{\epsilon}^A \int_0^B (B_{\psi}f)(a,b) a^{\mu-\frac{1}{2}} \psi\left(\frac{x}{a}, \frac{b}{a}\right) \frac{dbda}{a^{2\mu+2}}, \phi(x) \right\rangle. \end{aligned} \quad (5.4.6)$$

Using structure formula (5.3.24) for f , the distributional problem (5.4.6) is converted into a classical one. If we take lower limit $\epsilon \rightarrow 0+$ and upper limits $A, B \rightarrow \infty$, then from (5.4.4) we obtain

$$\begin{aligned} & \lim_{\substack{A,B \rightarrow \infty \\ \epsilon \rightarrow 0+}} \langle F(x), \phi(x) \rangle \\ &= \left\langle \frac{1}{C_{\mu,\psi}} \int_0^{\infty} \int_0^{\infty} \left\{ \sum_{k=0}^r \int_0^{\infty} f_k(t) (1+t^2)^r t^{-\mu-\frac{1}{2}} a^{\mu-\frac{1}{2}} S_{\mu,t}^k \psi\left(\frac{t}{a}, \frac{b}{a}\right) dt \right\} \right. \\ & \quad \left. \times a^{\mu-\frac{1}{2}} \psi\left(\frac{x}{a}, \frac{b}{a}\right) \frac{dbda}{a^{2\mu+2}}, \phi(x) \right\rangle \\ &= \sum_{k=0}^r \left\langle \frac{1}{C_{\mu,\psi}} \int_0^{\infty} \int_0^{\infty} \left\{ \int_0^{\infty} f_k(t) (1+t^2)^r t^{-\mu-\frac{1}{2}} a^{\mu-\frac{1}{2}} S_{\mu,t}^k \psi\left(\frac{t}{a}, \frac{b}{a}\right) dt \right\} \right. \\ & \quad \left. \times a^{\mu-\frac{1}{2}} \psi\left(\frac{x}{a}, \frac{b}{a}\right) \frac{dbda}{a^{2\mu+2}}, \phi(x) \right\rangle \\ &= \sum_{k=0}^r \left\langle \frac{1}{C_{\mu,\psi}} \int_0^{\infty} \left[\int_0^{\infty} \left\{ \int_0^{\infty} f_k(t) (1+t^2)^r t^{-\mu-\frac{1}{2}} a^{\mu-\frac{1}{2}} S_{\mu,b}^k \psi\left(\frac{t}{a}, \frac{b}{a}\right) dt \right\} \right. \right. \\ & \quad \left. \left. \times a^{\mu-\frac{1}{2}} \psi\left(\frac{x}{a}, \frac{b}{a}\right) db \right] \frac{da}{a^{2\mu+2}}, \phi(x) \right\rangle. \end{aligned}$$

From Lemma 5.4.3, $S_{\mu,t}^k \psi\left(\frac{t}{a}, \frac{b}{a}\right) = S_{\mu,b}^k \psi\left(\frac{t}{a}, \frac{b}{a}\right)$ and using integration by parts, we get

$$\begin{aligned}
& \lim_{\substack{A, B \rightarrow \infty \\ \epsilon \rightarrow 0^+}} \langle F(x), \phi(x) \rangle \\
&= \sum_{k=0}^r \left\langle \frac{1}{C_{\mu, \psi}} \int_0^\infty \left[\int_0^\infty \left\{ \int_0^\infty f_k(t) (1+t^2)^r t^{-\mu-\frac{1}{2}} a^{\mu-\frac{1}{2}} \psi\left(\frac{t}{a}, \frac{b}{a}\right) dt \right\} \right. \right. \\
&\quad \left. \left. \times a^{\mu-\frac{1}{2}} S_{\mu, b}^k \psi\left(\frac{x}{a}, \frac{b}{a}\right) db \right] \frac{da}{a^{2\mu+2}}, \phi(x) \right\rangle \\
&= \sum_{k=0}^r \left\langle \frac{1}{C_{\mu, \psi}} \int_0^\infty \left[\int_0^\infty \left\{ \int_0^\infty f_k(t) (1+t^2)^r t^{-\mu-\frac{1}{2}} a^{\mu-\frac{1}{2}} \psi\left(\frac{t}{a}, \frac{b}{a}\right) dt \right\} \right. \right. \\
&\quad \left. \left. \times a^{\mu-\frac{1}{2}} S_{\mu, x}^k \psi\left(\frac{x}{a}, \frac{b}{a}\right) db \right] \frac{da}{a^{2\mu+2}}, \phi(x) \right\rangle
\end{aligned}$$

$$\text{as } S_{\mu, b}^k \psi\left(\frac{t}{a}, \frac{b}{a}\right) = S_{\mu, x}^k \psi\left(\frac{t}{a}, \frac{b}{a}\right).$$

Now using distributional derivative and the Fubini's theorem, we get

$$\begin{aligned}
& \lim_{\substack{A, B \rightarrow \infty \\ \epsilon \rightarrow 0^+}} \langle F(x), \phi(x) \rangle \\
&= \sum_{k=0}^r \left\langle \frac{1}{C_{\mu, \psi}} \int_0^\infty \left[\int_0^\infty \left\{ \int_0^\infty f_k(t) (1+t^2)^r t^{-\mu-\frac{1}{2}} a^{\mu-\frac{1}{2}} \psi\left(\frac{t}{a}, \frac{b}{a}\right) dt \right\} \right. \right. \\
&\quad \left. \left. \times a^{\mu-\frac{1}{2}} \psi\left(\frac{x}{a}, \frac{b}{a}\right) db \right] \frac{da}{a^{2\mu+2}}, S_{\mu, x}^k \phi(x) \right\rangle \\
&= \sum_{k=0}^r \frac{1}{C_{\mu, \psi}} \int_0^\infty \int_0^\infty \left[\int_0^\infty \left\{ \int_0^\infty f_k(t) (1+t^2)^r t^{-\mu-\frac{1}{2}} a^{\mu-\frac{1}{2}} \psi\left(\frac{t}{a}, \frac{b}{a}\right) dt \right\} \right. \\
&\quad \left. \times a^{\mu-\frac{1}{2}} \psi\left(\frac{x}{a}, \frac{b}{a}\right) db \right] \frac{da}{a^{2\mu+2}} S_{\mu, x}^k \phi(x) dx \\
&= \sum_{k=0}^r \frac{1}{C_{\mu, \psi}} \int_0^\infty \left[\int_0^\infty \int_0^\infty \int_0^\infty \left\{ S_{\mu, x}^k \phi(x) \right\} \psi\left(\frac{t}{a}, \frac{b}{a}\right) \psi\left(\frac{x}{a}, \frac{b}{a}\right) \frac{dx db da}{a^3} \right] \\
&\quad \times f_k(t) (1+t^2)^r t^{-\mu-\frac{1}{2}} dt.
\end{aligned}$$

Now, using the continuous Bessel wavelet transform inversion formula, Lemma 5.4.1 and Lemma 5.4.2, the expression in the bracket

$$\int_0^\infty \int_0^\infty \int_0^\infty \left\{ S_{\mu,x}^k \phi(x) \right\} \psi\left(\frac{t}{a}, \frac{b}{a}\right) \psi\left(\frac{x}{a}, \frac{b}{a}\right) \frac{dx db da}{a^3},$$

converges to $S_{\mu,t}^k \phi(t)$

(i) in $L^2(\mathbb{R}^+)$

(ii) pointwise $\forall t \in \mathbb{R}^+$

(iii) uniformly $\forall t \in \mathbb{R}^+$

Therefore,

$$\begin{aligned} \lim_{\substack{A, B \rightarrow \infty \\ \epsilon \rightarrow 0^+}} \left\langle F(x), \phi(x) \right\rangle &= \sum_{k=0}^r \int_0^\infty \left\{ S_{\mu,t}^k \phi(t) \right\} f_k(t) (1+t^2)^r t^{-\mu-\frac{1}{2}} dt \\ &= \sum_{k=0}^r \left\langle f_k(t), (1+t^2)^r t^{-\mu-\frac{1}{2}} S_{\mu,t}^k \phi(t) \right\rangle \\ &= \left\langle f(x), \phi(x) \right\rangle, \quad \forall \phi(x) \in H_\mu(\mathbb{R}^+) \text{ and } f \in H'_{\mu,F}(\mathbb{R}^+) \end{aligned}$$

using Theorem 5.3.7.

Thus

$$\lim_{\substack{A, B \rightarrow \infty \\ \epsilon \rightarrow 0^+}} \left\langle \frac{1}{C_{\mu,\psi}} \int_\epsilon^A \int_0^B (B_\psi f)(a, b) a^{\mu-\frac{1}{2}} \psi\left(\frac{x}{a}, \frac{b}{a}\right) \frac{db da}{a^{2\mu+2}}, \phi \right\rangle = \langle f, \phi \rangle, \quad (5.4.7)$$

for all $\phi \in H_\mu(\mathbb{R}^+)$, and $f \in H'_{\mu,F}(\mathbb{R}^+)$. This inversion formula is valid uniquely if $f \in H'_{\mu,F}(\mathbb{R}^+)$. If $f \in H'_\mu(\mathbb{R}^+)$, then $f = h + Cx^{\mu+\frac{1}{2}}$, where $h \in H'_{\mu,F}(\mathbb{R}^+)$ and C is constant. Therefore the Bessel wavelet transform of f is

$$\begin{aligned} \left\langle f(x), a^{\mu-\frac{1}{2}} \psi\left(\frac{x}{a}, \frac{b}{a}\right) \right\rangle &= \left\langle h(x), a^{\mu-\frac{1}{2}} \psi\left(\frac{x}{a}, \frac{b}{a}\right) \right\rangle + \int_0^\infty C x^{\mu+\frac{1}{2}} a^{\mu-\frac{1}{2}} \psi\left(\frac{x}{a}, \frac{b}{a}\right) dx \\ &= \left\langle h(x), a^{\mu-\frac{1}{2}} \psi\left(\frac{x}{a}, \frac{b}{a}\right) \right\rangle, \end{aligned} \quad (5.4.8)$$

as the second term will be zero.

Hence

$$(B_\psi f)(a, b) = (B_\psi h)(a, b),$$

and so, using (5.4.7), we find

$$\begin{aligned} & \lim_{\substack{A, B \rightarrow \infty \\ \epsilon \rightarrow 0^+}} \left\langle \frac{1}{C_{\mu, \psi}} \int_\epsilon^A \int_0^B (B_\psi f)(a, b) a^{\mu - \frac{1}{2}} \psi\left(\frac{x}{a}, \frac{b}{a}\right) \frac{db da}{a^{2\mu + 2}}, \phi \right\rangle \\ &= \lim_{\substack{A, B \rightarrow \infty \\ \epsilon \rightarrow 0^+}} \left\langle \frac{1}{C_{\mu, \psi}} \int_\epsilon^A \int_0^B (B_\psi h)(a, b) a^{\mu - \frac{1}{2}} \psi\left(\frac{x}{a}, \frac{b}{a}\right) \frac{db da}{a^{2\mu + 2}}, \phi \right\rangle \\ &= \langle h, \phi \rangle \\ &= \left\langle f - C x^{\mu + \frac{1}{2}}, \phi \right\rangle \\ &= \langle f, \phi \rangle + \left\langle -C x^{\mu + \frac{1}{2}}, \phi \right\rangle. \end{aligned}$$

Thus for the validity of the uniqueness in our inversion formula 5.4.7, f must belong to $H'_{\mu, F}(\mathbb{R}^+)$. \square

5.5 Calderón reproducing formula and the applications of the continuous Bessel wavelet transform of distributions

In the present section, the Calderón reproducing formula associated with inversion formula of the continuous Bessel wavelet transform of distributions is given. The application of the continuous Bessel wavelet transform of distributions through heat equation is given and the solution of heat equation is expressed in form of the continuous Bessel wavelet transform of distributions in $H'_\mu(\mathbb{R}^+)$ space.

Theorem 5.5.1. *If $f \in H'_{\mu,F}(\mathbb{R}^+)$ and $\psi \in H_\mu(\mathbb{R}^+)$, then the Calderón reproducing formula associated with the inversion formula of the continuous Bessel wavelet transform can be expressed in the following form*

$$\lim_{\substack{A \rightarrow \infty \\ \epsilon \rightarrow 0^+}} \left\langle \frac{1}{C_{\mu,\psi}} \int_\epsilon^A (f \# \overline{\psi_a} \# \psi_a)(x) \frac{da}{a^{2\mu+2}}, \phi \right\rangle = \langle f, \phi \rangle, \quad \forall \phi \in H_\mu(\mathbb{R}^+) \quad (5.5.1)$$

Proof. From [56, p. 318] and Definition 5.2.2, for $f \in H'_\mu(\mathbb{R}^+)$ and $\psi \in H_\mu(\mathbb{R}^+)$, we have

$$\begin{aligned} (B_\psi f)(a, b) &= \left\langle f(x), a^{\mu-\frac{1}{2}} \psi\left(\frac{x}{a}, \frac{b}{a}\right) \right\rangle \\ &= \left\langle f(x), h_\mu \left\{ (b\omega)^{\frac{1}{2}} J_\mu(b\omega) \omega^{-\mu-\frac{1}{2}} (h_\mu \psi)(a\omega) \right\} \right\rangle. \end{aligned} \quad (5.5.2)$$

From [63, p. 142] and [33, p. 364], the above expression will be

$$\begin{aligned} (B_\psi f)(a, b) &= \left\langle h'_\mu f, (b\omega)^{\frac{1}{2}} J_\mu(b\omega) \omega^{-\mu-\frac{1}{2}} (h_\mu \psi)(a\omega) \right\rangle \\ &= \int_0^\infty (b\omega)^{\frac{1}{2}} J_\mu(b\omega) \omega^{-\mu-\frac{1}{2}} (h'_\mu f)(\omega) \overline{(h_\mu \psi)(a\omega)} d\omega \\ &= \int_0^\infty (b\omega)^{\frac{1}{2}} J_\mu(b\omega) h'_\mu \{f \# \overline{\psi_a}\}(\omega) d\omega. \end{aligned} \quad (5.5.3)$$

From Theorem 5.4.4, we have

$$\langle f, \phi \rangle = \lim_{\substack{A, B \rightarrow \infty \\ \epsilon \rightarrow 0^+}} \left\langle \frac{1}{C_{\mu,\psi}} \int_\epsilon^A \int_0^B (B_\psi f)(a, b) a^{\mu-\frac{1}{2}} \psi\left(\frac{x}{a}, \frac{b}{a}\right) \frac{db da}{a^{2\mu+2}}, \phi \right\rangle.$$

Using the same technique as in the proof of Theorem 5.4.4 and from (5.5.3), we get

$$\begin{aligned} \langle f, \phi \rangle &= \left\langle \frac{1}{C_{\mu,\psi}} \int_0^\infty \int_0^\infty \left(\int_0^\infty (b\omega)^{\frac{1}{2}} J_\mu(b\omega) h'_\mu \{f \# \overline{\psi_a}\}(\omega) d\omega \right) a^{\mu-\frac{1}{2}} \psi\left(\frac{x}{a}, \frac{b}{a}\right) \frac{db da}{a^{2\mu+2}}, \phi \right\rangle \\ &= \left\langle \frac{1}{C_{\mu,\psi}} \int_0^\infty \int_0^\infty h'_\mu \{f \# \overline{\psi_a}\}(\omega) (x\omega)^{\frac{1}{2}} J_\mu(x\omega) \omega^{-\mu-\frac{1}{2}} (h_\mu \psi_a)(\omega) \frac{d\omega da}{a^{2\mu+2}}, \phi \right\rangle \end{aligned}$$

$$\begin{aligned}
&= \left\langle \frac{1}{C_{\mu,\psi}} \int_0^\infty \int_0^\infty (x\omega)^{\frac{1}{2}} J_\mu(x\omega) \omega^{-\mu-\frac{1}{2}} h'_\mu \{f \# \overline{\psi_a}\}(\omega) (h_\mu \psi_a)(\omega) d\omega \frac{da}{a^{2\mu+2}}, \phi \right\rangle \\
&= \left\langle \frac{1}{C_{\mu,\psi}} \int_0^\infty \int_0^\infty (x\omega)^{\frac{1}{2}} J_\mu(x\omega) h'_\mu \{f \# \overline{\psi_a} \# \psi_a\}(\omega) d\omega \frac{da}{a^{2\mu+2}}, \phi \right\rangle \\
&= \left\langle \frac{1}{C_{\mu,\psi}} \int_0^\infty (f \# \overline{\psi_a} \# \psi_a)(x) \frac{da}{a^{2\mu+2}}, \phi \right\rangle.
\end{aligned}$$

If ψ satisfies orthonormal condition

$$C_{\mu,\psi} := \int_0^\infty \frac{|(h_\mu f)(\omega)|^2}{\omega^{2\mu+2}} d\omega = 1. \quad (5.5.4)$$

Then

$$\langle f, \phi \rangle = \left\langle \int_0^\infty (f \# \overline{\psi_a} \# \psi_a)(x) \frac{da}{a^{2\mu+2}}, \phi \right\rangle. \quad (5.5.5)$$

□

Theorem 5.5.2. Consider heat equation

$$\frac{\partial u(x,t)}{\partial t} = S_{\mu,x} u(x,t), \quad 0 < x < \infty, \quad t > 0 \quad (5.5.6)$$

with initial condition

$$u(x,0) = S_{\mu,x} f(x), \quad (5.5.7)$$

where $S_{\mu,x} = x^{-\mu-\frac{1}{2}} D_x x^{2\mu+1} D_x x^{-\mu-\frac{1}{2}}$ is the Bessel operator. Then the solution of (5.5.6) and (5.5.7) in $H'_\mu(\mathbb{R}^+)$ space can be obtained in the following form:

$$u(x,t) = \frac{1}{2^{\frac{3}{2}} t^{\frac{1}{2}(\mu+\frac{5}{2})}} (B_\psi f)(2\sqrt{t}, x). \quad (5.5.8)$$

Proof. Exploiting (1.7.8) and using Hankel transformation on (5.5.6) and (5.5.7), we get

$$\begin{aligned}\frac{\partial U(\omega, t)}{\partial t} &= h_\mu(S_{\mu, x}u)(\omega, t) \\ &= -\omega^2 U(\omega, t),\end{aligned}\tag{5.5.9}$$

and

$$\begin{aligned}U(\omega, 0) &= h_\mu(S_{\mu, x}f)(\omega) \\ &= -\omega^2(h_\mu f)(\omega),\end{aligned}\tag{5.5.10}$$

where $U(\omega, t) = (h_\mu u)(\omega, t)$.

Now, from (5.5.9) the solution will be

$$U(\omega, t) = Ce^{-\omega^2 t}.\tag{5.5.11}$$

Taking $t = 0$ in (5.5.11), we get

$$U(\omega, 0) = C.\tag{5.5.12}$$

So that,

$$U(\omega, t) = -\omega^2(h_\mu f)(\omega)e^{-\omega^2 t}.\tag{5.5.13}$$

Taking the inversion formula of Hankel transformation (1.4.2), we get

$$\begin{aligned}u(x, t) &= \int_0^\infty (x\omega)^{\frac{1}{2}} J_\mu(x\omega)(-\omega^2)(h_\mu f)(\omega)e^{-\omega^2 t} d\omega \\ &= \int_0^\infty (x\omega)^{\frac{1}{2}} J_\mu(x\omega)\omega^{-\mu-\frac{1}{2}}\omega^{\mu+\frac{1}{2}}(-\omega^2)(h_\mu f)(\omega)e^{-\omega^2 t} d\omega\end{aligned}$$

$$\begin{aligned}
&= \int_0^\infty (x\omega)^{\frac{1}{2}} J_\mu(x\omega) \omega^{-\mu-\frac{1}{2}} (h_\mu f)(\omega) (-\omega^2) \omega^{\mu+\frac{1}{2}} e^{-\omega^2 t} d\omega \\
&= \int_0^\infty (x\omega)^{\frac{1}{2}} J_\mu(x\omega) \omega^{-\mu-\frac{1}{2}} (h_\mu f)(\omega) \frac{1}{2^{\frac{3}{2}} t^{\frac{1}{2}} (\mu+\frac{5}{2})} h_\mu \left\{ S_{\mu,s} s^{\mu+\frac{1}{2}} e^{-s^2} \right\} (2\sqrt{t}\omega) d\omega,
\end{aligned}$$

since $h_\mu \left\{ S_{\mu,s} s^{\mu+\frac{1}{2}} e^{-s^2} \right\} (2\sqrt{t}\omega) = 2^{\frac{3}{2}} t^{\frac{1}{2}} (\mu+\frac{5}{2}) (-\omega^2) \omega^{\mu+\frac{1}{2}} e^{-\omega^2 t}$.

Let $\psi(s) = S_{\mu,s} s^{\mu+\frac{1}{2}} e^{-s^2}$. Since $(h_\mu \psi)(a\omega) = (h_\mu \psi_a)(\omega)$, $\psi_a(s) = \frac{1}{a} \psi\left(\frac{s}{a}\right)$; taking $a = 2\sqrt{t}$ we get

$$\begin{aligned}
u(x, t) &= \int_0^\infty (x\omega)^{\frac{1}{2}} J_\mu(x\omega) \omega^{-\mu-\frac{1}{2}} (h_\mu f)(\omega) \frac{1}{2^{\frac{3}{2}} t^{\frac{1}{2}} (\mu+\frac{5}{2})} (h_\mu \psi)(2\sqrt{t}\omega) d\omega \\
&= \frac{1}{2^{\frac{3}{2}} t^{\frac{1}{2}} (\mu+\frac{5}{2})} \int_0^\infty (x\omega)^{\frac{1}{2}} J_\mu(x\omega) \omega^{-\mu-\frac{1}{2}} (h_\mu f)(\omega) (h_\mu \psi)(2\sqrt{t}\omega) d\omega \\
&= \frac{1}{2^{\frac{3}{2}} t^{\frac{1}{2}} (\mu+\frac{5}{2})} \int_0^\infty (x\omega)^{\frac{1}{2}} J_\mu(x\omega) h_\mu \{f \# \psi_{2\sqrt{t}}\}(\omega) d\omega \\
&= \frac{1}{2^{\frac{3}{2}} t^{\frac{1}{2}} (\mu+\frac{5}{2})} (f \# \psi_{2\sqrt{t}})(x).
\end{aligned}$$

Using (1.4.3) and (1.4.4), we get

$$\begin{aligned}
u(x, t) &= \frac{1}{2^{\frac{3}{2}} t^{\frac{1}{2}} (\mu+\frac{5}{2})} \int_0^\infty f(s) \psi_{2\sqrt{t}}(s, x) ds \\
&= \frac{1}{2^{\frac{3}{2}} t^{\frac{1}{2}} (\mu+\frac{5}{2})} \int_0^\infty f(s) (2\sqrt{t})^{\mu-\frac{1}{2}} \psi\left(\frac{s}{2\sqrt{t}}, \frac{x}{2\sqrt{t}}\right) ds \\
&= \frac{1}{2^{\frac{3}{2}} t^{\frac{1}{2}} (\mu+\frac{5}{2})} \left\langle f(s), (2\sqrt{t})^{\mu-\frac{1}{2}} \psi\left(\frac{s}{2\sqrt{t}}, \frac{x}{2\sqrt{t}}\right) \right\rangle. \tag{5.5.14}
\end{aligned}$$

From [33], $\psi\left(\frac{s}{2\sqrt{t}}, \frac{x}{2\sqrt{t}}\right)$ will be an element of $H_\mu(\mathbb{R}^+)$ and from [63, p. 130, p. 137] it can be easily seen that $\psi(s) = S_{\mu,s} s^{\mu+\frac{1}{2}} e^{-s^2} \in H_\mu(\mathbb{R}^+)$.

In view of Definition 5.2.2, we have

$$f \in H'_\mu(\mathbb{R}^+).$$

Therefore (5.5.14) yields

$$u(x, t) = \frac{1}{2^{\frac{3}{2}} t^{\frac{1}{2}(\mu + \frac{1}{2})}} (B_\psi f)(2\sqrt{t}, x). \quad (5.5.15)$$

□

Theorem 5.5.3. *If*

$$(B_\psi f)(2\sqrt{t}, x) = \left\langle f(s), (2\sqrt{t})^{\mu - \frac{1}{2}} \psi\left(\frac{s}{2\sqrt{t}}, \frac{x}{2\sqrt{t}}\right) \right\rangle.$$

Then

$$\lim_{\substack{A, B \rightarrow \infty \\ \epsilon \rightarrow 0^+}} \left\langle \frac{2}{C_{\mu, \psi}} \int_\epsilon^A \int_0^B (B_\psi f)(2\sqrt{t}, x) (2\sqrt{t})^{\mu - \frac{1}{2}} \psi\left(\frac{y}{2\sqrt{t}}, \frac{x}{2\sqrt{t}}\right) \frac{dx dt}{(2\sqrt{t})^{2\mu + 3}}, \phi(y) \right\rangle = \langle f, \phi \rangle,$$

for all $\phi \in H_\mu(\mathbb{R}^+)$.

Proof. From Definition 5.2.2, for $\psi \in H_\mu(\mathbb{R}^+)$, the Bessel wavelet and $f \in H'_{\mu, F}(\mathbb{R}^+)$

$$(B_\psi f)(2\sqrt{t}, x) = \left\langle f(s), (2\sqrt{t})^{\mu - \frac{1}{2}} \psi\left(\frac{s}{2\sqrt{t}}, \frac{x}{2\sqrt{t}}\right) \right\rangle.$$

From Theorem 5.3.7, the above expression can be written as

$$\begin{aligned} (B_\psi f)(2\sqrt{t}, x) &= \left\langle \sum_{k=0}^r S_{\mu, s}^k (1 + s^2)^r s^{-\mu - \frac{1}{2}} f_k(s), (2\sqrt{t})^{\mu - \frac{1}{2}} \psi\left(\frac{s}{2\sqrt{t}}, \frac{x}{2\sqrt{t}}\right) \right\rangle \\ &= \sum_{k=0}^r \left\langle S_{\mu, s}^k (1 + s^2)^r s^{-\mu - \frac{1}{2}} f_k(s), (2\sqrt{t})^{\mu - \frac{1}{2}} \psi\left(\frac{s}{2\sqrt{t}}, \frac{x}{2\sqrt{t}}\right) \right\rangle. \end{aligned}$$

Applying distributional differentiation, we find

$$(B_\psi f)(2\sqrt{t}, x) = \sum_{k=0}^r \left\langle f_k(s), (1 + s^2)^r s^{-\mu - \frac{1}{2}} S_{\mu, s}^k (2\sqrt{t})^{\mu - \frac{1}{2}} \psi\left(\frac{s}{2\sqrt{t}}, \frac{x}{2\sqrt{t}}\right) \right\rangle$$

$$= \sum_{k=0}^r \left\langle f_k(s), (1+s^2)^r s^{-\mu-\frac{1}{2}} (2\sqrt{t})^{\mu-\frac{1}{2}} S_{\mu,s}^k \psi\left(\frac{s}{2\sqrt{t}}, \frac{x}{2\sqrt{t}}\right) \right\rangle.$$

Let us consider

$$\begin{aligned} F(y) &= \frac{1}{C_{\mu,\psi}} \int_{\epsilon}^A \int_0^B (B_{\psi}f)(2\sqrt{t}, x) (2\sqrt{t})^{\mu-\frac{1}{2}} \psi\left(\frac{y}{2\sqrt{t}}, \frac{x}{2\sqrt{t}}\right) \frac{dx d(2\sqrt{t})}{(2\sqrt{t})^{2\mu+2}} \\ &= \frac{2}{C_{\mu,\psi}} \int_{\epsilon}^A \int_0^B (B_{\psi}f)(2\sqrt{t}, x) (2\sqrt{t})^{\mu-\frac{1}{2}} \psi\left(\frac{y}{2\sqrt{t}}, \frac{x}{2\sqrt{t}}\right) \frac{dx dt}{(2\sqrt{t})^{2\mu+3}}. \end{aligned}$$

Using the proof of Theorem 5.4.4, we can conclude that

$$\begin{aligned} &\lim_{\substack{A, B \rightarrow \infty \\ \epsilon \rightarrow 0^+}} \langle F(y), \phi(y) \rangle \\ &= \lim_{\substack{A, B \rightarrow \infty \\ \epsilon \rightarrow 0^+}} \left\langle \frac{2}{C_{\mu,\psi}} \int_{\epsilon}^A \int_0^B (B_{\psi}f)(2\sqrt{t}, x) (2\sqrt{t})^{\mu-\frac{1}{2}} \psi\left(\frac{y}{2\sqrt{t}}, \frac{x}{2\sqrt{t}}\right) \frac{dx dt}{(2\sqrt{t})^{2\mu+3}}, \phi(y) \right\rangle \\ &= \langle f, \phi \rangle. \end{aligned}$$

□

Example 5.2. For $f(x) = x^{\mu+\frac{1}{2}} e^{-x^2}$ the solution of the heat equation (5.5.6) and (5.5.7) is given by

$$u(x, t) = \int_0^{\infty} (x\omega)^{\frac{1}{2}} J_{\mu}(x\omega) (-\omega)^2 (h_{\mu} x^{\mu+\frac{1}{2}} e^{-x^2})(\omega) e^{-\omega^2 t} d\omega.$$

Using (1.7.8) and Erdélyi et al. [19, p.30], the above expression yields

$$\begin{aligned} u(x, t) &= S_{\mu,x} \left\{ \int_0^{\infty} (x\omega)^{\frac{1}{2}} J_{\mu}(x\omega) (h_{\mu} x^{\mu+\frac{1}{2}} e^{-x^2})(\omega) e^{-\omega^2 t} d\omega \right\} \\ &= S_{\mu,x} \left\{ \int_0^{\infty} (x\omega)^{\frac{1}{2}} J_{\mu}(x\omega) \frac{\omega^{\mu+\frac{1}{2}}}{2^{\mu+1}} e^{-\frac{\omega^2}{4}} e^{-\omega^2 t} d\omega \right\} \\ &= S_{\mu,x} \left\{ \int_0^{\infty} (x\omega)^{\frac{1}{2}} J_{\mu}(x\omega) \frac{\omega^{\mu+\frac{1}{2}}}{2^{\mu+1}} e^{-\frac{\omega^2}{4}} e^{-\omega^2 t} d\omega \right\}. \end{aligned}$$

Again, using [19, p.30], we get

$$u(x, t) = S_{\mu, x} \left\{ \frac{x^{\mu+\frac{1}{2}}}{(1+4t)^{\mu+1}} e^{-\frac{x^2}{1+4t}} \right\},$$

so that

$$u(x, t) = \frac{4x^{\mu+\frac{1}{2}}}{(1+4t)^{\mu+3}} \{x^2 - (\mu+1)(1+4t)\} e^{-\frac{x^2}{1+4t}}. \quad (5.5.16)$$

With the help of Matlab, the graphical representations of the solution (5.5.16) of the heat equation at $\mu = 0$ and $\mu = 1$ are given in Fig. 5.1 and Fig. 5.2, respectively.

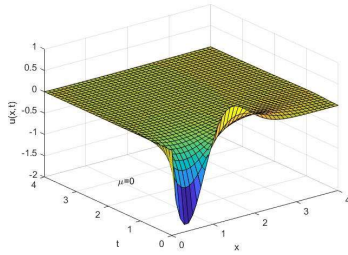


FIGURE 5.1: $u(x, t)$
at $\mu=0$.

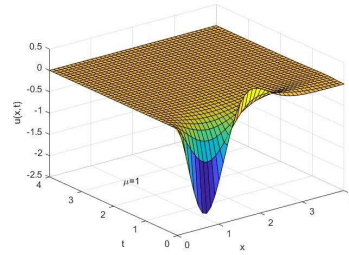


FIGURE 5.2: $u(x, t)$
at $\mu=1$.

5.6 Conclusions

In the present chapter, author discussed the various properties of the Bessel wavelet transform exploiting the theory of the Hankel transform, Hankel convolution and derived the inversion formula and its associated results of the continuous Bessel wavelet transform of distributions $f \in H'_\mu(\mathbb{R}^+)$ modulo $C.x^{\mu+\frac{1}{2}}$ with the Bessel wavelet kernel $\psi \in H_\mu(\mathbb{R}^+)$ having all the moments weighted by $x^{\mu+\frac{1}{2}}$ non-zero. Using the inversion formula, the Calderón reproducing formula on $H'_\mu(\mathbb{R}^+)$ is also obtained. This research work is useful in mathematical, physical and engineering

sciences. The aforesaid theory is also relevant for those researchers who are working in the area of image processing, signal processing and other computational types of problems.
