Chapter 3

Prescribed-time adaptive backstepping control for a class of nonlinear systems

3.1 Introduction

In this chapter, we introduce an adaptation law-based prescribed-time control methodology tailored for a class of uncertain nonlinear dynamical systems with unknown control coefficients. Leveraging the prescribed-time control theory proposed in [81], we design the control and parameter updating law within the framework of the integrator backstepping technique. The proposed control law ensures prescribed-time convergence of the system's states to the equilibrium point. By employing the Lyapunov stability method, we rigorously prove the boundedness of all signals within the closed-loop system. To demonstrate the effectiveness of the approach, we implement the proposed controller on a practical system, specifically the single-link manipulator with a flexible joint. The results exhibit superior convergence time compared to other existing methods while guaranteeing the boundedness of all signals in the closed-loop system, even in the presence of parametric uncertainties.

This chapter is structured as follows: Section 3.2 offers preliminary information that will aid in understanding the key outcomes of the study. Section 3.3 presents the primary findings. Section 3.4 validates the efficacy of the proposed regulation method with a practical academic example. Finally, Section 3.5 concludes this chapter.

3.2 Preliminaries

3.2.1 On prescribed-time stability

The following definitions and Lemmas are provided to facilitate the understanding of convergence within a specified time frame.

Consider a non-autonomous forced nonlinear system

$$\dot{z}(t) = f(t, z(t), \Phi, u(t)), \quad z(t_0) = z_0 \tag{3.1}$$

where $z(t) \in \mathbb{R}^n$ represents the system state, $\Phi \in \mathbb{R}^p$ denotes the uncertain system parameter vector and $u(t) \in \mathbb{R}^m$ represents the control input. The function $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^n$ satisfies the conditions $f(t, 0, \Phi, 0) = 0$, is Lipschitz in $z(t)$, and continuous with respect to t.

Definition 3.1 *[81]* (Prescribed-time stability).
The considered nonlinear system (3.1) is said to be prescribed time stable for a control $u(t) := u(t, z(t), T_p)$ *if*

1. *it is finite-time stable,*
2. *there exists a time* $T_p \geq t_0$*, which is irrespective of the initial conditions and system parameters and can be chosen a priori, and*
3. *the condition* $T_p \geq t_{ac}$ *holds, where* t_{ac} *represents the actual time of convergence, within which the trajectories of the plant converge to the origin.*

Now, we consider the following time-varying scalar system

$$\dot{z}(t) = \begin{cases} -\frac{\gamma(e^{z(t)}-1)}{e^{z(t)}(t_p+t_0-t)}, & \forall\ t_0 \leq t < t_p + t_0 \\ 0, & \text{otherwise} \end{cases} \tag{3.2}$$

where $z(t) \in \mathbb{R}$ denotes the system state, $t_0 \in \mathbb{R}_{>0}$ is the initial time such that $T_p = t_p + t_0$ where $t_p \in \mathbb{R}_{>t_0}$ is the desired time of convergence and $\gamma \in \mathbb{R}_{\geq 1}$ is the design parameter. The existence and uniqueness of the solution for this system can be proven, ensuring that $z(t) = 0$ and $\dot{z}(t) = 0$ for all $t \geq t_p + t_0$ [81]. For simplicity, we can consider the initial time to be $t_0 = 0$ in our further analysis.

Lemma 3.2 *[81] Let there exist a continuously differentiable function* $V(t) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *and a design constant* $\gamma \in \mathbb{R}_{\geq 1}$ *such that for all* $t \in [0, \infty)$ *and for all* $z(t) \in \mathbb{R}^n \backslash \{0\}$*, the following conditions hold:*

$$\dot{V}(t) \leq \begin{cases} -\frac{\gamma(e^{V(t)}-1)}{e^{V(t)}(t_p - t)}, & \forall\, 0 \leq t < t_p \\ 0, & otherwise \end{cases} \tag{3.3}$$

where $\gamma \in \mathbb{R}_{\geq 1}$ *is a design parameter then the considered system* (3.1) *is classified as prescribed-time stable with the time* t_p *chosen a priori.*

Lemma 3.3 *[108] Suppose* $\mu(x) \geq 0$ *is a continuous function defined on* $x \in [a, b)$*, including the point* $x = b$ *as a flaw. If the limit* $\lim_{x \to b^-} (b - x)\mu(x) = d$ *holds, where* d *is a positive value or* $+\infty$*, and the improper integral* $\int_a^b \mu(x)dx = +\infty$ *diverges, then the integral is considered divergent.*

3.2.2 Problem formulation

In this chapter, our objective is to propose an adaptive backstepping-based prescribed-time regulation algorithm with tuning functions for a specific class of nonlinear systems. These systems are characterized by uncertain parameters and can be described in the following strict-feedback form

$$\begin{cases} \dot{z}_1(t) = & z_2(t) + \Theta_1^\top \Phi + \Xi_1 \\ \dot{z}_2(t) = & z_3(t) + \Theta_2^\top \Phi + \Xi_2 \\ \quad \vdots & \quad \vdots \\ \dot{z}_{n-1}(t) = & z_n(t) + \Theta_{n-1}^\top \Phi + \Xi_{n-1} \\ \dot{z}_n(t) = & \mathrm{b}u(t) + \Theta_n^\top \Phi + \Xi_n \end{cases} \tag{3.4}$$

where $z(t)$ represents the system state with $z(t) = [z_1(t), z_2(t), \ldots, z_n(t)]^\top \in \mathbb{R}^n$. The parameter vector $\Phi \in \mathbb{R}^r$ denotes the unknown constant parameters of the system. The high frequency gain is represented by the unknown constant b, and the functions $\Theta_k \in \mathbb{R}^r$ and $\Xi_k \in \mathbb{R}$ for $k = 1, \ldots, n$ are known nonlinear functions. Our focus is on developing a prescribed-time regulation algorithm that can adaptively handle these uncertain parameters and achieve the desired system behavior.

Assumption 1 *The sign of the control coefficient* b *is well-known.*

3.3 Prescribed-time adaptive backstepping control

In this section, we discuss the treatment of unknown parameters that appear linearly in the system equations. We propose an adaptive control approach by combining a control law with a parameter estimator, which provides estimates of the unknown parameters. The adaptive controller dynamically adjusts its parameters while the system is operating. By utilizing this adaptive control scheme, the closed-loop signals can achieve prescribed-time stabilization and remain bounded even in the presence of parametric uncertainties.

Lemma 3.4 *Consider the system* (3.1) *and assume the existence of class $\mathcal{K}$ functions α_1, α_2, α_3, and α_4 defined on $\mathbb{R}^n$. Let there exist a continuously differentiable function $V(t) : [0, \infty) \times D \to \mathbb{R}_{\geq 0}$ and a design constant $\gamma \in \mathbb{R}_{\geq 1}$ such that $\forall t \in [0, \infty)$ and $\forall z(t) \in \mathbb{R}^n \backslash \{0\}$, the following conditions hold:*

$$V(t) = V_1(z) + V_2(\tilde{\Phi}) \tag{3.5}$$

$$\alpha_1(\|z\|) \leq V_1(z) \leq \alpha_2(\|z\|) \tag{3.6}$$

$$\alpha_3(\|\tilde{\Phi}\|) \leq V_2(\tilde{\Phi}) \leq \alpha_4(\|\tilde{\Phi}\|) \tag{3.7}$$

$$\dot{V}(t) \leq -\frac{\gamma(e^{V_1(z)} - 1)}{e^{V_1(z)}(t_p - t)}; \ \forall \ t \in [0, t_p) \tag{3.8}$$

then the uncertain nonlinear system (3.1) *is considered to be prescribed-time stable, where $\tilde{\Phi}$ represents the error between the true parameter value Φ and the estimated parameter value $\hat{\Phi}$ (i.e., $\tilde{\Phi} = \Phi - \hat{\Phi}$). The prescribed time t_p denotes the time within which convergence can be achieved. Importantly, the prescribed time t_p is independent of the system's initial conditions, indicating that the system will converge within a known time regardless of its starting state.*

Proof. Here, we will provide a proof that the state trajectories of the system (3.1) will converge to the equilibrium point within the predetermined time t_p.

From (3.8), we have the following inequality for the interval $[0, t_p)$

$$\dot{V}(t) \leq -\frac{\gamma(e^{V_1(z)} - 1)}{e^{V_1(z)}(t_p - t)} \tag{3.9}$$

Integrating both sides of (3.9) over the interval $[0, t_p)$, we have

$$\int_0^{t_p} \dot{V}(t)dt \leq -\int_0^{t_p} \frac{\gamma(e^{V_1(z)} - 1)}{e^{V_1(z)}(t_p - t)}dt$$

$$V(t_p) - V(0) \leq -\int_0^{t_p} \frac{\gamma(e^{V_1(z)} - 1)}{e^{V_1(z)}(t_p - t)}dt \tag{3.10}$$

To establish the boundedness of (3.10), we focus on analyzing the integral $\int_0^{t_p} \frac{\gamma(e^{V_1(z)}-1)}{e^{V_1(z)}(t_p-t)}dt$. Since $V_1(z)$ is a positive definite function, we can deduce that $e^{V_1(z)} > 1$. Thus, we can rewrite the integrand as follows:

$$\frac{\gamma(e^{V_1(z)} - 1)}{e^{V_1(z)}(t_p - t)} = \frac{\gamma}{(t_p - t)} - \frac{\gamma}{V_1(z)(t_p - t)}.$$

Let's consider the first term, $\frac{\gamma}{(t_p-t)}$. Since $t \in [0, t_p]$, the denominator $(t_p - t)$ is always positive. Consequently, the first term is bounded. Now, let's examine the second term, $\frac{\gamma}{V_1(z)(t_p-t)}$. We know that $e^{V_1(z)} > 1$, which ensures that the denominator is always positive. Furthermore, as $V_1(z)$ is a positive definite function, the term $e^{V_1(z)}$ is bounded below by a positive constant. Therefore, the second term is also bounded. Since both terms in the integrand are bounded, we conclude that the integral $\int_0^{t_p} \frac{\gamma(e^{V_1(z)}-1)}{e^{V_1(z)}(t_p-t)}dt$ is bounded.

Now, in order to establish the prescribed-time convergence of $z(t)$ to the equilibrium point, we will utilize the method of contradiction.

Let's assume that there exists a constant $\epsilon > 0$ such that

$$\lim_{t \to t_p^-} V_1(z) = \epsilon \neq 0. \tag{3.11}$$

Since, from Lemma 1, we know that $\lim_{t \to t_p^-} \frac{1}{(t_p-t)} = +\infty$, the improper integral $\int_0^{t_p} \frac{\gamma(e^{V_1(z)}-1)}{e^{V_1(z)}(t_p-t)}dt$ is unbounded at t_p. However, we observe that $\frac{\gamma(e^{V_1(z)}-1)}{e^{V_1(z)}(t_p-t)} \geq 0$ for $t \in [0, t_p)$. Consequently, the integral $\int_0^{t_p} \frac{\gamma(e^{V_1(z)}-1)}{e^{V_1(z)}(t_p-t)}dt$ is monotonically increasing on $[0, t_p)$, and according to (3.10), it is bounded. Therefore, $\int_0^{t_p} \frac{\gamma(e^{V_1(z)}-1)}{e^{V_1(z)}(t_p-t)}dt$ is convergent. Next, considering $\lim_{t \to t_p^-} (t_p - t) \frac{\gamma(e^{V_1(z)}-1)}{e^{V_1(z)}(t_p-t)} > 0$, we can conclude from Lemma 3.3 that the improper integral $\int_0^{t_p} \frac{\gamma(e^{V_1(z)}-1)}{e^{V_1(z)}(t_p-t)}dt$ is divergent.

By comparing the two aspects, we observe a contradiction in their conclusions. Thus, the assumption (3.11) cannot hold true, implying that $\lim_{t \to t_p^-} V_1(z) = \epsilon = 0$. From (3.2), it follows that $\lim_{t \to t_p^-} z(t) = 0$. By considering the continuity and existence properties of the solution $z(t)$, we can conclude that $z(t_p) = 0$ and $u(t_p) = 0$. This further verifies the prescribed-time regulation of the system states in the presence of unknown system parameters. ■

3.3.1 Controller design

The design procedure of the proposed control algorithm, which aims to achieve the prescribed-time convergence of all states $z(t)$ of the uncertain nonlinear system (3.4) to an equilibrium point within a predetermined time t_p, is outlined in this subsection.

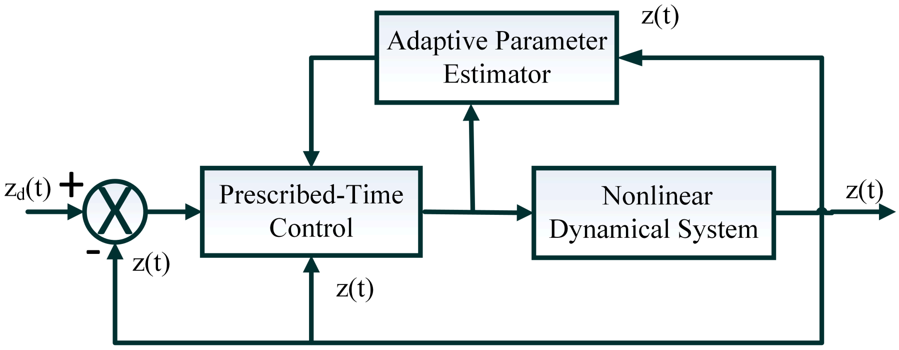

Figure 3.1: Schematic block diagram of the proposed adaptive prescribed-time stabilization scheme.

Now we introduce the following change of coordinates:

$$\begin{cases} x_1(t) &= z_1(t) \\ x_2(t) &= z_2(t) - \sigma_1(t) \\ \quad\vdots \\ x_k(t) &= z_k(t) - \sigma_{k-1}(t), \quad k = 3, 4, \ldots, n \end{cases} \tag{3.12}$$

where $\sigma_k(t)$ are virtual controllers. In order to design the adaptive law-based prescribed-time control algorithm for the considered uncertain nonlinear system (3.4), n design stages are needed. At each step, an error variable $x_k(t)$, a stabilizing function $\sigma_k(t)$ and a tuning function Υ_k are generated. Additionally, the prescribed-time control $u(t)$ and an uncertain parameter estimator $\hat{\Phi}$ are created. The schematic block diagram of the proposed adaptive backstepping-based prescribed-time stabilization scheme is depicted in Figure 3.1.

The design procedure is elaborated in the following subsequent steps.

Step 1. Let's begin with the first equation of (3.12) taking $z_2(t)$ into account as fictitious control variable. The time derivative of error variable $x_1(t)$ is obtained as

$$\begin{aligned} \dot{x}_1(t) &= \dot{z}_1(t) \\ &= x_2(t) + \sigma_1(t) + \Theta_1^T \Phi + \Xi_1 \end{aligned} \tag{3.13}$$

The first prescribed-time stabilizing function $\sigma_1(t)$ being designed as

$$\sigma_1(t) = -\frac{\eta_1 (e^{x_1(t)} - 1)}{e^{x_1(t)} (t_p - t)} - \Theta_1^T \hat{\Phi} - \Xi_1 \tag{3.14}$$

where $\eta_1 > n$ is a positive design constant and $\hat{\Phi}$ represents estimated value of the unknown constant parameter Φ.

To achieve the prescribed-time stabilization for the subsystem (3.13), the following Lyapunov candidate function is considered

$$V_1(t) = \frac{1}{2}x_1^2(t) + \frac{1}{2}\tilde{\Phi}^T\Gamma^{-1}\tilde{\Phi} \tag{3.15}$$

where Γ is a definite positive matrix with the proper dimensions. $\tilde{\Phi}$ represents the error between the true value Φ and estimated value $\hat{\Phi}$ (i.e., $\tilde{\Phi} = \Phi - \hat{\Phi}$).

Now taking the time derivative of (3.15), we get

$$\begin{aligned}\dot{V}_1(t) &= x_1(t)\dot{x}_1(t) - \tilde{\Phi}^T\Gamma^{-1}\dot{\hat{\Phi}} \\ &= x_1(t)\left(x_2(t) + \sigma_1(t) + \Theta_1^T\Phi + \Xi_1\right) - \tilde{\Phi}^T\left(\Gamma^{-1}\dot{\hat{\Phi}}_1 - \Theta_1 x_1(t)\right) \\ &= -\frac{\eta_1 x_1(t)(e^{x_1(t)} - 1)}{e^{x_1(t)}(t_p - t)} + \tilde{\Phi}^T\left(\Upsilon_1 - \Gamma^{-1}\dot{\hat{\Phi}}\right) + x_1(t)x_2(t)\end{aligned} \tag{3.16}$$

with the tuning function

$$\Upsilon_1 = \Theta_1 x_1(t) \tag{3.17}$$

where, Υ_1 represents the first tuning function.

Step 2. Now, treating $z_3(t)$ as a virtual control variable, we examine the second equation of (3.4). By utilizing equation (3.12), we derive the dynamics of $x_2(t)$ as

$$\begin{aligned}\dot{x}_2(t) =& \ \dot{z}_2(t) - \dot{\sigma}_1(t) \\ =& \ z_3(t) + \Theta_2^T\Phi + \Xi_2 - \frac{\partial\sigma_1(t)}{\partial z_1(t)}\left(z_2(t) + \Theta_1^T\Phi + \Xi_1\right) - \frac{\partial\sigma_1(t)}{\partial t} - \frac{\partial\sigma_1(t)}{\partial\hat{\Phi}}\dot{\hat{\Phi}} \\ =& \ x_3(t) + \sigma_2(t) + \Xi_2 - \frac{\partial\sigma_1(t)}{\partial z_1(t)}\left(z_2(t) + \Xi_1\right) - \frac{\partial\sigma_1(t)}{\partial t} \\ &+ \left(\Theta_2 - \frac{\partial\sigma_1(t)}{\partial z_1(t)}\Theta_1\right)^T\Phi - \frac{\partial\sigma_1(t)}{\partial\hat{\Phi}}\dot{\hat{\Phi}}\end{aligned} \tag{3.18}$$

Now, our task is to stabilize the subsystem (3.13) and (3.18). To do so the following Lyapunov candidate function $V_2(t)$ is taken into account

$$V_2(t) = V_1(t) + \frac{1}{2}x_2^2(t) \tag{3.19}$$

The resulting time derivative of $V_2(t)$ can be obtain as

$$\begin{aligned}\dot{V}_2(t) =& \dot{V}_1(t) + x_2(t)\dot{x}_2(t) \\ =& -\frac{\eta_1 x_1(t)(e^{x_1(t)} - 1)}{e^{x_1(t)}(t_p - t)} + \tilde{\Phi}^T\left(\Upsilon_1 - \Gamma^{-1}\dot{\hat{\Phi}}\right) + x_1(t)x_2(t) + x_2(t)\Bigg(x_3(t) + \sigma_2(t) + \Xi_2 \\ &- \frac{\partial\sigma_1(t)}{\partial z_1(t)}\left(z_2(t) + \Xi_1\right) - \frac{\partial\sigma_1(t)}{\partial t} + \left(\Theta_2 - \frac{\partial\sigma_1(t)}{\partial z_1(t)}\Theta_1\right)^T\Phi - \frac{\partial\sigma_1(t)}{\partial\hat{\Phi}}\dot{\hat{\Phi}}\Bigg)\end{aligned} \tag{3.20}$$

Further simplifying, we get

$$\begin{aligned}\dot{V}_2(t) =& -\frac{\eta_1 x_1(t)(e^{x_1(t)}-1)}{e^{x_1(t)}(t_p-t)} + x_2(t)\Bigg(x_3(t)+\sigma_2(t)+x_1(t)+\Xi_2+\hat{\Phi}^T\big(\Theta_2-\frac{\partial\sigma_1(t)}{\partial z_1(t)}\Theta_1\big)\\ &-\frac{\partial\sigma_1(t)}{\partial z_1(t)}(z_2(t)+\Xi_1)-\frac{\partial\sigma_1(t)}{\partial t}-\frac{\partial\sigma_1(t)}{\partial\hat{\Phi}}\Bigg)\\ &+\tilde{\Phi}^T\left(\Upsilon_1+\big(\Theta_2-\frac{\partial\sigma_1(t)}{\partial z_1(t)}\Theta_1\big)x_2(t)-\Gamma^{-1}\dot{\hat{\Phi}}\right)\end{aligned} \tag{3.21}$$

For all $t\in[0,t_p)$, the virtual control input $\sigma_2(t)$ is designed as

$$\begin{aligned}\sigma_2(t) =& -x_1(t)-\frac{\eta_2 x_2(t)(e^{x_2(t)}-1)}{e^{x_2(t)}(t_p-t)}-\Xi_2+\frac{\partial\sigma_1(t)}{\partial z_1(t)}(z_2(t)+\Xi_1)+\frac{\partial\sigma_1(t)}{\partial t}\\ &-\hat{\Phi}^T\left(\Theta_2-\frac{\partial\sigma_1(t)}{\partial z_1(t)}\Theta_1\right)+\frac{\partial\sigma_1(t)}{\partial\hat{\Phi}}\Gamma\Upsilon_2\end{aligned} \tag{3.22}$$

where $\eta_2 > n$ is a positive design constant. The second tuning function is designed as

$$\Upsilon_2 = \Upsilon_1+\left(\Theta_2-\frac{\partial\sigma_1(t)}{\partial z_1(t)}\Theta_1\right)x_2(t) \tag{3.23}$$

Further considering (3.22) and (3.23), (3.21) can be written as

$$\begin{aligned}\dot{V}_2(t) =& -\frac{\eta_1 x_1(t)(e^{x_1(t)}-1)}{e^{x_2(t)}(t_p-t)}-\frac{\eta_2 x_2(t)(e^{x_2(t)}-1)}{e^{x_2(t)}(t_p-t)}+x_2(t)x_3(t)+x_2(t)\frac{\partial\sigma_1}{\partial\hat{\Phi}}\left(\Gamma\Upsilon_2-\dot{\hat{\Phi}}\right)\\ &+\tilde{\Phi}^T\left(\Upsilon_2-\Gamma^{-1}\dot{\hat{\Phi}}\right)\end{aligned} \tag{3.24}$$

Step k, (k $= 3,4,\ldots,n$): Repeating the above procedure in a recursive approach, we obtain the dynamics of k-th tracking error for $x_k(t)$

$$\begin{aligned}\dot{x}_k(t) =& x_{k+1}(t)+\sigma_k(t)+\Xi_k-\sum_{l=1}^{k-1}\frac{\partial\sigma_{k-1}(t)}{\partial z_l(t)}(z_{l+1}(t)+\Xi_l)\\ &-\frac{\partial\sigma_{l-1}(t)}{\partial t}+\hat{\Phi}^T\left(\Theta_k-\sum_{l=1}^{k-1}\frac{\partial\sigma_{k-1}(t)}{\partial z_l(t)}\Theta_l\right)-\frac{\partial\sigma_{k-1}(t)}{\partial\hat{\Phi}}\dot{\hat{\Phi}}\end{aligned} \tag{3.25}$$

For the k-th subsystem (3.25), the following Lyapunov candidate function is considered

$$V_k(t) = V_{k-1}(t)+\frac{1}{2}x_k^2(t) \tag{3.26}$$

Now, taking the time derivative of $V_k(t)$

$$\dot{V}_k(t) = \dot{V}_1(t)+\dot{V}_2(t)+\ldots+\dot{V}_{k-1}(t)+x_k(t)\dot{x}_k(t) \tag{3.27}$$

The following prescribed-time stabilizing function $\sigma_k(t)$ is selected

$$\sigma_k(t) = -\frac{\eta_k x_k(t)(e^{x_k(t)}-1)}{e^{x_k(t)}(t_p-t)} - x_{k-1}(t) - \Xi_k + \sum_{l=1}^{k-1}\frac{\partial\sigma_{k-1}(t)}{\partial z_l(t)}\left(z_{l+1}(t)+\Xi_l\right)$$
$$- \hat{\Phi}^T\left(\Theta_k - \sum_{l=1}^{k-1}\frac{\partial\sigma_{k-1}(t)}{\partial z_l(t)}\Theta_l\right) + \frac{\partial\sigma_{k-1}(t)}{\partial t} + \frac{\partial\sigma_{k-1}(t)}{\partial\hat{\Phi}}\Gamma\Upsilon_k$$
$$+ \left(\sum_{l=2}^{k-1}x_l(t)\frac{\partial\sigma_{l-1}(t)}{\partial\hat{\Phi}}\right)\Gamma\times\left(\Theta_k - \sum_{l=1}^{k-1}\frac{\partial\sigma_{k-1}(t)}{\partial z_l(t)}\Theta_l\right) \tag{3.28}$$

where $\eta_k > n$ is the positive design constant and the k-th tuning function is chosen as

$$\Upsilon_k = \Upsilon_{k-1} + \left(\Theta_k - \sum_{l=1}^{k-1}\frac{\partial\sigma_{k-1}(t)}{\partial z_l(t)}\Theta_l\right)x_k(t) \tag{3.29}$$

Further considering designed prescribed-time stabilizing function (3.28) along with the tuning function (3.29), $\dot{V}_k(t)$ can be written as

$$\dot{V}_k(t) = -\frac{\eta_1 x_1(t)(e^{x_1(t)}-1)}{e^{x_1(t)}(t_p-t)} - \frac{\eta_2 x_2(t)(e^{x_2(t)}-1)}{e^{x_2(t)}(t_p-t)} - \ldots - \frac{\eta_k x_k(t)(e^{x_k(t)}-1)}{e^{x_2(t)}(t_p-t)}$$
$$+ x_k(t)x_{k+1} + \left(\sum_{l=2}^{k}x_l\frac{\partial\sigma_{l-1}(t)}{\partial\hat{\Phi}}\right)\left(\Gamma\Upsilon_k - \dot{\hat{\Phi}}\right) + \tilde{\Phi}^T\left(\Upsilon_k - \Gamma^{-1}\dot{\hat{\Phi}}\right) \tag{3.30}$$

Step n: In the final design step n, the actual controlled input $u(t)$ arises and is available to us. Utilizing (3.12) and (3.28), the dynamics of $x_n(t)$ can be written as

$$\dot{z}_n(t) = \mathrm{b}u(t) + \Xi_n - \sum_{l=1}^{n-1}\frac{\partial\sigma_{k-1}(t)}{\partial z_l(t)}\left(z_{l+1}(t)+\Xi_l\right) + \hat{\Phi}^T\left(\Theta_n - \sum_{l=1}^{n-1}\frac{\partial\sigma_{k-1}(t)}{\partial z_l(t)}\Theta_l\right)$$
$$- \frac{\partial\sigma_{n-1}(t)}{\partial\hat{\Phi}}\dot{\hat{\Phi}} - \frac{\partial\sigma_{n-1}(t)}{\partial t} \tag{3.31}$$

Now, the prescribed-time control law $u(t)$ can be designed as

$$u(t) = \begin{cases}\hat{\Pi}\sigma_n(t), & \forall t\in[0,t_p)\\ 0, & \text{otherwise}\end{cases} \tag{3.32}$$

with $\sigma_n(t)$ as

$$\sigma_n(t) = -\frac{\eta_n x_n(t)(e^{x_n(t)}-1)}{e^{x_n(t)}(t_p-t)} - x_{n-1}(t) - \Xi_n + \sum_{l=1}^{n-1}\frac{\partial\sigma_{n-1}(t)}{\partial z_l(t)}\left(z_{l+1}(t)+\Xi_l\right)$$
$$- \hat{\Phi}^T\left(\Theta_n - \sum_{l=1}^{n-1}\frac{\partial\sigma_{n-1}(t)}{\partial z_l(t)}\Theta_l\right) + \frac{\partial\sigma_{n-1}(t)}{\partial t} + \frac{\partial\sigma_{n-1}(t)}{\partial\hat{\Phi}}\Gamma\Upsilon_n$$
$$+ \left(\sum_{l=2}^{n-1}x_l(t)\frac{\partial\sigma_{l-1}(t)}{\partial\hat{\Phi}}\right)\Gamma\times\left(\Theta_n - \sum_{l=1}^{n-1}\frac{\partial\sigma_{n-1}(t)}{\partial z_l(t)}\Theta_l\right)$$

where $\eta_n > n$ is a positive design constant and $\hat{\Pi}$ represents the estimated value of $\Pi = 1/b$

The uncertain parameter updating law for $\dot{\hat{\Phi}}$ and $\dot{\hat{\Pi}}$ designed as

$$\dot{\hat{\Phi}} =\Gamma\bigg(\Upsilon_{n-1} + \big(\Theta_n - \sum_{l=1}^{n-1} \frac{\partial \sigma_{n-1}(t)}{\partial z_l(t)} \Theta_l\big) x_n(t) \bigg) \tag{3.33}$$

$$\dot{\hat{\Pi}} =-\beta \, \mathrm{sign(b)} x_n(t) \sigma_n(t) \tag{3.34}$$

where β is a positive design constant.

Next, the prescribed-time stabilization of system (3.4) i.e., $\lim_{t\to t_p} z(t) = 0$ and boundedness of all the closed-loop signals are ensured. The following Theorem explicitly states the concept.

3.3.2 Stability analysis

Theorem 3.5 *Consider the nonlinear dynamical system* (3.4) *under parametric uncertainties along with Assumption 1, if the continuous time-varying actual control input defined by* $u(t)$ *in* (3.32) *with adaptive laws* (3.33) *and* (3.34) *are utilized, then for any initial conditions* $z(0)$ *which is bounded, the n-th order strict-feedback system* (3.4) *is prescribed time stable. Additionally, all signals of the closed-loop system remain bounded for all future time.*

Proof. Two cases comprise the proof. 1) The state trajectories $z(t)$ of the uncertain nonlinear system (3.4) converges to the equilibrium point within a prescribed-time t_p irrespective of parameter uncertainties and initial conditions. 2) For all future time, all signals in the closed-loop system are guaranteed to remain bounded.

Case 1. All the working states of the system (3.4) converges to the equilibrium point within the prescribed-time t_p:

We choose the following candidate Lyapunov function

$$\begin{aligned} V_n(t) =& V_{n-1}(t) + \frac{1}{2} x_n^2(t) + \frac{|\mathrm{b}|}{2\beta} \tilde{\Pi}^2 \\ =& \sum_{k=1}^{n} \frac{1}{2} x_k^2(t) + \frac{1}{2} \tilde{\Phi}^T \Gamma^{-1} \tilde{\Phi} + \frac{|\mathrm{b}|}{2\beta} \tilde{\Pi}^2 \\ =& V_p(t) + V_q(t) \end{aligned} \tag{3.35}$$

where $\tilde{\Pi} = \Pi - \hat{\Pi}$, $V_p(t) = \sum_{k=1}^{n} \frac{1}{2} x_k^2(t)$ and $V_q(t) = \frac{1}{2} \tilde{\Phi}^T \Gamma^{-1} \tilde{\Phi} + \frac{|\mathrm{b}|}{2\beta} \tilde{\Pi}^2$

Now the time derivative of (3.35) along the solution of (3.4) is

$$\dot{V}_n(t) = -\sum_{k=1}^{n} \frac{\eta_n x_n(t)(e^{x_n(t)}-1)}{e^{x_n(t)}(t_p-t)} + \sum_{l=2}^{n} x_l(t)\frac{\partial \sigma_{l-1}}{\partial \hat{\Theta}}\left(\Gamma \Upsilon_n - \dot{\hat{\Theta}}\right)$$
$$+ \tilde{\Theta}^T\left(\Upsilon_n - \Gamma^{-1}\dot{\hat{\Theta}}\right) - \frac{|\mathrm{b}|}{\beta}\tilde{\Pi}\left(\dot{\hat{\Pi}} + \beta\,\mathrm{sign}(\mathrm{b})x_n(t)\sigma_n(t)\right) \tag{3.36}$$

Considering the adaptation laws (3.33) and (3.34), (3.36) reduced to

$$\dot{V}_n(t) = -\frac{\eta_1 x_1(t)(e^{x_1(t)}-1)}{e^{x_1(t)}(t_p-t)} - \frac{\eta_2 x_2(t)(e^{x_2(t)}-1)}{e^{x_2(t)}(t_p-t)},\dots,-\frac{\eta_n x_n(t)(e^{x_n(t)}-1)}{e^{x_n(t)}(t_p-t)}$$
$$\le -\frac{\eta_1 |x_1(t)|(e^{|x_1(t)|}-1)}{e^{|x_1(t)|}(t_p-t)} - \frac{\eta_2 |x_2(t)|(e^{|x_2(t)|}-1)}{e^{|x_2(t)|}(t_p-t)},\dots,-\frac{\eta_n |x_n(t)|(e^{|x_n(t)|}-1)}{e^{|x_n(t)|}(t_p-t)} \tag{3.37}$$

From (3.35) we have

$$V_p(t) = \sum_{k=1}^{n}\frac{1}{2}x_k^2(t)$$
$$= \frac{1}{2}x_1^2(t) + \frac{1}{2}x_2^2(t),\dots,\frac{1}{2}x_n^2(t) \tag{3.38}$$

Then, utilizing (3.38) we have

$$V_p(t) \le \frac{n}{2}\left(\max\{|x_1(t)|, |x_2(t)|,\dots,|x_n(t)|\}\right)^2$$

So one can write

$$\sqrt{\frac{2V_p(t)}{n}} \le \max\{|x_1(t)|, |x_2(t)|,\dots,|x_n(t)|\}$$

Now for a particular instance, let's assume that

$$\max\{|x_1(t)|, |x_2(t)|,\dots,|x_n(t)|\}$$

returns $|x_n(t)|$ (for other possibilities the results are obvious), then

$$\sqrt{\frac{2V_p(t)}{n}} \le |x_n(t)|$$

Now (3.37) can be written as

$$\dot{V}_n(t) \le -\frac{\eta_n |x_n(t)|(e^{|x_n(t)|}-1)}{e^{|x_n(t)|}(t_p-t)} \tag{3.39}$$

Since $\sqrt{\frac{2V_p(t)}{n}} \le |x_n(t)|$, we get

$$\dot{V}_n(t) \le -\frac{\eta_n\sqrt{\frac{2V_p(t)}{n}}\left(e^{\sqrt{\frac{2V_p(t)}{n}}}-1\right)}{e^{\sqrt{\frac{2V_p(t)}{n}}}(t_p-t)} \tag{3.40}$$

Let $\Psi(t)=\sqrt{\frac{2V_p(t)}{n}}$, then we get $\dot{\Psi}(t)=\frac{\dot{V}_p(t)}{n\Psi(t)}$. Furthermore, from (3.36) and (3.37) we have $\dot{V}_n(t)=\dot{V}_p(t)$. So (3.40) can be written as

$$\dot{\Psi}(t)\leq-\frac{\bar{\gamma}_n(e^{\Psi(t)}-1)}{e^{\Psi(t)}(t_p-t)} \tag{3.41}$$

where $\bar{\gamma}_n=\frac{\eta_n}{n}$. By utilizing Lemma 3.2, we can say that (3.41) leads to prescribed-time convergent dynamics, which implies that $\lim_{t\to t_p}=\Psi(t)=0$. Since we have assumed $\Psi(t)=\sqrt{\frac{2V_p(t)}{n}}$, hence we get $\lim_{t\to t_p}V_p(t)=0$. Further, from (3.38), we can conclude that $\lim_{t\to t_p}x(t)=0$, which ensures the prescribed-time convergence of the states of the considered uncertain nonlinear system (3.4) i.e., $\lim_{t\to t_p}z(t)=0$. Through the use of the existence and persistence characteristics of the solution $z(t)$, one may obtain that $z(t_p)=0$, $u(t_p)=0$. Additionally, because $f(t,z(t),\Phi,u(t))$ disappear at the equilibrium point, and then choosing $u(t)=0$ for all $t\geq t_p$, we can guarantee $z(t)=0$ for all future time.

Case 2. The evolution of the states $z(t)$, and the adaptive parameter $\hat{\Phi}$ with the aid of the proposed control law (3.32) with adaptive laws (3.33) and (3.34) are bounded:

From (3.8), $\forall t\in[0,t_p)$ one has

$$\dot{V}(t)\leq-\frac{\gamma(e^{V_1(z)}-1)}{e^{V_1(z)}(t_p-t)}\leq 0 \tag{3.42}$$

Hence, $V(t)$ is diminishing continuously on the interval $[0,t_p)$.

Further, utilizing (3.5) we can get

$$\begin{aligned}V_1(z(t))&\leq V(t)\leq V(0)\\ V_2(\tilde{\Phi}(t))&\leq V(t)\leq V(0)\end{aligned} \tag{3.43}$$

Therefore, in keeping with (3.6), (3.7) and (3.43), we can further write

$$\|z(t)\|\leq\alpha_1^{-1}\left(V_1(z(t))\right)\leq\alpha_1^{-1}(V(0)) \tag{3.44}$$

$$\|\tilde{\Phi}(t)\|\leq\alpha_3^{-1}\left(V_2(\tilde{\Phi}(t))\right)\leq\alpha_3^{-1}(V(0)) \tag{3.45}$$

which further implies that the system state $z(t)$ and the parameter estimate $\tilde{\Phi}(t)$ are bounded. Because $\hat{\Phi}(t)=\Phi-\tilde{\Phi}(t)$, one can easily guarantee that $\hat{\Phi}(t)$ is also bounded for all future time which also satisfy Lemma 3.4 Which completes the proof. ■

3.4 Illustrative example

In order to verify the efficiency of the developed prescribed-time convergent adaptive regulation algorithm, a single-link manipulator with a flexible joint that operates on a vertical plane is considered in this section. Figure 3.2 depicts the free-body diagram of a single-link manipulator with a flexible joint.

The mathematical model of the single-link manipulator with a flexible joint is derived using the Euler-Lagrangian formula, resulting in the following equations:

$$\begin{aligned} &I\ddot{\theta}_1 + MgL\sin(\theta_1) + K(\theta_1 - \theta_2) = 0 \\ &J\ddot{\theta}_2 - K(\theta_1 - \theta_2) = \tau(t) \end{aligned} \tag{3.46}$$

where I and J are moments of inertia, θ_1 and θ_2 are angular positions, K is a spring constant, M is the total mass, L is a distance, and $\tau(t)$ is a torque input.

Now, the state variables and control input is defined as

$$\begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \\ \zeta_3(t) \\ \zeta_4(t) \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}, \quad \begin{bmatrix} u(t) \end{bmatrix} = \begin{bmatrix} \tau(t) \end{bmatrix} \tag{3.47}$$

Applying the nonlinear state coordinate transformation of the form $z(t) = T(\zeta)$ as given in [105], the nonlinear dynamics (3.46) can be rewritten in the strict-feedback form as

$$\begin{aligned} &\dot{z}_1(t) = z_2(t) \\ &\dot{z}_2(t) = z_3(t) \\ &\dot{z}_3(t) = z_4(t) \\ &\dot{z}_4(t) = \frac{K}{IJ}u(t) - \left(\frac{K}{I} + \frac{K}{J} + \frac{MgL}{I}\cos(z_1)\right) z_3(t) + \frac{MgL}{I}\sin(z_1)\left(z_2^2(t) - \frac{K}{J}\right) \end{aligned} \tag{3.48}$$

To further facilitate the control proposed design, we can rewrite (3.48) as a system of first-order differential equations in matrix form:

$$\begin{aligned} \dot{z}_1(t) =& z_2(t) \\ \dot{z}_2(t) =& z_3(t) \\ \dot{z}_3(t) =& z_4(t) \\ \dot{z}_4(t) =& \Theta^\top(z)\Phi + \mathrm{b}u(t) \end{aligned} \tag{3.49}$$

Table 3.1: Single-link flexible-joint manipulator parameter values.

Parameters	**Value**	**Unit**
I	0.5	$kg.m^2$
J	0.8	$kg.m^2$
L	1	m
K	0.3	N/m
M	1	kg
g	9.8	m/s^2

where the nonlinear function Θ, which is known, is defined as follows:

$$\Theta = \begin{bmatrix} -z_3(t) \\ -z_3(t) \\ \sin(z_1)z_2^2(t) - \cos(z_1)z_3(t) \\ -\sin(z_1) \end{bmatrix}$$

Furthermore, we specify the unknown constant vector Φ as

$$\Phi = \begin{bmatrix} \frac{K}{I} \\ \frac{K}{J} \\ \frac{MgL}{I} \\ \frac{MgLK}{IJ} \end{bmatrix}$$

and the unknown constant control coefficient b is defined as

$$\mathrm{b} = \frac{K}{IJ}$$

The goal is to design a control law $u(t)$ that causes all the states of the considered single-link manipulator with a flexible joint, converge to the origin within a prescribed time, which can be specified in advance. To achieve this control objective, we make the following assumptions:

Assumption 2 *The unknown parameters* Φ*, and* b *are all positive constants.*

Next, through the use of the adaptive backstepping strategy described in Section 3.3, the virtual control signals, the actual control inputs, and the adaptive laws are designed.

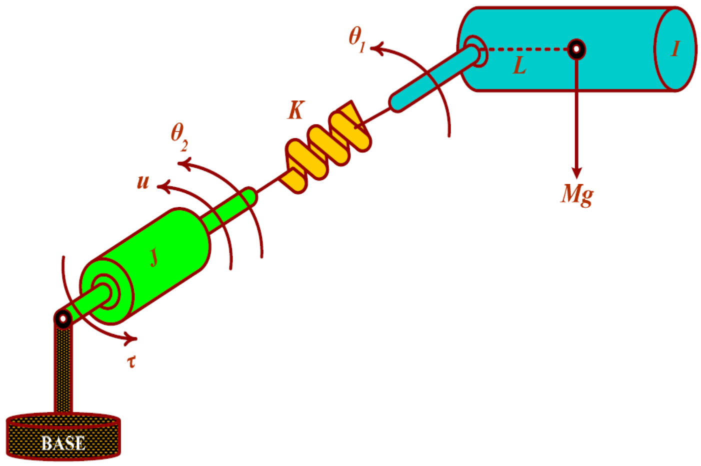

Figure 3.2: Free-body diagram of single-link flexible-joint manipulator.

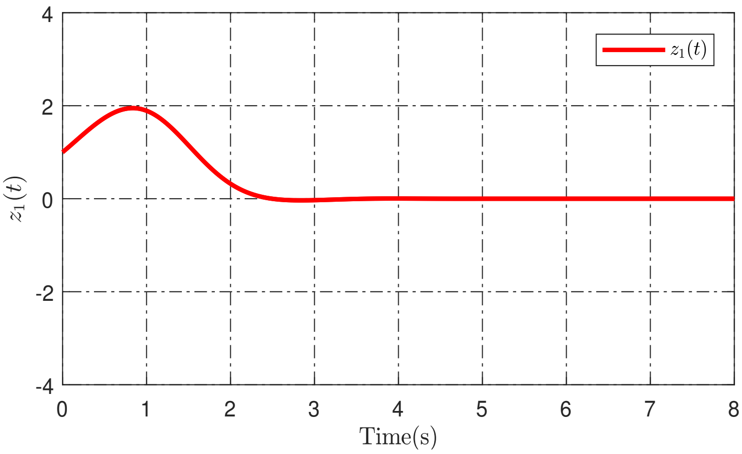

Figure 3.3: Time evolution of angular position $z_1(t)$ for $t_p = 6s$.

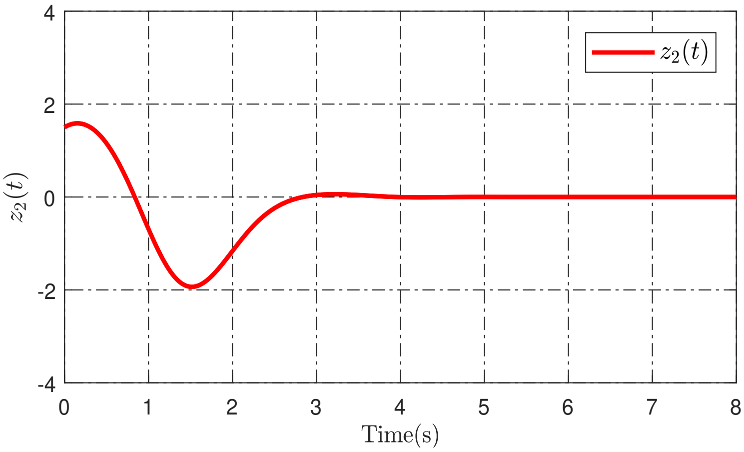

Figure 3.4: Time evolution of angular velocity $z_2(t)$ for $t_p = 6s$.

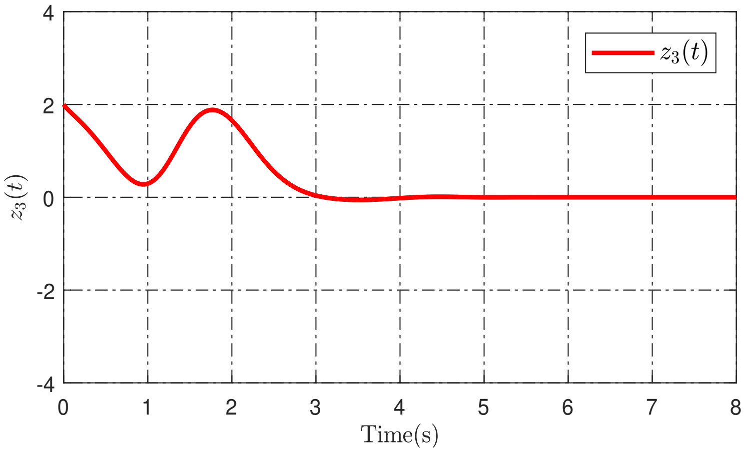

Figure 3.5: Time evolution of the angular position $z_3(t)$ for $t_p = 6s$.

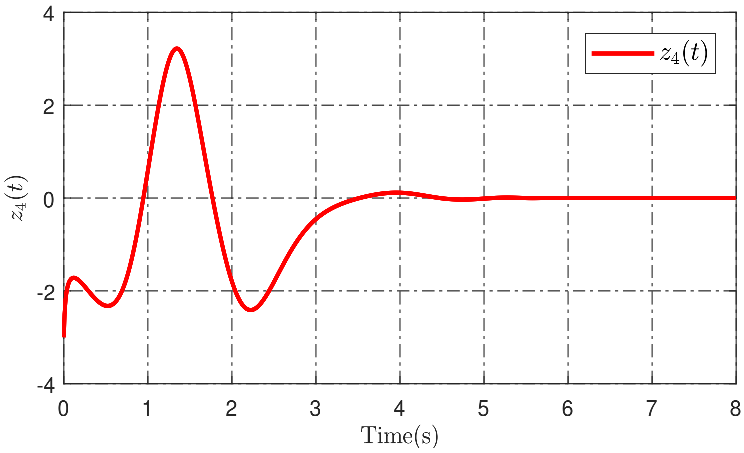

Figure 3.6: Time evolution of the angular velocity $z_4(t)$ for $t_p = 6s$.

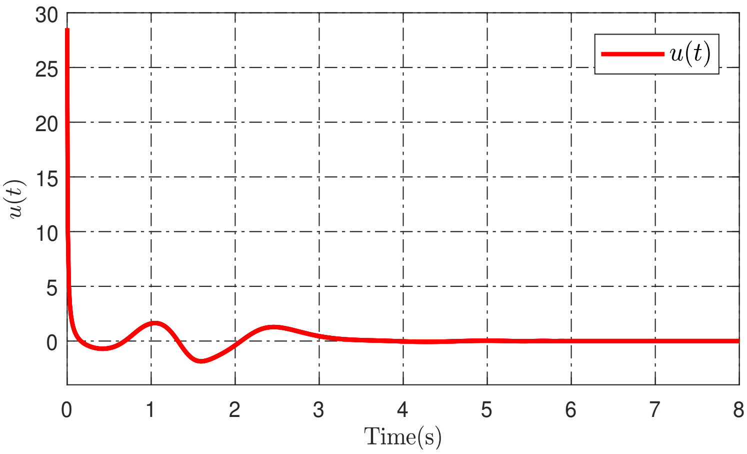

Figure 3.7: Time evolution of the required control input $u_1(t)$ for $t_p = 6s$.

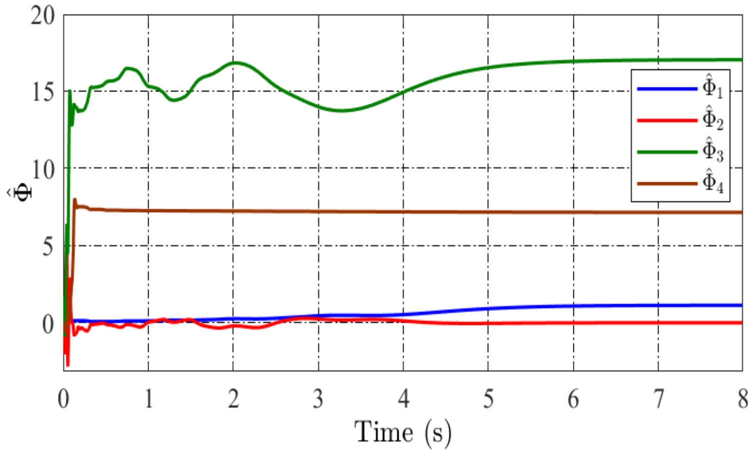

Figure 3.8: Time evolution of $\hat{\Phi}$ for $t_p = 6s$.

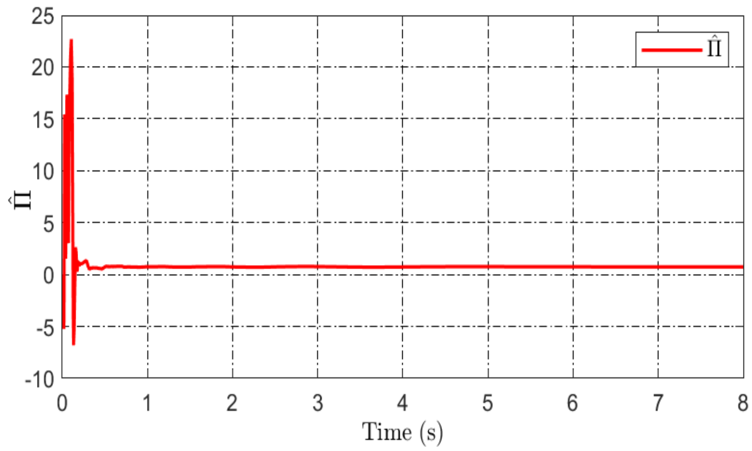

Figure 3.9: Time evolution of $\hat{\Pi}$ for $t_p = 6s$.

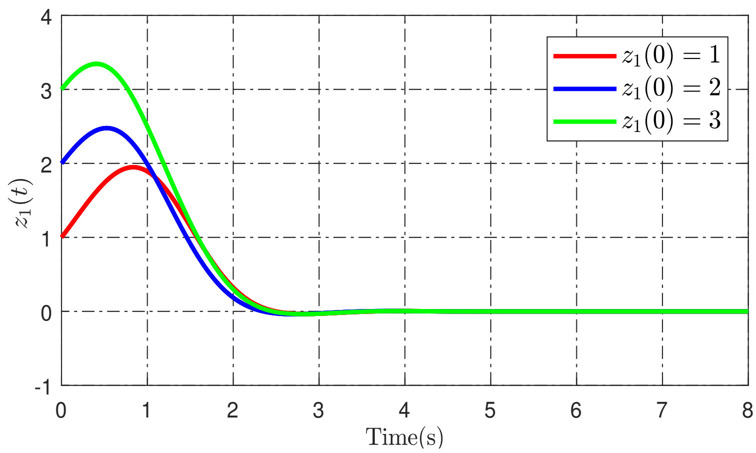

Figure 3.10: Time evolution of angular position $z_1(t)$ under different initial conditions for $t_p = 6s$.

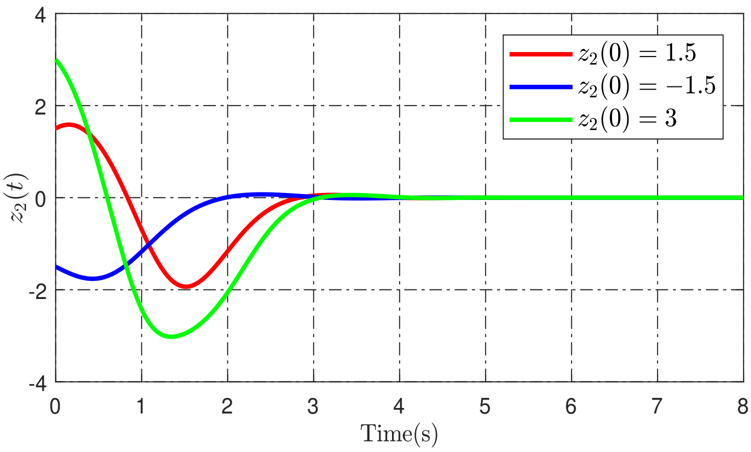

Figure 3.11: Time evolution of angular velocity $z_2(t)$ under different initial conditions for $t_p = 6s$.

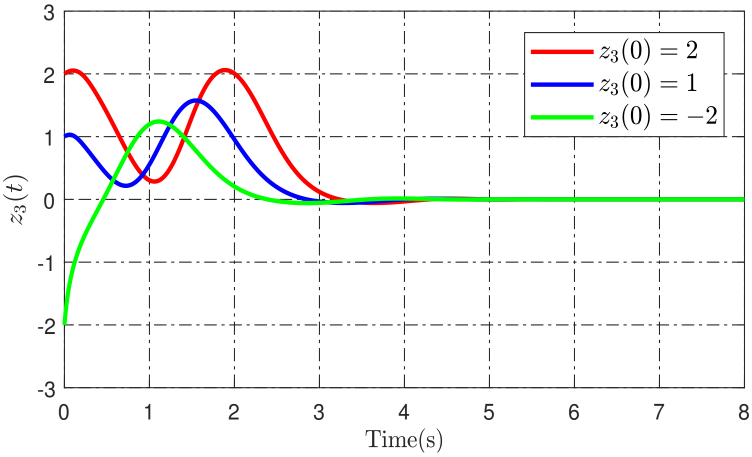

Figure 3.12: Time evolution of the angular position $z_3(t)$ under different initial conditions for $t_p = 6s$.

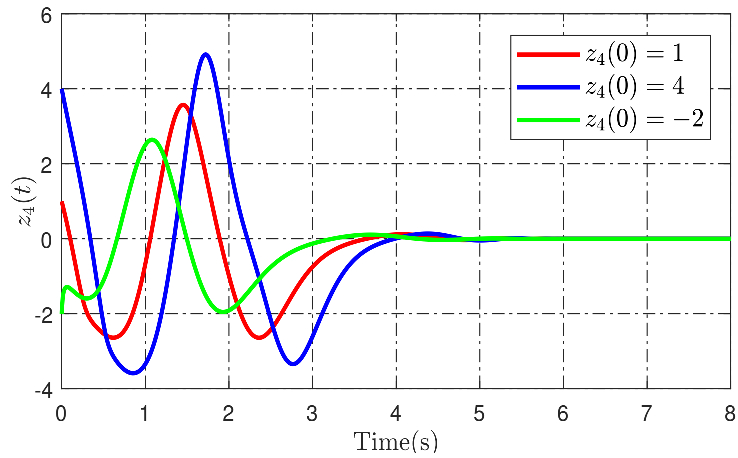

Figure 3.13: Time evolution of the angular velocity $z_4(t)$ under different initial conditions for $t_p = 6s$.

To start, we apply a change of coordinates to the state variables as follows:

$$x_1(t) = z_1(t) \tag{3.50}$$

$$x_2(t) = z_2(t) - \sigma_1(t) \tag{3.51}$$

$$x_3(t) = z_3(t) - \sigma_2(t) \tag{3.52}$$

$$x_4(t) = z_4(t) - \sigma_3(t) \tag{3.53}$$

where $\sigma_1(t)$, $\sigma_2(t)$ and $\sigma_3(t)$ are the virtual controllers and set them to be:

$$\sigma_1(t) = -\frac{\eta_1(e^{x_1(t)} - 1)}{e^{x_1(t)}(t_p - t)} \tag{3.54}$$

$$\sigma_2(t) = -\frac{\eta_2(e^{x_2(t)} - 1)}{e^{x_2(t)}(t_p - t)} \tag{3.55}$$

$$\sigma_3(t) = -\frac{\eta_3(e^{x_3(t)} - 1)}{e^{x_3(t)}(t_p - t)} \tag{3.56}$$

where $\eta_1 > 4$, $\eta_2 > 4$ and $\eta_3 > 4$ are the positive constants to be designed later.

Now the transformed dynamics can be rewritten as:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + \sigma_1(t) \\ \dot{x}_2(t) &= x_3(t) + \sigma_2(t) \\ \dot{x}_3(t) &= x_4(t) + \sigma_3(t) \\ \dot{x}_4(t) &= \Theta^\top(z)\Phi + \mathrm{b}u(t) - \dot{\sigma}_3(t) \end{aligned} \tag{3.57}$$

Now, we propose the following adaptation law-based prescribed-time control law

$$u(t) = \begin{cases} \hat{\Pi}\sigma_4(t), & \forall t \in [0, 6) \\ 0, & \forall t \in [6, 8] \end{cases} \tag{3.58}$$

with

$$\sigma_4(t) = -x_3(t) - \Theta^\top(z)\hat{\Phi} - \frac{\eta_4(e^{x_4(t)} - 1)}{e^{x_4(t)}(t_p - t)} + \dot{\alpha}_3(t) \tag{3.59}$$

and the parameter updating laws are chosen as

$$\dot{\hat{\Phi}} = \Gamma\Theta x_4(t) \tag{3.60}$$

$$\dot{\hat{\Pi}} = -\beta \mathrm{sign}(\mathrm{b}) x_4(t)\sigma_4(t) \tag{3.61}$$

To conduct the simulation, we refer to the system parameters specified in Table 3.1. From the values presented in Table 3.1, we compute the constant parameters as

$\Phi = [0.6\ 0.375\ 19.6\ 7.35]^\top$ and $\Pi = 0.75$. For the proposed adaptive prescribed-time control algorithm, the design parameters are set as $\eta_1 = 5$, $\eta_2 = 5$, $\eta_3 = 5$, $\eta_4 = 5$, $\Gamma = 2$, $\beta = 1.5$, and the prescribed time is defined as $t_p = 6$ seconds. The initial conditions of the system state are chosen as $z_1(0) = 1$, $z_2(0) = 1.5$, $z_3(0) = 2$, $z_4(0) = -3$, while the initial conditions for the unknown parameters are $\hat{\Phi} = 0.5$ and $\hat{\Pi} = 0.2$.

The obtained simulation results are shown in Figures 3.3-3.13. Figures 3.3-3.6 shows the time evolution of $z_1(t)$, $z_2(t)$, $z_3(t)$ and $z_4(t)$. Figure 3.7 depict the time evolution of required control effort $u(t)$, which also shows that the control input is bounded and continuous over the terminal time. From the obtained results, it is simple to notice that all the states and control input steer to the equilibrium point within the prescribed time. The estimate of parameters $\hat{\Phi}$ and $\hat{\Pi}$ are shown in Figure 3.8 and 3.9.

In addition, using the same control parameters under various initial conditions of the states, we examined the convergence characteristics of system states to the equilibrium point. The evolution of the system states $z(t)$ under various initial conditions are shown in Figure 3.10-3.13. From the obtained results, one can infer that the convergence time is irrespective of the initial conditions.

3.5 Conclusion

This chapter discussed an adaptive prescribed-time control scheme for a class of uncertain nonlinear systems in strict feedback form. In particular, we employed the adaptive law-based integrator backstepping technique to obtain the proposed prescribed-time controller recursively. Through extensive theoretical analysis, it is proven that the proposed scheme can ensure convergence of the system states to the equilibrium point within a priori chosen time, which is a noteworthy feature of the proposed control. The boundedness of all the signals of the closed-loop system can be guaranteed throughout all future times. In the end, to testify the superiority of the designed time-varying control scheme, a practical system of the single-link manipulator with a flexible joint is taken into consideration.

In the next chapter, the prescribed-time adaptive backstepping control methodology is extended to tackle the control challenges specific to the twin-rotor helicopter.