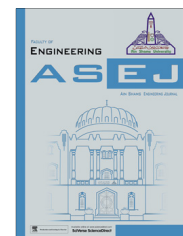




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ENGINEERING PHYSICS AND MATHEMATICS

# Stable numerical solutions of fractional partial differential equations using Legendre scaling functions operational matrix



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## KEYWORDS

Two dimensional Legendre scaling function;  
Operational matrix;  
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System of linear algebraic equations

**Abstract** In this paper we solve initial and boundary value problem for non-homogeneous fractional order partial differential equations. Here we use operational matrix approach to construct approximate solutions using Legendre scaling functions as basis. We also give the error analysis of the proposed method. Some numerical examples are given to verify the theoretical bound of error and to show the stability of the proposed method. Results are also compared with some known methods and it is observed that our method is more easy to implement and accurate.

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## 1. Introduction

Instantaneous change in application sometime depends on the past. That is why it is important to study fractional order differentiation. Nowadays people are adopting the fractional order differentiation in the modelling in order to better understand the physical processes of the various problems such as fractional diffusion equations [1] and fractional diffusion-wave equations [2–4]. Such problems are being used widely in many branches of science and engineering. For example, these equations represent propagation of mechanical waves

in visco-elastic media [5,6] and many more could be found in [7–12]. These applications of the fractional partial differential equations motivate us to consider such equations and propose a numerical method to solve them. Here, we consider the fractional partial differential equation defined as

$$\frac{\partial^{\alpha} y(x, t)}{\partial x^{\alpha}} + \frac{\partial^{\beta} y(x, t)}{\partial t^{\beta}} = g(x, t), \quad 0 < \alpha, \beta \leq 2, \quad (1)$$

with initial conditions  $y(x, 0) = a(x)$ ,  $D_t y(x, 0) = c(x)$  and boundary conditions  $y(0, t) = b(t)$ ,  $D_x y(0, t) = d(t)$ , where  $g(x, t)$  is known continuous function and  $y(x, t)$  is the unknown function.

Analytical solutions of such equations are rare in many cases; hence, numerical methods become important to solve them. The most commonly used methods are Variational Iteration Method [13], Generalized Differential Transform Method [14,15], Adomian Decomposition Method [16,17], Finite Difference Method [18], Wavelet Method [19] and operational matrix method [20–22].

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In [20], Eq. (1) is solved using block pulse functions as basis with constant initial and boundary conditions.

In our paper we study the above equation for general initial and boundary data. The method we use here is the operational matrix approach taking two dimensional Legendre scaling functions as a basis which is more efficient and convenient. Our proposed method is stable.

The present paper is arranged as follows. In Section 2, we describe basic preliminaries for fractional calculus and Legendre scaling functions. In Section 3, we construct operational matrix of fractional integration using Legendre scaling functions as basis. In Section 4, we describe algorithm for the approximate solution. In Section 5, we give the error analysis of the proposed method. In Section 6, we discuss the stability of our method based on maximum absolute error and root mean square error. In Section 7, we present numerical experiment to show the effectiveness of the proposed method.

**2. Preliminaries**

There are several definitions of fractional order derivatives and integrals. These are not necessarily equivalent. In this paper, the fractional order differentiations and integrations are Caputo and Riemann–Liouville sense respectively.

**Definition 2.1.** The Riemann–Liouville fractional order integral operator is given by [23]

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt \quad \alpha > 0, x > 0,$$

$$I^0 f(x) = f(x).$$

For the fractional Riemann–Liouville integration

$$I^\alpha x^k = \frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)} x^{k+\alpha}$$

**Definition 2.2.** The Caputo fractional derivative of order  $\beta$  is defined as

$$D^\beta f(x) = I^{m-\beta} D^m f(x) = \frac{1}{\Gamma(m-\beta)} \int_0^x (x-t)^{m-\beta-1} \frac{d^m}{dt^m} f(t) dt,$$

$$m - 1 < \beta < m, \quad x > 0.$$

For the Caputo derivative, we have

$$D^\beta A = 0 \quad (A \text{ is a constant}), \quad [24]$$

$$D^\beta x^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\beta)} x^{k-\beta} \text{ for } k \in \mathbb{N}_0 \text{ and } k \geq \lceil \beta \rceil \text{ or } k \notin \mathbb{N} \text{ and } k > \lfloor \beta \rfloor,$$

$$= 0 \text{ for } k \in \mathbb{N}_0 \text{ and } k < \lceil \beta \rceil$$

where  $\lceil \beta \rceil$  and  $\lfloor \beta \rfloor$  are the ceiling and floor functions respectively, while  $N = \{1, 2, 3, \dots\}$  and  $N_0 = \{0, 1, 2, \dots\}$ .

The Caputo fractional differentiation is a linear operator similar to the integer order differential operator. The Legendre scaling functions  $\{\phi_i(t)\}$  in one dimension are defined by

$$\phi_i(t) = \begin{cases} \sqrt{(2i+1)} P_i(2t-1), & \text{for } 0 \leq t < 1. \\ 0, & \text{otherwise} \end{cases}$$

where  $P_i(t)$  is Legendre polynomials of order  $i$  on the interval  $[-1, 1]$ , given explicitly by the following formula:

$$P_i(t) = \sum_{k=0}^i (-1)^{i+k} \frac{(i+k)!}{(i-k)!} \frac{t^k}{(k!)^2}. \tag{2}$$

Using one dimensional Legendre scaling functions, we construct two dimensional Legendre scaling function  $\phi_{i_1, i_2}$ ,

$$\phi_{i_1, i_2}(x, t) = \phi_{i_1}(x) \phi_{i_2}(t), \quad i_1, i_2 \in N_0.$$

An explicit expression of two dimensional Legendre scaling functions is given as

$$\phi_{i_1, i_2}(x, t) = \begin{cases} \sqrt{(2i_1+1)} \sqrt{(2i_2+1)} P_{i_1}(2x-1) P_{i_2}(2t-1), & \text{for } 0 \leq x < 1, \quad 0 \leq t < 1. \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

From the above formula it is clear that two dimensional Legendre scaling functions are orthogonal:

$$\int_0^1 \int_0^1 \phi_{i_1, i_2}(x, t) \phi_{j_1, j_2}(x, t) dx dt = \begin{cases} 1, & i_1 = j_1 \text{ and } i_2 = j_2, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

$\{\phi_{i_1, i_2}\}$  forms a complete orthonormal basis (proof is given in Section 5).

So a function  $f(x, t) \in L^2([0, 1] \times [0, 1])$ , can be approximated as

$$f(x, t) \cong \sum_{i_1=0}^{n_1} \sum_{i_2=0}^{n_2} c_{i_1, i_2} \phi_{i_1, i_2}(x, t) = C^T \phi(x, t), \tag{5}$$

where  $C = [c_{0,0}, \dots, c_{0,n_2}, \dots, c_{n_1,1}, \dots, c_{n_1,n_2}]^T$ ;

$$\phi(x, t) = [\phi_{0,0}(x, t), \dots, \phi_{0,n_2}(x, t), \dots, \phi_{n_1,1}(x, t), \dots, \phi_{n_1,n_2}(x, t)]^T.$$

The coefficients  $c_{i_1, i_2}$  in the Fourier expansions of  $f(x, t)$  are given by the formula

$$c_{i_1, i_2} = \int_0^1 \int_0^1 f(x, t) \phi_{i_1, i_2}(x, t) dx dt. \tag{6}$$

Using matrix notation Eq. (4) can be written as

$$f(x, t) \cong \phi^T(x) C \phi(t), \tag{7}$$

where  $\phi(x) = [\phi_0(x), \dots, \phi_{n_1}(x)]^T$ ,  $\phi(t) = [\phi_0(t), \dots, \phi_{n_2}(t)]^T$  and  $C = (c_{i_1, i_2})_{(n_1+1) \times (n_2+1)}$ .

**3. Operational matrix**

**Theorem 3.1.** Let  $\phi(x) = [\phi_0(x), \phi_1(x), \dots, \phi_n(x)]^T$ , be Legendre scaling vector and consider  $\alpha > 0$ , then

$$I^\alpha \phi_i(x) = I^{(\alpha)} \phi(x), \tag{8}$$

where  $I^{(\alpha)}$  is  $(n+1) \times (n+1)$  operational matrix of fractional integral of order  $\alpha$  and its entries are given as

$$\omega(i, j) = (2i + 1)^{1/2} (2j + 1)^{1/2} \sum_{k=0}^i \sum_{l=0}^j (-1)^{i+j+k+l} \frac{(i+k)!(j+l)!}{(i-k)!(j-l)!(k)!(l)!^2 (\alpha+k+l+1)!(\alpha+k+1)}$$

$$0 \leq i, j \leq n.$$

**Proof.** Using the Legendre scaling function of degree  $i$ , we get

$$\begin{aligned} I^\alpha \phi_i(x) &= (2i + 1)^{1/2} \sum_{k=0}^i (-1)^{i+k} \frac{(i+k)!}{(i-k)!(k)!^2} I^\alpha x^k \\ &= (2i + 1)^{1/2} \sum_{k=0}^i (-1)^{i+k} \frac{(i+k)!}{(i-k)!(k)!(\alpha+k+1)} x^{\alpha+k} \end{aligned}$$

using Legendre scaling function approximation for  $x^{\alpha+k}$ , we have

$$x^{\alpha+k} = \sum_{j=0}^n c_j \phi_j(x), \quad \text{where}$$

$$c_j = (2j + 1)^{1/2} \sum_{l=0}^j (-1)^{l+j} \frac{(j+l)!}{(j-l)!} \frac{1}{(l)!^2} \frac{1}{(\alpha+k+l+1)} \quad (9)$$

then

$$I^\alpha \phi_i(x) = \sum_{j=0}^n \omega(i, j) \phi_j(x), \quad (10)$$

where

$$\omega(i, j) = (2i + 1)^{1/2} (2j + 1)^{1/2} \sum_{k=0}^i \sum_{l=0}^j (-1)^{i+j+k+l} \frac{(i+k)!(j+l)!}{(i-k)!(j-l)!(k)!(l)!^2 (\alpha+k+l+1)!(\alpha+k+1)} \quad (11)$$

Hence

$$I^\alpha \phi_i(x) = [\omega(i, 0), \omega(i, 1), \omega(i, 2), \dots, \omega(i, n)] \Phi(x) \quad \square. \quad (12)$$

#### 4. Method of solution

In this section we take  $n_1 = n_2 = n$ , for any approximation.

Using (7), let solution of (1) be approximated as

$$y(x, t) \cong \phi^T(x) C \phi(t). \quad (13)$$

Taking fractional integral of order  $\beta$  and  $\alpha$  in Eq. (13), with respect to  $t$  and  $x$  respectively, we get

$$\begin{aligned} I_t^\beta y(x, t) &= I_t^\beta (\phi^T(x) C \phi(t)) = \phi^T(x) C I_t^\beta \phi(t) \\ &= \phi^T(x) C I^{(\beta)} \phi(t), \end{aligned} \quad (14)$$

$$\begin{aligned} I_x^\alpha y(x, t) &= I_x^\alpha (\phi^T(x) C \phi(t)) = I_x^\alpha (\phi^T(x)) C \phi(t) \\ &= \phi^T(x) (I^{(\alpha)})^T C \phi(t). \end{aligned} \quad (15)$$

where  $I^{(\beta)}$  and  $I^{(\alpha)}$  are  $(n + 1) \times (n + 1)$  operational matrices of fractional integration of order  $\beta$  and  $\alpha$  respectively.

Similarly approximating non homogeneous term,

$$g(x, t) \cong \phi^T(x) G \phi(t) \quad (16)$$

where

$$G = (g_{ij})_{0 \leq i, j \leq n}. \quad (17)$$

Given partial differential equation is

$$\frac{\partial^\alpha y(x, t)}{\partial x^\alpha} + \frac{\partial^\beta y(x, t)}{\partial t^\beta} = g(x, t), \quad (18)$$

with initial conditions  $y(x, 0) = a(x)$ ,  $D_t^\alpha y(x, 0) = c(x)$  and boundary conditions  $y(0, t) = b(t)$ ,  $D_x^\alpha y(0, t) = d(t)$ .

Taking fractional integral of order  $\alpha$  with respect to  $x$  in Eq. (18), we get

$$y(x, t) - y(0, t) - x D_x^\alpha y(0, t) + I_x^\alpha (D_t^\beta) y(x, t) = I_x^\alpha g(x, t). \quad (19)$$

Taking fractional integral of order  $\beta$  with respect to  $t$  in Eq. (19), we get

$$\begin{aligned} I_t^\beta y(x, t) - I_t^\beta (y(0, t) + x D_x^\alpha y(0, t)) + I_t^\beta I_x^\alpha (D_t^\beta) y(x, t) \\ = I_t^\beta I_x^\alpha g(x, t). \end{aligned} \quad (20)$$

Eq. (20), can be written as

$$\begin{aligned} I_t^\beta y(x, t) - I_t^\beta (y(0, t) + x D_x^\alpha y(0, t)) + I_x^\alpha y(x, t) - I_x^\alpha (y(x, 0) \\ + t D_t^\alpha y(x, 0)) = I_t^\beta I_x^\alpha g(x, t). \end{aligned} \quad (21)$$

$$\begin{aligned} \text{Let } I_t^\beta (y(0, t) + x D_x^\alpha y(0, t) + I_x^\alpha (y(x, 0) + t D_t^\alpha y(x, 0))) \\ = f(x, t), \end{aligned} \quad (22)$$

Using (7), approximating  $f(x, t)$ ,

$$f(x, t) = \phi^T(x) A \phi(t). \quad (23)$$

Now using (14)–(16), (22) and (23) in (21) we get

$$\begin{aligned} \phi^T(x) C I^{(\beta)} \phi(t) + \phi^T(x) (I^{(\alpha)})^T C \phi(t) \\ = \phi^T(x) (I^{(\alpha)})^T G I^{(\beta)} \phi(t) + \phi^T(x) A \phi(t), \end{aligned} \quad (24)$$

where matrices  $A$  and  $G$  are known. Eq. (24) can be written as

$$C I^{(\beta)} + (I^{(\alpha)})^T C = (I^{(\alpha)})^T G I^{(\beta)} + A. \quad (25)$$

Eq. (25) gives a system of linear algebraic equations which can be solved easily using Sylvester's approach.

5. Error analysis

**Lemma 5.1.**  $(\phi_{i_1, i_2})$  form a complete orthonormal basis for the Hilbert space  $L^2([0, 1] \times [0, 1])$ .

**Proof.** Space of continuous functions on  $C([0, 1] \times [0, 1])$  is a dense subspace of  $L^2([0, 1] \times [0, 1])$ . Now consider the following sub algebra  $A$  of  $C([0, 1] \times [0, 1])$ :

$$A = \left\{ \sum_{i_1=1}^n \sum_{i_2=1}^m c_{i_1 i_2} \phi_{i_1 i_2} \mid c_{i_1 i_2} \in \mathbb{R}; n, m \in \mathbb{N}_0 \right\}, \tag{26}$$

$A$  contains constants and separates points. So by Weierstrass theorem  $A$  is a dense in  $C([0, 1] \times [0, 1])$ .

Using orthogonality of one dimensional Legendre scaling functions, It is easy to see  $(\phi_{i_1, i_2})$  forms an orthogonal set.

Now to show this is complete, let  $f(x, t)$  be any function in  $L^2([0, 1] \times [0, 1])$

$$\int_0^1 \int_0^1 f(x, t) \phi_{ij}(x, t) dx dt = 0 \quad \forall i, j, \tag{27}$$

By linearity,

$$\int_0^1 \int_0^1 f(x, t) \phi(x, t) dx dt = 0 \quad \forall \phi \in A, \tag{28}$$

Since  $A$  is dense in  $C([0, 1] \times [0, 1])$ , we get

$$\int_0^1 \int_0^1 f(x, t) \psi(x, t) dx dt = 0 \quad \forall \psi \in C[0, 1] \times [0, 1], \tag{29}$$

Again since  $C[0, 1] \times [0, 1]$  is dense in  $L^2([0, 1] \times [0, 1])$

$$\int_0^1 \int_0^1 f(x, t) \psi(x, t) dx dt = 0 \quad \forall \psi \in L^2[0, 1] \times [0, 1], \tag{30}$$

Chose  $\psi = f(x, t)$ , so  $\int_0^1 \int_0^1 f^2(x, t) dx dt = 0$

$$f(x, t) = 0 \quad \text{a.e.} \tag{31}$$

So  $(\phi_{i_1, i_2})$  form a complete orthonormal basis for the Hilbert space  $L^2([0, 1] \times [0, 1])$ .

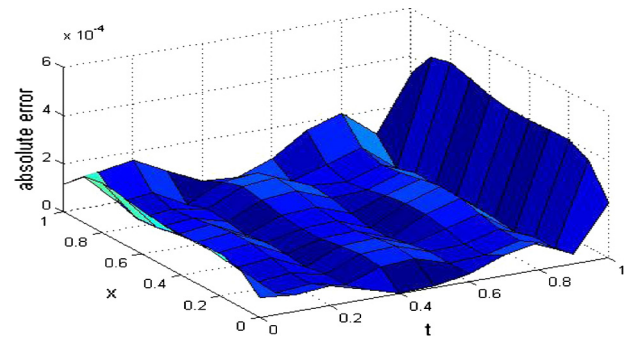
**Theorem 5.1.** Let  $f(x, t) \in L^2([0, 1] \times [0, 1])$ , and  $f_n(x, t)$  be its approximation obtained by using  $(n + 1)^2$ , 2-dimensional

**Table 1** Comparison of exact solution with numerical values obtained from our method and method in [20].

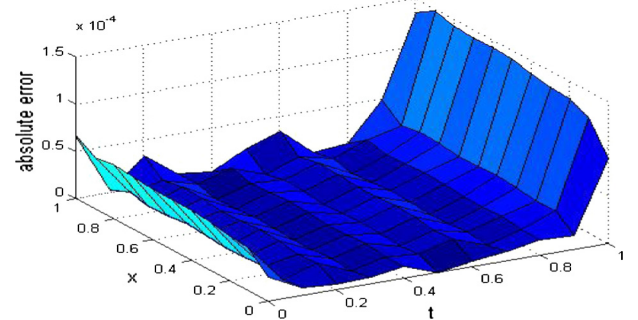
$(x, t)$	Exact solution	Present method		Method in [20]
		$n = 4$	$n = 8$	$n = 8$
(0,0)	0	0	0	0
(1/8, 1/8)	0.0155	0.0155	0.0155	0.0155
(2/8, 2/8)	0.0612	0.0612	0.0612	0.0611
(3/8, 3/8)	0.1342	0.1342	0.1342	0.1340
(4/8, 4/8)	0.2298	0.2298	0.2298	0.2295
(5/8, 5/8)	0.3423	0.3423	0.3423	0.3419
(6/8, 6/8)	0.4646	0.4646	0.4646	0.4640
(7/8, 7/8)	0.5891	0.5891	0.5891	0.5884

**Table 2** Comparison of exact solution with numerical values obtained from our method and method in [20].

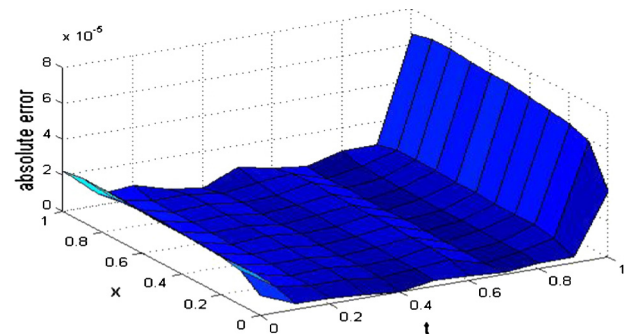
$(x, t)$	Exact solution	Present method		Method in [20]
		$n = 4$	$n = 8$	$n = 8$
(0,0)	0	0	0	0
(1/8, 1/8)	0.0156	0.0156	0.0156	0.0150
(2/8, 2/8)	0.0625	0.0625	0.0625	0.0620
(3/8, 3/8)	0.1406	0.1407	0.1406	0.1401
(4/8, 4/8)	0.2500	0.2501	0.2500	0.2494
(5/8, 5/8)	0.3906	0.3906	0.3906	0.3900
(6/8, 6/8)	0.5625	0.5623	0.5625	0.5619
(7/8, 7/8)	0.7656	0.7655	0.7656	0.7650



**Figure 1** Absolute error for  $n = 4$ .



**Figure 2** Absolute error for  $n = 7$ .



**Figure 3** Absolute error for  $n = 10$ .

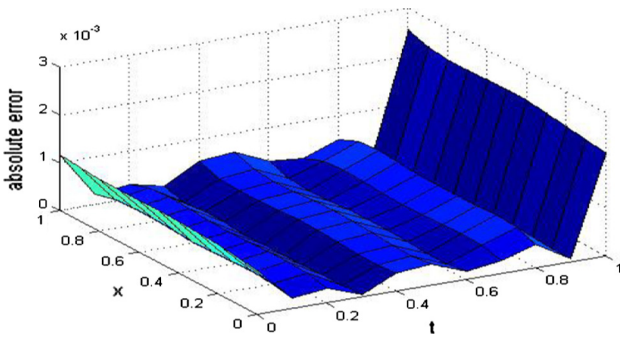


Figure 4 Absolute error for  $n = 4$ .

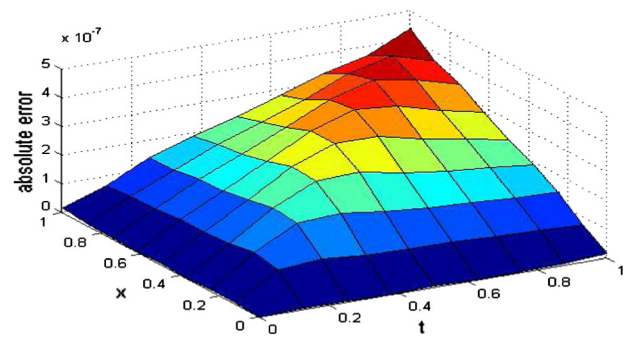


Figure 8 Absolute error for  $n = 7$ .

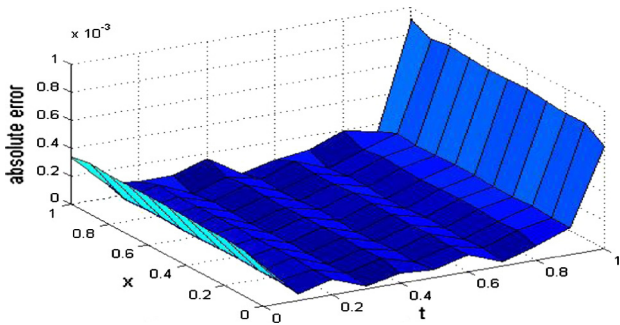


Figure 5 Absolute error for  $n = 7$ .

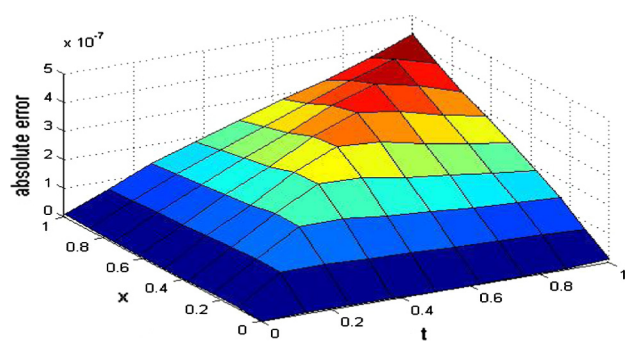


Figure 9 Absolute error for  $n = 10$ .

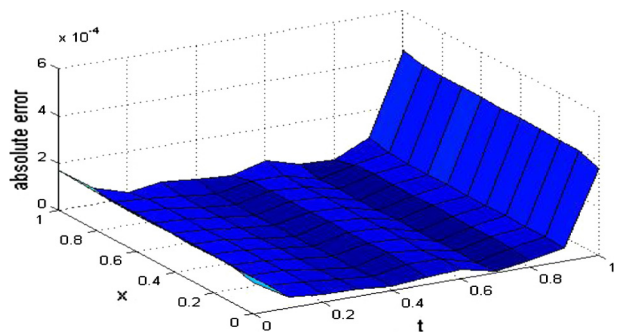


Figure 6 Absolute error for  $n = 10$ .

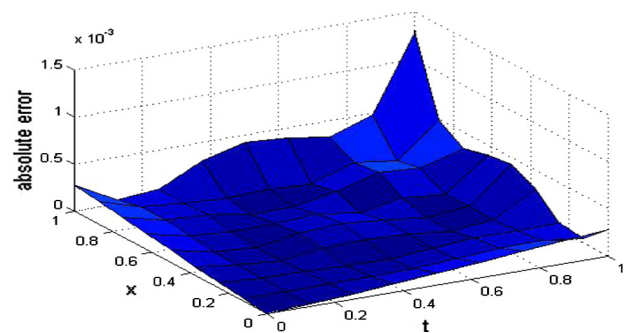


Figure 10 Absolute error for  $n = 4$ .

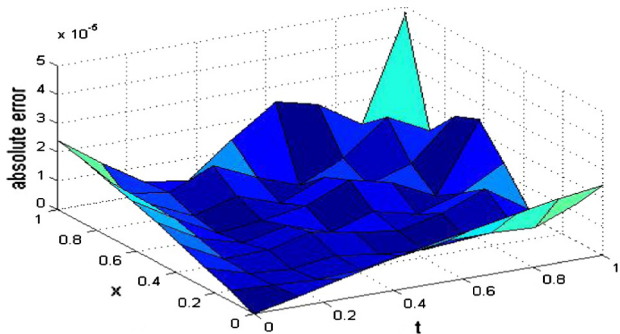


Figure 7 Absolute error for  $n = 4$ .

Legendre scaling vectors. Assuming  $\left| \frac{\partial^4 f(x,t)}{\partial x^2 \partial t^2} \right| \leq K$ , we have the following upper bound for error:

$$\|f(x,t) - f_n(x,t)\|_{L^2} < \left(\frac{K}{256}\right) \left(F_3\left(-\frac{1}{2} + n\right)\right) \tag{32}$$

where

$$\|f(x,t)\|_{L^2} = \left(\int_0^1 \int_0^1 |f(x,t)|^2 dx dt\right)^{\frac{1}{2}}$$

where  $F_n(z)$  is given as

$$F_n(z) = (-1)^{n+1} n \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}} \tag{33}$$

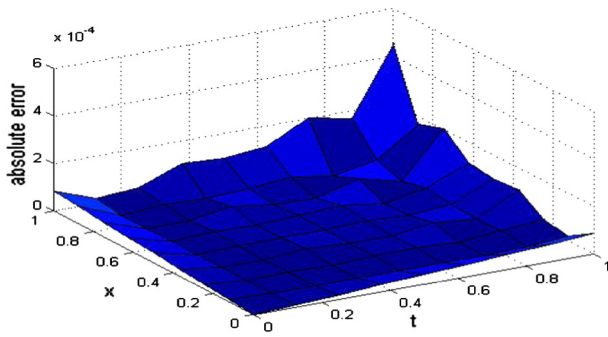


Figure 11 Absolute error for  $n = 7$ .

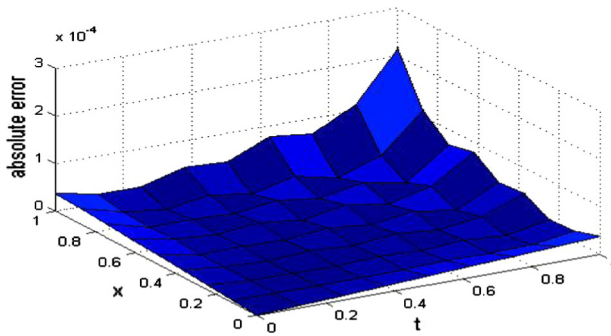


Figure 12 Absolute error for  $n = 10$ .

**Proof.** Let  $f(x, t) = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} c_{i_1, i_2} \phi_{i_1, i_2}(x, t)$ . truncating it to level  $n$ , we get  $f_n(x, t) = \sum_{i_1=0}^n \sum_{i_2=0}^n c_{i_1, i_2} \phi_{i_1, i_2}(x, t)$ ; thus,

$$f(x, t) - f_n(x, t) = \sum_{i_1=n+1}^{\infty} \sum_{i_2=n+1}^{\infty} c_{i_1, i_2} \phi_{i_1, i_2}(x, t), \tag{34}$$

$$\begin{aligned} \|f(x, t) - f_n(x, t)\|_{L^2}^2 &= \int_0^1 \int_0^1 (f(x, t) - f_n(x, t))^2 dx dt \\ &= \int_0^1 \int_0^1 \left( \sum_{i_1=n+1}^{\infty} \sum_{i_2=n+1}^{\infty} c_{i_1, i_2} \phi_{i_1, i_2}(x, t) \right)^2 dx dt \\ &= \sum_{i_1=n+1}^{\infty} \sum_{i_2=n+1}^{\infty} c_{i_1, i_2}^2, \end{aligned} \tag{35}$$

where

$$\begin{aligned} c_{i_1, i_2} &= \int_0^1 \int_0^1 f(x, t) \phi_{i_1, i_2}(x, t) dx dt \\ &= \int_0^1 \int_0^1 f(x, t) \phi_{i_1}(x) \phi_{i_2}(t) dx dt \end{aligned}$$

Now using similar process as in [25], we get

$$|c_{i_1, i_2}|^2 < \frac{9K^2}{64(2i_1 - 3)^4(2i_2 - 3)^4}. \tag{36}$$

Using (36), we can write

$$\begin{aligned} \sum_{i_1=n}^{\infty} \sum_{i_2=n}^{\infty} c_{i_1, i_2}^2 &< \sum_{i_1=n+1}^{\infty} \sum_{i_2=n+1}^{\infty} \frac{9K^2}{64(2i_1 - 3)^4(2i_2 - 3)^4} \\ &= \frac{9K^2}{64} \left( \frac{1}{9216} \right) \left( F_3 \left( -\frac{1}{2} + n \right) \right)^2 \\ &= \left( \frac{K^2}{65,536} \right) \left( F_3 \left( -\frac{1}{2} + n \right) \right)^2 \end{aligned}$$

So from (35),

$$\|f(x, t) - f_n(x, t)\|_{L^2} < \left( \frac{K}{256} \right) \left( F_3 \left( -\frac{1}{2} + n \right) \right) \quad \square \tag{37}$$

### 6. Stability analysis

The accuracy of proposed method is demonstrated by calculating absolute error, and average deviation  $\sigma$  also known as root mean square error (RMS). They are calculated using the following equations:

$$\Delta y(x_i, t_j) = |y_e(x_i, t_j) - y_a(x_i, t_j)|, \tag{38}$$

and

$$\sigma_{(N+1)^2} = \left\{ \frac{1}{(N+1)^2} \sum_{i=0}^N \sum_{j=0}^N [y_e(x_i, t_j) - y_a(x_i, t_j)]^2 \right\}^{1/2}, \tag{39}$$

where  $y_e(x_i, t_j)$  is the exact value of output function at point  $(x_i, t_j)$  and  $y_a(x_i, t_j)$  is the approximate value of output function at the same point.

In all the examples, the input function with and without noise is denoted by  $g^\delta(x, t)$  and  $g(x, t)$  respectively, where  $g^\delta(x, t)$  is obtained by adding a noise  $\delta$  to  $g(x, t)$  such that  $g^\delta(x_i, t_j) = g(x_i, t_j) + \delta\theta_{ij}$ , where  $x_i = ih, i = 1, 2, \dots, N, Nh = 1; t_j = jk, j = 1, 2, \dots, N, Nk = 1$  and  $\theta_{ij}$  is the uniform random variable with values in  $[-1, 1]$  such that

$$\max_{\substack{1 \leq i \leq N \\ 1 \leq j \leq N}} |g^\delta(x_i, t_j) - g(x_i, t_j)| \leq \delta. \tag{40}$$

In Section 7, five examples are solved with and without noise to illustrate the stability of the proposed method. In all the five examples, we add the noise  $\delta = \sigma_{(N+1)^2}$ , for two different values of  $N = 10, 20$ . For different values of  $N$  we calculate maximum absolute error and root mean square errors denoted by  $E_1$  and  $E_2$  respectively for input functions without noise term. Similarly, these respective errors are denoted by  $E_1^*$  and  $E_2^*$  for input function with noise respectively. In Table 3, we have listed the different values of  $E_1, E_2, E_1^*$  and  $E_2^*$  for  $N = 10, 20$ . From the table it is clear that there is a very small change in errors when we add noise term in input function showing the stability of our method.

### 7. Numerical results

In this section we discuss the implementation of our proposed numerical methods based on fractional operational matrix of integration for Legendre scaling function. We show the accuracy of our method by the graph of absolute errors in various examples. In Tables 1 and 2 we have compared numerical solutions obtained by our method and method given in [20], and

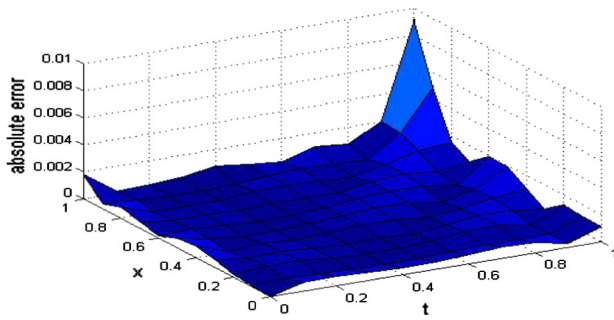


Figure 13 Absolute error for  $n = 4$ .

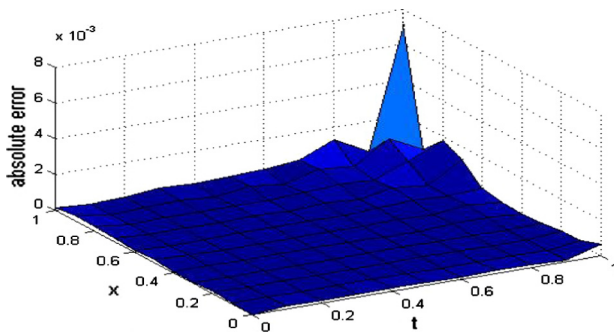


Figure 14 Absolute error for  $n = 7$ .

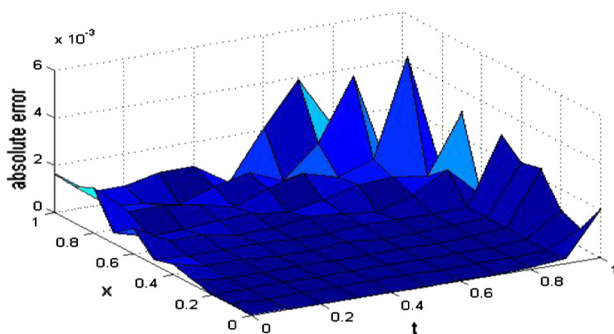


Figure 15 Absolute error for  $n = 10$ .

it is observed our method gives better approximation. It is also observed that desired accuracy is obtained even for  $n = 4$ .

**Example 1.** Consider the following partial differential equation:

$$\frac{\partial^{1/4}y(x,t)}{\partial x^{1/4}} + \frac{\partial^{1/5}y(x,t)}{\partial t^{1/5}} = g(x,t), \quad 0 \leq x, t \leq 1, \quad (41)$$

with initial-boundary conditions as  $y(x,0) = x^2$ ,  $y(0,t) = t$  respectively, where  $g(x,t) = \frac{\sqrt{3}}{\Gamma(1/4)}x^{7/4} + \frac{\sqrt{2}}{\Gamma(1/5)}t^{4/5}$  and the exact solution is  $y(x,t) = x^2 + t$ .

Figs. 1–3 represent absolute error of Eq. (41) for  $n = 4, 7, 10$  respectively.

**Example 2.** Consider the following partial differential equation:

$$\frac{\partial^{1/8}y(x,t)}{\partial x^{1/8}} + \frac{\partial^{1/3}y(x,t)}{\partial t^{1/3}} = g(x,t), \quad 0 \leq x, t \leq 1, \quad (42)$$

with initial-boundary conditions as  $y(x,0) = x$ ,  $y(0,t) = 2t$  respectively, where  $g(x,t) = \frac{\sqrt{2}}{\Gamma(1/8)}x^{7/8} + \frac{2\sqrt{2}}{\Gamma(1/3)}t^{2/3}$  and the exact solution is  $y(x,t) = x + 2t$ .

Figs. 4–6 represent absolute error of Eq. (42) for  $n = 4, 7, 10$  respectively.

**Example 3.** Consider the following non homogeneous partial differential equation [20]:

$$\frac{\partial y(x,t)}{\partial x} + \frac{\partial y(x,t)}{\partial t} = \sin(x+t), \quad 0 \leq x, t \leq 1, \quad (43)$$

With initial-boundary conditions as  $y(x,0) = y(0,t) = 0$  respectively, and the exact solution is  $y(x,t) = \sin x \sin t$ .

Figs. 7–9 represent absolute error of Eq. (43) for  $n = 4, 7, 10$  respectively.

In Table 1, given below, we compare our result with the method of [20].

**Example 4.** Consider the following non homogeneous fractional partial differential equation [20]:

$$\frac{\partial^{1/4}y(x,t)}{\partial x^{1/4}} + \frac{\partial^{1/4}y(x,t)}{\partial t^{1/4}} = g(x,t), \quad 0 \leq x, t \leq 1, \quad (44)$$

with initial-boundary conditions as  $y(x,0) = y(0,t) = 0$  respectively, where  $g(x,t) = \frac{4(x^{3/4}t + xt^{3/4})}{3\sqrt[3]{4}}$  and the exact solution is  $y(x,t) = xt$ .

Figs. 10–12 represent absolute error of Eq. (44) for  $n = 4, 7, 10$  respectively.

In Table 2, given below, we compare our result with that of [20].

**Example 5.** Consider the following non homogeneous fractional partial differential equation:

$$\frac{\partial^{1.5}y(x,t)}{\partial x^{1.5}} + \frac{\partial^{1.2}y(x,t)}{\partial t^{1.2}} = g(x,t), \quad 0 \leq x, t \leq 1, \quad (45)$$

with initial-boundary conditions as  $y(x,0) = x^2$ ,  $D_t^1 y(x,0) = 0$  and  $y(0,t) = t^2$ ,  $D_x^1 y(0,t) = 0$  respectively, where  $g(x,t) = \frac{\sqrt{3}}{\Gamma(1.5)}x^{1/2} + \frac{\sqrt{3}}{\Gamma(1.8)}t^{4/5}$  and the exact solution is  $y(x,t) = x^2 + t^2$ .

Figs. 13–15 represent absolute error of Eq. (45) for  $n = 4, 7, 10$  respectively.

In Table 3, given below, we list the errors  $E_1, E_2, E_1^*$  and  $E_2^*$  to show the stability of our method.

The computational order for the numerical results is calculated in the paper [26]. The computational order for the numerical results is given as follows:

**Table 3** Absolute and RMS errors with and without noise.

Example No.	$N$	Errors without noise		Errors with noise $\delta = \sigma_{(N+1)^2}$	
		$E_1$	$E_2$	$E_1^*$	$E_2^*$
1	10	$9.9693 \times 10^{-6}$	$2.0190 \times 10^{-7}$	$1.0058 \times 10^{-5}$	$2.0284 \times 10^{-7}$
	20	$1.4386 \times 10^{-5}$	$7.3441 \times 10^{-8}$	$1.4352 \times 10^{-5}$	$7.3495 \times 10^{-8}$
2	10	$6.3873 \times 10^{-5}$	$1.2155 \times 10^{-6}$	$6.4411 \times 10^{-5}$	$1.2174 \times 10^{-6}$
	20	$1.1114 \times 10^{-4}$	$5.0195 \times 10^{-7}$	$1.1091 \times 10^{-4}$	$5.0176 \times 10^{-7}$
3	10	$4.2034 \times 10^{-7}$	$9.4835 \times 10^{-9}$	$4.2710 \times 10^{-7}$	$9.6074 \times 10^{-9}$
	20	$4.2181 \times 10^{-7}$	$3.4345 \times 10^{-9}$	$4.2441 \times 10^{-7}$	$3.4510 \times 10^{-9}$
4	10	$2.6613 \times 10^{-5}$	$3.4135 \times 10^{-7}$	$4.9749 \times 10^{-5}$	$1.7266 \times 10^{-7}$
	20	$4.9905 \times 10^{-5}$	$1.7277 \times 10^{-7}$	$4.9826 \times 10^{-5}$	$1.7271 \times 10^{-7}$
5	10	$7.1002 \times 10^{-3}$	$1.6594 \times 10^{-5}$	$7.1735 \times 10^{-3}$	$1.7150 \times 10^{-5}$
	20	$7.1002 \times 10^{-3}$	$8.9995 \times 10^{-6}$	$7.1735 \times 10^{-3}$	$9.1354 \times 10^{-6}$

**Table 4** Computational order obtained for all examples.

Example No.	$n$	$E_n$	Order
1	2	$1.5789 \times 10^{-3}$	–
	4	$4.6130 \times 10^{-4}$	1.7751
	8	$1.1369 \times 10^{-4}$	2.0206
2	2	$7.2824 \times 10^{-3}$	–
	4	$2.5544 \times 10^{-3}$	1.5114
	8	$6.9316 \times 10^{-4}$	1.8817
3	2	$1.2215 \times 10^{-2}$	–
	4	$4.7945 \times 10^{-5}$	7.9931
	8	$4.3849 \times 10^{-7}$	6.7727
4	2	$3.8175 \times 10^{-3}$	–
	4	$1.2933 \times 10^{-3}$	1.5616
	8	$3.4485 \times 10^{-4}$	1.9070
5	2	$9.4023 \times 10^{-3}$	–
	4	$9.2421 \times 10^{-3}$	0.0248
	8	$2.1573 \times 10^{-3}$	2.0989

Order =  $\log_2 \left[ \frac{E_n}{E_{2n}} \right]$  where  $E_n$  is maximum absolute error ( $\max_{1 \leq i, j \leq N} E(x_i, t_j)$ ) for approximation having  $n$  number of basis elements.

In Table 4, given below, we list the computational order for the numerical results.

## 8. Conclusions

From Figs. 1–15, it is observed that absolute error decreases with the increasing  $n$ . Similarly as we increase the dimension of basis function, we obtain more accurate numerical solution. Operational matrix of Haar wavelet in [21] is used to solve fractional partial differential equation of the same type but the procedure of construction of such operational matrix is very complicated. Our method based on two dimensional Legendre scaling functions is easy in comparison with that of [21]. Further our method is more accurate than that in [20]. The stability with respect to the data is restored and accuracy is good even for high noise levels in the data. An error analysis and stability analysis are also given.

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