Chapter 6

Conclusions and Future Work

The concluding chapter of the thesis consists of two segments: a summary and a discussion on potential future avenues for research.

6.1 Conclusions

The primary contribution of this thesis lies in the advancement of approximation techniques for FVPs and FSLPs. Within this thesis, we have formulated numerical methods developed to solve FVPs wherein the problem formulations involve Caputo fractional derivatives and generalized fractional derivatives. These FSLPs are defined in terms of both Riemann-Liouville and Caputo fractional derivatives, particularly for higher-order systems and dimensions.

Chapter 1 serves as the introduction to the thesis, encompassing a presentation of fractional operators and their fundamental characteristics. The chapter includes a comprehensive literature review on FVPs and FSLPs, offering insights into both past and contemporary research in these domains. Additionally, the chapter elucidated the motivation of the research work.

Chapter 2 presented a Rayleigh-Ritz method for solving a class of FVPs with several dependent variables defined in terms of CFDs. The method employs Jacobi poly-fractonomials as basis functions for the approximation of solutions. The choice of Jacobi poly-fractonomials as basis functions is strategic, aiming to enhance the precision of the approximated solutions. The proposed method is also effective when replacing CFDs with RLFDs.

In Chapter 3, we proposed three numerical schemes, i.e. linear, quadratic, and quadratic-linear approximation, for solving a class GICFVP defined in terms of GFD. Regarding the approximation, the addressed problem transformed into an eigenvalue problem. By solving this eigenvalue problem, the minimum eigenvalue and its corresponding eigenfunction are determined. The eigenfunction associated with the minimum eigenvalue serves to minimize the considered cost functional. The demonstrated simulation results showcase a convergence order of $2 - \alpha$ and $3-\alpha$ for schemes S1 and S2, respectively. The provided tables substantiate that the theoretical findings are well-established. Additionally, it is observed from the graphs that as α approaches 1, the numerical solutions align with the analytical solution of the corresponding integer-order problem. It was observed that among the schemes considered, the S2 exhibited superior convergence in both achieving an approximate solution $y(t)$ and minimizing the cost functional when compared to S1 and S3.

In Chapter 4, we developed a variational and numerical approximations for higher-order regular FSLPs. We have illustrated that the higher-order FSLP possesses an infinite set of eigenvalues and corresponding unique eigenfunctions for fractional order $(0, 1)$. Additionally, we have established that the first eigenvalue of the FSLP serves as the minimizer of the functional. To validate our theoretical findings, a numerical method is introduced, and we discuss the convergence of this method. The presented results encompass various fractional order derivatives, revealing that as the order α approaches 1, the eigenvalues converge to the analytical results of the corresponding integer-order problem.

In Chapter 5, variational methods were discussed for regular N-DFSLP of fractional order $(0, 1]$ expressed in terms of fractional gradient operators associated with both left and right CFDs. For FSLP, we established the existence of an infinite set of monotonically increasing eigenvalues. Similarly, analogous results can be established for N-DFSLPs formulated with Riemann-Liouville Fractional Derivatives (RLFDs) alone or in combination with CFDs. Additionally, our theoretical results were implemented through an example, and eigenvalues were computed for various values of α. We validated that the approximated results align with our analytical predictions.

6.2 Future Work

The primary objective of this thesis is to formulate an approximation method for FVPs and to gain a comprehensive understanding of the theoretical results of FSLPs in higher orders and dimensions. Very little research has been conducted on FVPs in higher dimensions. Therefore, there is an opportunity to expand this theory and devise numerical methods for addressing such problems. The domain of numerical methods is still open to solving FSLPs in higher orders and dimensions. Approximation methods like the finite element and spectral methods can be developed for considered FSLPs. In this thesis, we exclusively focus on regular FSLPs. However, it is possible to extend this theory to encompass singular FSLPs.

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