Chapter 4

Nonsmooth Interval Optimization with gH-Dini Hadamard Subdifferential and Its Applications in Control Systems

4.1 Introduction

The area of nonsmooth analysis primarily deals with differential calculus developed for nondifferentiable functions. In areas such as optimization [34], differential equations [39], calculus of variations [32], and control theory [32], one has to deal with nondifferentiable functions on a regular basis. In dealing with such functions, one useful tool in nonsmooth analysis is the concept of subdifferential. However, for nonsmooth and nonconvex function, the idea of subdifferential from conventional literature cannot be directly borrowed and applied. Therefore, over the years, researchers have developed the concepts of Clarke subdifferential [178], Fréchet subdifferential [146], Mordukhovich subdifferential [112], Proximial subdifferential [92], Weak subdifferential [41] etc. as generalization of subdifferential for nonsmooth and nonconvex functions. Weak subdifferential, which is studied in the previous chapter as well, is an immediate consequence of Fréchet subdifferential. Fréchet subdifferential is also a basis for Mordukhovich subdifferential. Dini-Hadamard subdifferential is at the core of the so-called A-subdifferential. These generalizations are also closely linked to generalized directional derivatives.

Among them, Dini-Hadamard subdifferential is the extension of conventional subdifferential on a Banach space that is separable. First appearing in 1970's, Dini-Hadamard subdifferntial was developed using Dini-Hadamard generalized directional derivative. As its application, Dini-Hadamard subdifferential's exact difference rule is employed to construct the cone-constrained optimization problem where the objective is the difference between two functions and the difference adheres to the property of calmness. For lower-semicontinuous and Lipschitz function on a separable Banach space, the nonemptiness of Dini-Hadamard subdifferential set is especially useful for obtaining optimality conditions for lower semicontinuous function [11, 17, 90]. Under practical applications, Dini-Hadamard subdifferential is used for studying asymptotic controllability of nonsmooth control system.

Motivated by the usefulness of Dini-Hadamard subdifferential in optimization and nonsmooth control theory, and our goal, in this chapter, is to extend the concept of Dini-Hadamard subdifferential and its associated results to IVFs so that these can become important tools for solving problems that involve interval-valued functions.

4.2 Motivation

Similar to the classical case, gH-subdifferential for IVFs may not be directly applicable for nonsmooth and nonconvex IVFs. Thus, we need the concepts of generalized gH- subdifferentials for IVFs to be defined. Accordingly, we also need the concepts of gH-directional derivatives, which are closely connected to generalized subdifferentials. Particularly in terms of Dini-Hadamard class, Dini-Hadamard derivatives have already been studied in the literature. Dini-Hadamard derivative although useful for identifying monotonic property of lower-semi continuous functions, it does not hold the basic rules of algebra such as sum rule, product rule, difference rule, chain rule. Thus, gH-Dini-Hadamard derivative could be very restrictive in studying optimality conditions for lower semi-continuous IVFs. Dini-Hadamard subdifferential, on the other hand, does not have any problem with these standard algebraic properties. Hence, for the application of identifying optimality conditions for nonsmooth IVFs, it will be more useful to generalize Dini-Hadamard subdifferential to define gH-Dini-Hadamard subdifferential for IVFs.

4.3 Contributions

In this chapter, we propose the notions of gH-Dini Hadamard subdifferential and gH-Dini Hadamard superdifferential for nonsmooth interval-valued functions. We observe that both subdifferential and superdifferential coincide for convex IVFs. We also find that gH-Dini Hadamard subdifferential is closed and convex. Next, we show gH-Dini Hadamard subdifferential to hold the properties of one-sided sum rule, partial chain rule, linear transformation rule. We also present a requirement for an IVF to be gH-Dini Hadamard subdifferentiable.

Further, we propose a smooth variational description of gH-Dini Hadamard subgradients where all gH-Dini Hadamard subgradients can be equivalently identified using the gH-Dini Hadamard derivative of a smooth IVF. This description is useful for better understanding of the gH-Dini Hadamard subdifferential set for nonsmooth and nonconvex IVFs.

Apart from the above points, other major contributions of this chapter are as follows:

(i) With the help of gH-Dini Hadamard subgradients, we derive Fritz-John-type (FJ) and KKT-type optimality conditions for constrained nonsmooth interval optimization problem.

(ii) We present an asymptotic controllability theory for the interval-valued nonsmooth control system comprised of an interval-valued Lyapunov function. We hope that this becomes useful for advancing the fields of control theory and nonsmooth analysis.

4.4 gH-Dini Hadamard subdifferential and superdifferential for IVFs

In this section, we establish the gH-Dini Hadamard subdifferential and superdifferential for IVFs, employing gH-lower and gH-upper Hadamard derivatives along with their respective key properties. We also discuss a smooth variational-type characterization of gH-Dini Hadamard subgradients to establish optimality conditions. Additionally, we aim to derive certain calculus rules such as the partial chain rule, subadditive formula, and others.

Definition 4.1 (gH-upper Hadamard derivative). *Consider an IVF $\boldsymbol{\Upsilon} : \mathcal{Z} \to I(\mathbb{R})$. For $\bar{z} \in \mathcal{Z}$ and $v \in \mathbb{R}^n$, the gH-upper Hadamard derivative of $\boldsymbol{\Upsilon}$ at $\bar{z}$ along the direction $v \in \mathbb{R}^n$ is given by*

$$\boldsymbol{\Upsilon}^+_{\mathscr{H}}(\bar{z})(v) = \limsup_{\substack{u \to v \\ t \to 0+}} \tfrac{1}{t} \odot \{\boldsymbol{\Upsilon}(\bar{z} + tu) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z})\}, \text{ given that the limit exists.}$$

The gH-Dini Hadamard superdifferential of $\boldsymbol{\Upsilon}$ at $\bar{z}$ is the collection

$$\partial^+_{\mathscr{H}} \boldsymbol{\Upsilon}(\bar{z}) = \Big\{ \widehat{\boldsymbol{K}} \in I(\mathbb{R})^n : v^\top \odot \widehat{\boldsymbol{K}} \preceq \boldsymbol{\Upsilon}^+_{\mathscr{H}}(\bar{z})(v) \ \forall \ v \in \mathbb{R}^n, \\ \text{where } \widehat{\boldsymbol{K}} : \mathcal{Z} \to I(\mathbb{R})^n \text{ is a linear and } gH\text{-continuous IVF} \Big\}.$$

Definition 4.2 (gH-lower Hadamard derivative). *Consider an IVF $\boldsymbol{\Upsilon} : \mathcal{Z} \to I(\mathbb{R})$. Then, for $\bar{z} \in \mathcal{Z}$ and $v \in \mathbb{R}^n$, the gH-lower Hadamard derivative of $\boldsymbol{\Upsilon}$ at $\bar{z}$ along the* *direction $v \in \mathbb{R}^n$ is given by*

$$\boldsymbol{\Upsilon}^-_{\mathscr{H}}(\bar{z})(v) = \liminf_{\substack{u \to v\\ t \to 0+}} \tfrac{1}{t} \odot \{\boldsymbol{\Upsilon}(\bar{z} + tu) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z})\}, \text{ given that the limit exists.}$$

The gH-Dini Hadamard subdifferential of $\boldsymbol{\Upsilon}$ at $\bar{z}$ is the collection

$$\partial^-_{\mathscr{H}}\boldsymbol{\Upsilon}(\bar{z}) = \{\widehat{\boldsymbol{K}} \in I(\mathbb{R})^n : v^\top \odot \widehat{\boldsymbol{K}} \preceq \boldsymbol{\Upsilon}^-_{\mathscr{H}}(\bar{z})(v) \ \forall\ v \in \mathbb{R}^n,$$
$$\text{where } \widehat{\boldsymbol{K}} : \mathcal{Z} \to I(\mathbb{R})^n \text{ is a linear and } gH\text{-continuous IVF}\}.$$

The definition above can be equivalently restated as $\widehat{\mathbf{K}} \in \partial^-_{\mathscr{H}}\boldsymbol{\Upsilon}(\bar{z})$ if and only if

$$\mathbf{0} \preceq \liminf_{\substack{u \to v\\ t \to 0+}} \tfrac{1}{t} \odot \{(\boldsymbol{\Upsilon}(\bar{z} + tu) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z})) \ominus_{gH} (tu)^\top \odot \widehat{\mathbf{K}}\} \text{ for all } \ v \in \mathbb{R}^n. \tag{4.1}$$

On the other hand, for every $\epsilon > 0$ there exists some $\delta > 0$ such that for all $t \in (0, \delta)$,

$$- t \odot [\epsilon, \epsilon] \preceq (\boldsymbol{\Upsilon}(\bar{z} + tu) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z})) \ominus_{gH} (tu)^\top \odot \widehat{\mathbf{K}} \text{ whenever } \|u - v\| \leq \delta. \tag{4.2}$$

Remark 4.4.1 *By (ii) of Lemma 1.1 and Lemma 1.4, notice that*

$$\partial^+_{\mathscr{H}}\boldsymbol{\Upsilon}(\bar{z}) = -1 \odot \partial^-_{\mathscr{H}}(\boldsymbol{0} \ominus_{gH} \boldsymbol{\Upsilon})(\bar{z}) = -1 \odot \partial^-_{\mathscr{H}}(-1 \odot \boldsymbol{\Upsilon})(\bar{z}).$$

We can easily confirm that $\boldsymbol{\Upsilon}$ is gH-Dini Hadamard superdifferentiable at $\bar{z}$ if and only if $(\boldsymbol{0} \ominus_{gH} \boldsymbol{\Upsilon})$ is gH-Dini Hadamard subdifferentiable at $\bar{z}$.

In the example that follows, we calculate the gH-Dini Hadamard subdifferential of three IVFs, denoted as $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, and $\boldsymbol{\Upsilon}_3$, at the point 0. This example illustrates that the gH-Dini Hadamard subgradients can intersect the image set of the IVFs from above, in the middle, or from below.

Example 4.4.1 *(a) Let the IVF $\boldsymbol{\Upsilon}_1 : [-3, 3] \to I(\mathbb{R})$ be defined as*

$$\boldsymbol{\Upsilon}_1(z) = [\sin|z|, |z|] \,.$$

Here, we make note that

$$\partial^-_{\mathscr{H}}\boldsymbol{\Upsilon}_1(0) = \left\{\boldsymbol{K} \in I(\mathbb{R}) : \boldsymbol{0} \preceq \liminf_{\substack{t \to 0+\\ u \to v}} \tfrac{1}{t} \odot \{([\sin|tu|, |tu|] \ominus_{gH} \boldsymbol{0}) \ominus_{gH} (tu) \odot \boldsymbol{K}\} \right.$$
$$\left. \forall v \ \in \mathbb{R} \right\}.$$

Take $\boldsymbol{K} = [\underline{k}, \overline{k}] \in \partial^-_{\mathscr{H}}\boldsymbol{\Upsilon}_1(0)$. This means that

$$\boldsymbol{0} \preceq \liminf_{\substack{t\to 0+\\ u\to v}} \tfrac{1}{t}\odot\left([\sin|tu|, |tu|]\ominus_{gH}(tu)\odot[\underline{k},\overline{k}]\right) \;\; \textit{for all} \;\; v\in\mathbb{R}. \tag{4.3}$$

Using ((ii)) of Lemma 1.1 and Lemma 2.3 of [68], (4.3) is similar to

$$\begin{aligned}
&\boldsymbol{0} \preceq \liminf_{\substack{t\to 0+\\ u\to v}} \tfrac{1}{t}\odot\left([\sin t|u|, t|u|]\right)\ominus_{gH}\lim_{u\to v} u\odot[\underline{k},\overline{k}] \;\textit{for every}\; v\in\mathbb{R}\\
\iff &\boldsymbol{0} \preceq [|v|, |v|]\ominus_{gH} v\odot[\underline{k},\overline{k}] \;\textit{for every}\; v\in\mathbb{R}.
\end{aligned} \tag{4.4}$$

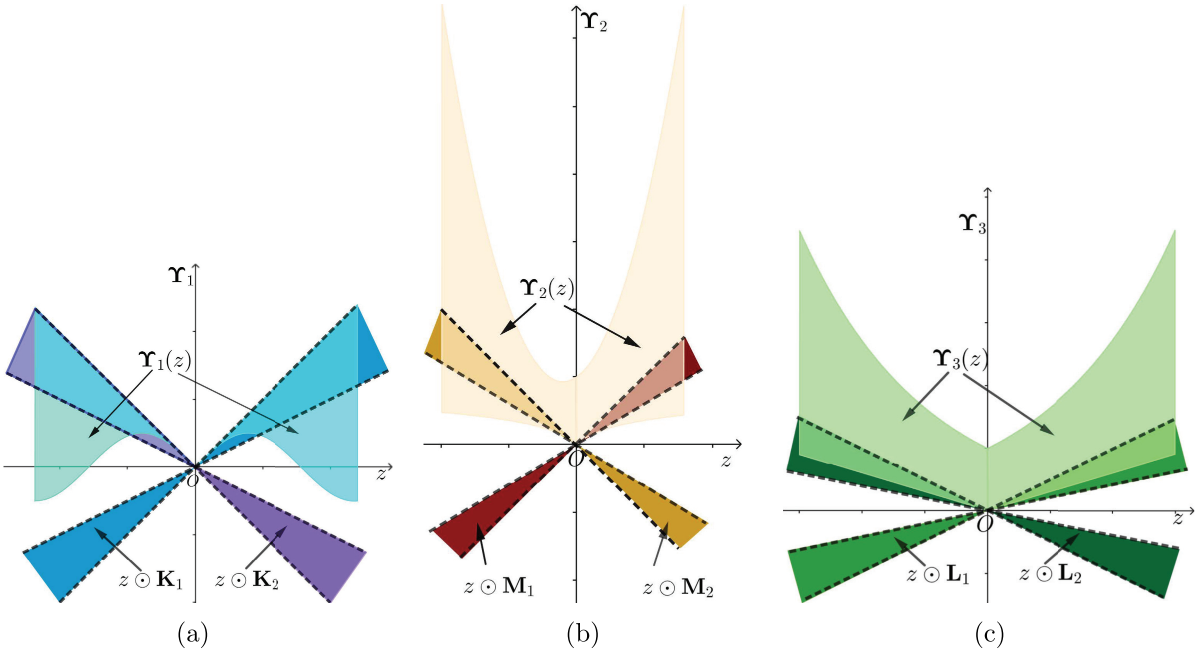


Figure 4.1: Geometrical illustration of gH-Dini Hadamard subgradients intersecting (a) $\boldsymbol{\Upsilon}_1$ from above, (b) $\boldsymbol{\Upsilon}_2$ in the middle, and (c) $\boldsymbol{\Upsilon}_3$ from below in Example 4.4.1

- *Case 1. For* $v \geq 0$, *(4.4) is same as*

$$\begin{aligned}
&\boldsymbol{0} \preceq [v, v]\ominus_{gH}[\underline{k}v, \overline{k}v] \iff [\underline{k}v, \overline{k}v] \preceq [v, v]\\
\iff &\underline{k}v \leq v \;\textit{and}\; \overline{k}v \leq v \iff \underline{k} \leq 1 \;\textit{and}\; \overline{k} \leq 1.
\end{aligned}$$

- *Case 2. For* $v < 0$, *the (4.4) is same as*

$$\begin{aligned}
&\boldsymbol{0} \preceq [-v, -v]\ominus_{gH}[\overline{k}v, \underline{k}v] \iff [\overline{k}v, \underline{k}v] \preceq [-v, -v]\\
\iff &\underline{k}v \leq -v \;\textit{and}\; \overline{k}v \leq -v \iff \underline{k} \geq -1 \;\textit{and}\; \overline{k} \geq -1.
\end{aligned}$$

Hence, taking Case 1 and Case 2 together, we have

$$\partial^-_{\mathscr{H}} \boldsymbol{\Upsilon}_1(0) = \{[\underline{k}, \overline{k}] \in I(\mathbb{R}) : -1 \leq \underline{k} \leq 1 \text{ and } -1 \leq \overline{k} \leq 1\}. \tag{4.5}$$

Using (4.5) in Figure 4.1(a), we depict two possible gH-Dini Hadamard subgradients of $\boldsymbol{\Upsilon}_1$ (shown in cyan-blue color), named as $\boldsymbol{K}_1$ (shaded in sky-blue color) and $\boldsymbol{K}_2$ (shaded in violet color). These two gH-Dini Hadamard subgradients intersect the graph of $\boldsymbol{\Upsilon}_1$ from above (as visible in Figure 4.1(a)). This is not true for other IVFs, as we show next for the following IVFs.

(b) Let the IVF $\boldsymbol{\Upsilon}_2 : [-10, 10] \to I(\mathbb{R})$ be defined as

$$\boldsymbol{\Upsilon}_2(z) = [\ln(1 + |z|), 0.35z^2 + 0.7z + 5].$$

The gH-Dini Hadamard subdifferential set of $\boldsymbol{\Upsilon}_2$ at 0 is given by

$$\partial^-_{\mathscr{H}} \boldsymbol{\Upsilon}_2(0) = \{[\underline{m}, \overline{m}] \in I(\mathbb{R}) : -1 \leq \underline{m} \leq 0.7 \text{ and } -0.7 \leq \overline{m} \leq 1\}. \tag{4.6}$$

Using (4.6) in Figure 4.1(b), we show two possible gH-Dini Hadamard subgradients $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ of $\boldsymbol{\Upsilon}_2$. These two gH-Dini Hadamard subgradients intersect the graph of $\boldsymbol{\Upsilon}_2$ in the middle. But this is generally not true as shown through the next IVF.

(c) Let the IVF $\boldsymbol{\Upsilon}_3 : [-4, 4] \to I(\mathbb{R})$ be defined as

$$\boldsymbol{\Upsilon}_3(z) = [0.3|z|, e^{0.5|z|}].$$

The gH-Dini Hadamard subdifferential set of $\boldsymbol{\Upsilon}_3$ at 0 is given by

$$\partial^-_{\mathscr{H}} \boldsymbol{\Upsilon}_3(0) = \{[\underline{l}, \overline{l}] \in I(\mathbb{R}) : -0.5 \leq \underline{l} \leq 0.3 \text{ and } -0.3 \leq \overline{l} \leq 0.5\}. \tag{4.7}$$

Using (4.7) in Figure 4.1(c), we show two possible gH-Dini Hadamard subgradients of $\boldsymbol{\Upsilon}_3$ (shown in light-green color), denoted by $\boldsymbol{L}_1$ (shaded in dark-green color) and $\boldsymbol{L}_2$ (shaded in forest-green color). We see that these two gH-Dini Hadamard subgradients intersect the graph of $\boldsymbol{\Upsilon}_3$ from below.

Theorem 4.1 *Consider a convex IVF $\boldsymbol{\Theta} : \mathscr{Z} \to I(\mathbb{R})$. Then, with $\bar{z} \in \mathscr{Z}$, the set $\partial^-_{\mathscr{H}} \boldsymbol{\Theta}(\bar{z})$ is convex.*

Proof: For $\partial^-_{\mathscr{H}} \boldsymbol{\Theta}(\bar{z}) = \emptyset$, it is trivial to show that it is convex. On the other hand, consider $\widehat{\mathbf{M}}$ and $\widehat{\mathbf{N}} \in \partial^-_{\mathscr{H}} \boldsymbol{\Theta}(\bar{z})$, where $\widehat{\mathbf{M}} = (\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_n)^\top$ and $\widehat{\mathbf{N}} = (\mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_n)^\top$.

With respect to (4.2) on gH-Dini Hadamard subdifferential and using Lemma 1.8, we see that for all $(t, u) \in (0, \delta) \times B_\delta(v)$

$$\begin{aligned} & (tu)^\top \odot \widehat{\mathbf{M}} \preceq \mathbf{\Theta}(\bar{z} + tu) \ominus_{gH} \mathbf{\Theta}(\bar{z}) \oplus t \odot [\epsilon, \epsilon] \\ \text{and} \quad & (tu)^\top \odot \widehat{\mathbf{N}} \preceq \mathbf{\Theta}(\bar{z} + tu) \ominus_{gH} \mathbf{\Theta}(\bar{z}) \oplus t \odot [\epsilon, \epsilon] \\ \Longrightarrow & \bigoplus_{j=1}^{n} \mathbf{M}_j \odot tu_j \preceq \mathbf{\Theta}(\bar{z} + tu) \ominus_{gH} \mathbf{\Theta}(\bar{z}) \oplus t \odot [\epsilon, \epsilon] \\ \text{and} \quad & \bigoplus_{j=1}^{n} \mathbf{N}_j \odot tu_j \preceq \mathbf{\Theta}(\bar{z} + tu) \ominus_{gH} \mathbf{\Theta}(\bar{z}) \oplus t \odot [\epsilon, \epsilon]. \end{aligned}$$

Now, for $\beta_1, \beta_2 \geq 0$ where $\beta_1 + \beta_2 = 1$ and for all $(t, u) \in (0, \delta) \times B_\delta(v)$, we get

$$\begin{aligned} & \beta_1 \odot \left(\bigoplus_{j=1}^{n} \mathbf{M}_j \odot tu_j \right) \oplus \beta_2 \odot \left(\bigoplus_{j=1}^{n} \mathbf{N}_j \odot tu_j \right) \preceq \mathbf{\Theta}(\bar{z} + tu) \ominus_{gH} \mathbf{\Theta}(\bar{z}) \oplus t \odot [\epsilon, \epsilon] \\ \Longrightarrow & \bigoplus_{j=1}^{n} (\beta_1 \odot \mathbf{M}_j \oplus \beta_2 \odot \mathbf{N}_j) \odot tu_j \preceq \mathbf{\Theta}(\bar{z} + tu) \ominus_{gH} \mathbf{\Theta}(\bar{z}) \oplus t \odot [\epsilon, \epsilon]. \end{aligned}$$

Hence, for every $\epsilon > 0$ there exists some $\delta > 0$ so that for $\|u - v\| \leq \delta$,

$$(tu)^\top \odot \left(\beta_1 \odot \widehat{\mathbf{M}} \oplus \beta_2 \odot \widehat{\mathbf{N}} \right) \preceq \mathbf{\Theta}(\bar{z} + tu) \ominus_{gH} \mathbf{\Theta}(\bar{z}) \oplus t \odot [\epsilon, \epsilon] \ \ \forall\, t \in (0, \delta),$$

which means that $\left(\beta_1 \odot \widehat{\mathbf{M}} \oplus \beta_2 \odot \widehat{\mathbf{N}} \right) \in \partial^-_{\mathscr{H}} \mathbf{\Theta}(\bar{z})$, and therefore $\partial^-_{\mathscr{H}} \mathbf{\Theta}(\bar{z})$ is convex. □

Theorem 4.2 *If $\mathbf{\Upsilon} : \mathcal{Z} \to I(\mathbb{R})$ is an IVF, then $\partial^-_{\mathscr{H}} \mathbf{\Upsilon}(\bar{z})$ is closed for every $\bar{z} \in \mathcal{Z}$.*

Proof: Take an arbitrary sequence $\{\widehat{\mathbf{K}}_j\}$ in $\partial^-_{\mathscr{H}} \mathbf{\Upsilon}(\bar{z})$ that converges to $\widehat{\mathbf{K}} \in I(\mathbb{R})^m$, where $\widehat{\mathbf{K}}_j = (\mathbf{K}_{j1}, \mathbf{K}_{j2}, \ldots, \mathbf{K}_{jm})^\top$ and $\widehat{\mathbf{K}} = (\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_m)^\top$. Then, for any $\epsilon > 0$ there is a $\delta > 0$ satisfying

$$(tu)^\top \odot \widehat{\mathbf{K}} \preceq \mathbf{\Upsilon}(\bar{z} + tu) \ominus_{gH} \mathbf{\Upsilon}(\bar{z}) \oplus t \odot [\epsilon, \epsilon] \ \forall\, t \in (0, \delta) \text{ whenever } \|u - v\| \leq \delta. \tag{4.8}$$

This again means that

$$\bigoplus_{l=1}^{m} (tu_l) \odot \mathbf{K}_{jl} \preceq \mathbf{\Upsilon}(\bar{z} + tu) \ominus_{gH} \mathbf{\Upsilon}(\bar{z}) \oplus t\epsilon \ \forall\, t \in (0, \delta) \text{ whenever } \|u - v\| \leq \delta. \tag{4.9}$$

With a rearrangement of terms, take the first p coordinates of u to be non-negative, while the remaining $(m - p)$ coordinates are negative. Thus, from (4.9), we get that for all $t \in (0, \delta)$ and $\|u - v\| \leq \delta$

$$\begin{aligned}
&\bigoplus_{l=1}^{p} (tu_l) \odot \mathbf{K}_{jl} \bigoplus_{n=p+1}^{m} (tu_n) \odot \mathbf{K}_{jn} \preceq \boldsymbol{\Upsilon}(\bar{z} + tu) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z}) \oplus t \odot [\epsilon, \epsilon] \\
\implies &\bigoplus_{l=1}^{p} (tu_l) \odot [\underline{k}_{jl}, \overline{k}_{jl}] \bigoplus_{n=p+1}^{m} (tu_n) \odot [\underline{k}_{jn}, \overline{k}_{jn}] \preceq \boldsymbol{\Upsilon}(\bar{z} + tu) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z}) \oplus t \odot [\epsilon, \epsilon].
\end{aligned}$$

Thus,

$$\sum_{l=1}^{p} \underline{k}_{jl} tu_l + \sum_{n=p+1}^{m} \overline{k}_{jn} tu_n \preceq \min \left\{ \underline{\gamma}(\bar{z} + tu) - \underline{\gamma}(\bar{z}) + t\epsilon, \overline{\gamma}(\bar{z} + tu) - \overline{\gamma}(\bar{z}) + t\epsilon \right\} \quad (4.10)$$

and

$$\sum_{l=1}^{p} \overline{k}_{jl} tu_l + \sum_{n=p+1}^{m} \underline{k}_{jn} tu_n \preceq \max \left\{ \underline{\gamma}(\bar{z} + tu) - \underline{\gamma}(\bar{z}) + t\epsilon, \overline{\gamma}(\bar{z} + tu) - \overline{\gamma}(\bar{z}) + t\epsilon \right\}. \quad (4.11)$$

Since $\{\widehat{\mathbf{K}}_j\}$ sequence converges to $\widehat{\mathbf{K}}$, we get $\{\underline{k}_{jl}\}$ and $\{\overline{k}_{jl}\}$ sequences converging to $\underline{k}_l$ and $\overline{k}_l$, respectively, for all l. Hence, by (4.10) and (4.11), we get

$$\begin{aligned}
&\left[\sum_{l=1}^{p} \underline{k}_l tu_l + \sum_{n=p+1}^{m} \overline{k}_n tu_n, \sum_{l=1}^{p} \overline{k}_l tu_l + \sum_{n=p+1}^{m} \underline{k}_n tu_n \right] \preceq \boldsymbol{\Upsilon}(\bar{z} + tu) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z}) \oplus t \odot [\epsilon, \epsilon] \\
\implies &\bigoplus_{l=1}^{p} [\underline{k}_l tu_l, \overline{k}_l tu_l] \bigoplus_{n=p+1}^{m} [\overline{k}_n tu_n, \underline{k}_n tu_n] \preceq \boldsymbol{\Upsilon}(\bar{z} + tu) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z}) \oplus t \odot [\epsilon, \epsilon] \\
\implies &\bigoplus_{l=1}^{p} (tu_l) \odot \mathbf{K}_l \bigoplus_{n=p+1}^{m} (tu_n) \odot \mathbf{K}_n \preceq \boldsymbol{\Upsilon}(\bar{z} + tu) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z}) \oplus t \odot [\epsilon, \epsilon] \\
\implies &(tu)^\top \odot \widehat{\mathbf{K}} \preceq \boldsymbol{\Upsilon}(\bar{z} + tu) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z}) \oplus t \odot [\epsilon, \epsilon] \text{ for each } t \in (0, \delta) \text{ whenever } \|u - v\| \leq \delta.
\end{aligned}$$

Thus, $\widehat{\mathbf{K}} \in \partial^{-}_{\mathscr{H}} \boldsymbol{\Upsilon}(\bar{z})$, and hence $\partial^{-}_{\mathscr{H}} \boldsymbol{\Upsilon}(\bar{z})$ is closed. □

Theorem 4.3 (Smooth variational type description of gH-Dini Hadamard subgradients). *Consider $\boldsymbol{\Upsilon} : \mathcal{Z} \to I(\mathbb{R})$ to be an IVF. Then, $\widehat{\boldsymbol{K}} \in \partial^{-}_{\mathscr{H}} \boldsymbol{\Upsilon}(\bar{z})$ if and only if there exists an IVF $\boldsymbol{\Theta} : \mathcal{Z} \to I(\mathbb{R})$ satisfying*

(a) $\boldsymbol{\Theta}(\bar{z}) = \boldsymbol{\Upsilon}(\bar{z})$, $\boldsymbol{\Theta}(z) \preceq \boldsymbol{\Upsilon}(z)$ for all $z \in \mathcal{Z}$, and

(b) the IVF $\boldsymbol{\Theta}$ is gH-Hadamard differentiable at $\bar{z}$ with gH-Hadamard derivative $\widehat{\boldsymbol{K}}$.

Proof: Let $\widehat{\mathbf{K}} \in \partial^-_{\mathscr{H}} \boldsymbol{\Upsilon}(\bar{z})$ and the IVF $\boldsymbol{\Theta} : \mathcal{Z} \to I(\mathbb{R})$ expressed by

$$\boldsymbol{\Theta}(z) = \inf \left\{ \boldsymbol{\Upsilon}(z), \boldsymbol{\Upsilon}(\bar{z}) \oplus (z - \bar{z})^\top \odot \widehat{\mathbf{K}} \right\}.$$

From the definition of IVF $\boldsymbol{\Theta}$, part (a) is clear. Because of Lemma 1.6, we have the inclusion

$$\begin{aligned} &\boldsymbol{\Theta}(\bar{z} + tu) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z}) \subseteq \inf \left\{ \boldsymbol{\Theta}(\bar{z} + tu) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z}), (tu)^\top \odot \widehat{\mathbf{K}} \right\} \\ \implies &(\boldsymbol{\Theta}(\bar{z} + tu) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z})) \ominus_{gH} (tu)^\top \odot \widehat{\mathbf{K}} \subseteq \inf \left\{ (\boldsymbol{\Upsilon}(\bar{z} + tu) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z})) \ominus_{gH} (tu)^\top \odot \widehat{\mathbf{K}}, \mathbf{0} \right\} \preceq \mathbf{0}. \end{aligned}$$

This results in

$$\limsup_{\substack{t \to 0+ \\ u \to v}} \tfrac{1}{t} \odot \left\{ (\boldsymbol{\Theta}(\bar{z} + tu) \ominus_{gH} \boldsymbol{\Theta}(\bar{z})) \ominus_{gH} (tu)^\top \odot \widehat{\mathbf{K}} \right\} \preceq \mathbf{0} \ \ \forall\ z \in \mathcal{Z} \setminus \{\bar{z}\}. \tag{4.12}$$

Moreover, the gH-Dini Hadamard subdifferentiability of $\boldsymbol{\Upsilon}$ at $\bar{z}$ means that for every $\epsilon > 0$, there is $\delta > 0$ satisfying

$$[-\epsilon, -\epsilon] \preceq \tfrac{1}{t} \odot \inf \left\{ (\boldsymbol{\Upsilon}(\bar{z} + tu) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z})) \ominus_{gH} (tu)^\top \odot \widehat{\mathbf{K}}, \mathbf{0} \right\} \ \ \forall\ t \in (0, \delta) \text{ whenever } \|u - v\| \leq \delta,$$

which results in

$$\mathbf{0} \preceq \liminf_{\substack{t \to 0+ \\ u \to v}} \tfrac{1}{t} \odot \left\{ (\boldsymbol{\Theta}(\bar{z} + tu) \ominus_{gH} \boldsymbol{\Theta}(\bar{z})) \ominus_{gH} (tu)^\top \odot \widehat{\mathbf{K}} \right\}, \tag{4.13}$$

since $\epsilon > 0$ is arbitrary. Hence, from (4.12) and (4.13), we get part (b).

In the converse sense, if (a) and (b) hold, then

$$\begin{aligned} &\boldsymbol{\Theta}(\bar{z} + tu) \preceq \boldsymbol{\Upsilon}(\bar{z} + tu) \\ \implies &\boldsymbol{\Theta}(\bar{z} + tu) \ominus_{gH} \boldsymbol{\Theta}(\bar{z}) \preceq \boldsymbol{\Upsilon}(\bar{z} + tu) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z}) \text{ by Note 2 of [65]} \\ \implies &\liminf_{\substack{t \to 0+ \\ u \to v}} \tfrac{1}{t} \odot \{ (\boldsymbol{\Theta}(\bar{z} + tu) \ominus_{gH} \boldsymbol{\Theta}(\bar{z})) \ominus_{gH} (tu)^\top \odot \widehat{\mathbf{K}} \} \\ &\preceq \liminf_{\substack{t \to 0+ \\ u \to v}} \tfrac{1}{t} \odot \{ (\boldsymbol{\Upsilon}(\bar{z} + tu) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z})) \ominus_{gH} (tu)^\top \odot \widehat{\mathbf{K}} \}. \end{aligned} \tag{4.14}$$

Because $\boldsymbol{\Theta}$ is gH-Hadamard differentiable at $\bar{z}$ with gH-Hadamard derivative $\widehat{\mathbf{K}}$, then for all $v \in \mathbb{R}^n$, we get that

$$\begin{aligned} &\lim_{\substack{t \to 0+ \\ u \to v}} \tfrac{1}{t} \odot \{ \boldsymbol{\Theta}(\bar{z} + tu) \ominus_{gH} \boldsymbol{\Theta}(\bar{z}) \} = v^\top \odot \widehat{\mathbf{K}} \\ \implies &\liminf_{\substack{t \to 0+ \\ u \to v}} \tfrac{1}{t} \odot \{ (\boldsymbol{\Theta}(\bar{z} + tu) \ominus_{gH} \boldsymbol{\Theta}(\bar{z})) \ominus_{gH} (tu)^\top \odot \widehat{\mathbf{K}} \} = \mathbf{0}. \end{aligned}$$

Thus, using the inequality (4.14), $\widehat{\mathbf{K}}$ is gH-Dini Hadamard subgradient of $\mathbf{\Upsilon}$ at $\bar{z}$. $\square$

Example 4.4.2 *In this example, we show geometrical illustration of the findings in Theorem 4.3. Consider an IVF* $\mathbf{\Upsilon} : \mathbb{R}^2 \rightarrow I(\mathbb{R})$ *given by*

$$\mathbf{\Upsilon}(z_1, z_2) = [2, 5] \odot |z_1| \oplus [3, 6] \odot |z_2|.$$

where the upper and lower functions of $\mathbf{\Upsilon}$ *are given by* $\overline{\gamma}(z_1, z_2) = 5|z_1| + 6|z_2|$ *and* $\underline{\gamma}(z_1, z_2) = 2|z_1| + 3|z_2|$*, respectively.*

For $\widehat{\boldsymbol{K}} = ([a, b], [c, d]) \in \partial^{-}_{\mathscr{H}} \mathbf{\Upsilon}(0, 0)$ *and for any* $(\tilde{u}, \tilde{v}) \in \mathbb{R}^2_+$*, we can show that*

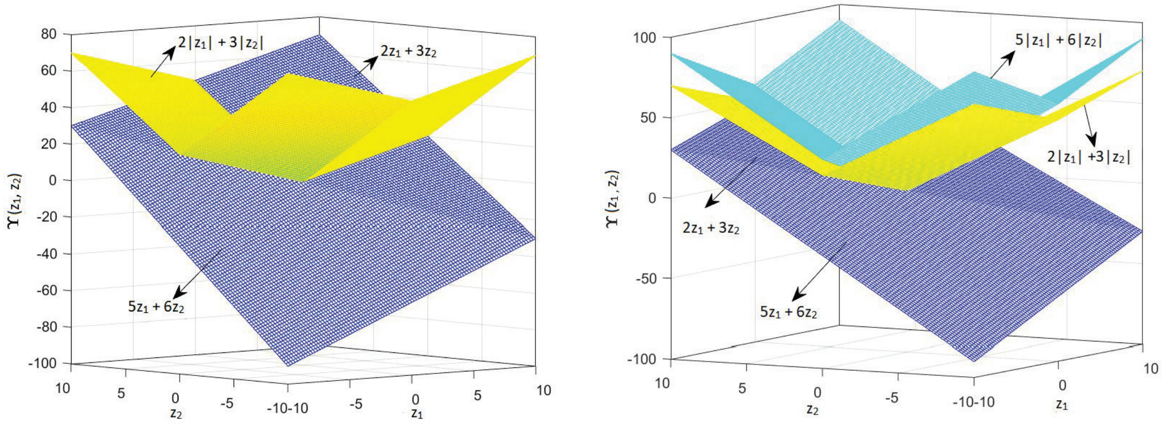
$$(\tilde{u}, \tilde{v}) \odot ([a, b], [c, d]) \preceq \liminf_{\substack{t \to 0+ \\ (u,v) \to (\tilde{u}, \tilde{v})}} \tfrac{1}{t} \odot \{\mathbf{\Upsilon}(ut, vt) \ominus_{gH} \mathbf{\Upsilon}(0, 0)\}$$

$$\Longrightarrow \tilde{u} \odot [a, b] \oplus \tilde{v} \odot [c, d] \preceq [2, 5] \odot |\tilde{u}| \oplus [3, 6] \odot |\tilde{v}|.$$

Hence, $([2, 5], [3, 6]) \in \partial^{-}_{\mathscr{H}} \mathbf{\Upsilon}(0, 0)$*. Here, using Theorem 4.3, we get a linear IVF* $\mathbf{\Theta} : \mathbb{R}^2 \rightarrow I(\mathbb{R})$ *expressed as*

$$\mathbf{\Theta}(z_1, z_2) = \begin{cases} [\underline{\theta}(z_1, z_2), \overline{\theta}(z_1, z_2)], & \textit{if } z_1 + z_2 \geq 0 \\ [\overline{\theta}(z_1, z_2), \underline{\theta}(z_1, z_2)], & \textit{if } z_1 + z_2 < 0, \end{cases}$$

with $\underline{\theta}(z_1, z_2) = 2z_1 + 3z_2$ *and* $\overline{\theta}(z_1, z_2) = 5z_1 + 6z_2$*. It is clear that* $\mathbf{\Theta}(0, 0) = \mathbf{\Upsilon}(0, 0)$



(a) The function $\underline{\gamma}$ and the IVF $\mathbf{\Theta}$ (b) The IVFs $\mathbf{\Upsilon}$ and $\mathbf{\Theta}$

Figure 4.2: Depiction of the IVF $\mathbf{\Upsilon}$ from Example 4.4.2 with its gH-Hadamard smooth support IVF $\mathbf{H}$.

and $\mathbf{\Theta}(z_1, z_2) \preceq \mathbf{\Upsilon}(z_1, z_2)$*, geometrically* $\mathbf{\Theta}$ *supports* $\mathbf{\Upsilon}$ *from bottom. Hence, part (a) of*

Theorem 4.3 is validated. Further, as $\boldsymbol{\Theta}$ is a linear IVF, it is gH-Hadamard differentiable with gH-Hadamard derivative given by $\boldsymbol{\Theta}_{\mathscr{H}}(0,0) = ([2,5],[3,6])$. Thus, part (b) of Theorem 4.3 is also validated.

Further, in figure 4.2(a), we see that the graph of $\underline{\gamma}$ remains either above or coincides with some section of the graph of $\boldsymbol{\Theta}$, i.e., $\boldsymbol{\Theta}(z_1, z_2) \preceq \boldsymbol{1} \odot \underline{\gamma}(z_1, z_2)$. Also evident is the fact that for at least one point $(0,0)$, the graph of $\boldsymbol{\Theta}$ coincides with the graph of lower function $\underline{\gamma}$ of $\boldsymbol{\Upsilon}$, i.e., $\boldsymbol{\Theta}(0,0) = \boldsymbol{1} \odot \underline{\gamma}(0,0)$. This shows that any tangent of the graph $\boldsymbol{\Theta}$ can be classical Dini Hadamard subgradient of $\underline{\gamma}$. Using figure 4.2(b) and following steps of reasoning similar to that in the proof of converse part of Theorem 4.3, the gH-Dini Hadamard derivative of $\boldsymbol{\Theta}$ at $(0,0)$, denoted by $\boldsymbol{\Theta}_{\mathscr{H}}(0,0)$, is the gH-Dini Hadamard subgradient of $\boldsymbol{\Upsilon}$ at $(0,0)$. This validates the converse of Theorem 4.3 as well.

In the following, we present a partial chain rule for gH-Dini Hadamard subdifferential.

Theorem 4.4 (Partial chain rule). *Consider $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ to be a Hadamard differentiable vector-valued function and $\boldsymbol{\Upsilon} : \mathbb{R}^n \to I(\mathbb{R})$ is gH-continuous at $\Theta(\bar{z})$. Then for a point $\bar{z} \in \mathbb{R}^n$ and a direction $v \in \mathbb{R}^n$, we have*

$$\Theta_{\mathscr{H}}(\bar{z}) \odot \partial^{-}_{\mathscr{H}} \boldsymbol{\Upsilon}(\bar{w}) \subseteq \partial^{-}_{\mathscr{H}} (\boldsymbol{\Upsilon} \circ \Theta)(\bar{z}), \text{ where } \bar{w} = \Theta(\bar{z}).$$

Proof: Consider $\widehat{\mathbf{K}} \in \partial^{-}_{\mathscr{H}} \boldsymbol{\Upsilon}(\bar{w})$ and $v \in \mathbb{R}^n$. Select a function W given by

$$W(t,u) = \tfrac{1}{t}\{\Theta(\bar{z} + tu) - \Theta(\bar{z})\},\ u \in \mathbb{R}^n,\ t \geq 0.$$

Observe that $\lim\limits_{\substack{t \to 0+ \\ h \to v}} W(t,u) = \Theta_{\mathscr{H}}(\bar{z})(v)$. Let us take $\Omega = \Theta_{\mathscr{H}}(\bar{z})(v)$. Using definition of $W(t,u)$, we get $\Theta(\bar{z} + tu) = \Theta(\bar{z}) + tW(t,u)$. As per the hypothesis, we have that $\partial^{-}_{\mathscr{H}} \boldsymbol{\Upsilon}(\bar{w})$ exists. Hence, we get

$$(\Theta_{\mathscr{H}}(\bar{z})(v))^{\top} \odot \widehat{\mathbf{K}} \preceq \liminf_{\substack{t \to 0+ \\ W' \to \Omega}} \tfrac{1}{t} \odot \{\boldsymbol{\Upsilon}(\Theta(\bar{z}) + tW') \ominus_{gH} \boldsymbol{\Upsilon}(\Theta(\bar{z}))\}. \tag{4.15}$$

Since function $\Theta_{\mathscr{H}}(\bar{z})$ is linear and $\widehat{\mathbf{K}} : \mathbb{R}^n \to I(\mathbb{R})^n$ is a linear IVF, using (4.15), we get

$$\begin{aligned} v^{\top} \odot \{(\Theta_{\mathscr{H}}(\bar{z})(1)) \odot \widehat{\mathbf{K}}\} &\preceq \liminf_{\substack{t \to 0+ \\ u \to v}} \tfrac{1}{t} \odot \{\boldsymbol{\Upsilon}(\Theta(\bar{z}) + tW(t,u)) \ominus_{gH} \boldsymbol{\Upsilon}(\Theta(\bar{z}))\} \\ &= \liminf_{\substack{t \to 0+ \\ u \to v}} \tfrac{1}{t} \odot \{\boldsymbol{\Upsilon}(\Theta(\bar{z} + tu)) \ominus_{gH} \boldsymbol{\Upsilon}(\Theta(\bar{z}))\} \\ &= \liminf_{\substack{t \to 0+ \\ u \to v}} \tfrac{1}{t} \odot \{(\boldsymbol{\Upsilon} \circ \Theta)(\bar{z} + tu) \ominus_{gH} (\boldsymbol{\Upsilon} \circ \Theta)(\bar{z})\} \end{aligned}$$

$$\implies \quad v^\top \odot \{(\Theta_{\mathscr{H}}(\bar{z}) \odot \widehat{\mathbf{K}})\} \preceq \liminf_{\substack{t\to 0+\\ u\to v}} \tfrac{1}{t} \odot \{(\boldsymbol{\Upsilon} \circ \Theta)(\bar{z}+tu) \ominus_{gH} (\boldsymbol{\Upsilon} \circ \Theta)(\bar{z})\}.$$

Because $\widehat{\mathbf{K}}$ is arbitrarily selected, $\Theta_{\mathscr{H}}(\bar{z}) \odot \partial^-_{\mathscr{H}} \boldsymbol{\Upsilon}(\bar{w}) \subseteq \partial^-_{\mathscr{H}}(\boldsymbol{\Upsilon} \circ \Theta)(\bar{z})$, where $\bar{w} = \Theta(\bar{z})$. □

Theorem 4.5 (Subadditive formula of gH-Dini Hadamard subgradients). *Consider $\boldsymbol{\Upsilon}_1, \boldsymbol{\Upsilon}_2 : \mathcal{Z} \to \overline{I(\mathbb{R})}$ to be two IVFs. Then, for any $\bar{z} \in \mathcal{Z}$, we have*

$$\partial^-_{\mathscr{H}} \boldsymbol{\Upsilon}_1(\bar{z}) \oplus \partial^-_{\mathscr{H}} \boldsymbol{\Upsilon}_2(\bar{z}) \subseteq \partial^-_{\mathscr{H}}(\boldsymbol{\Upsilon}_1 \oplus \boldsymbol{\Upsilon}_2)(\bar{z}).$$

Proof: Take $\widehat{\mathbf{K}} = \widehat{\mathbf{K}}_1 \oplus \widehat{\mathbf{K}}_2 \in \partial^-_{\mathscr{H}} \boldsymbol{\Upsilon}_1(\bar{z}) \oplus \partial^-_{\mathscr{H}} \boldsymbol{\Upsilon}_2(\bar{z})$, where $\widehat{\mathbf{K}}_1 \in \partial^-_{\mathscr{H}} \boldsymbol{\Upsilon}_1(\bar{z})$ and $\widehat{\mathbf{K}}_2 \in \partial^-_{\mathscr{H}} \boldsymbol{\Upsilon}_2(\bar{z})$. Hence, for $t > 0$ and any $v \in \mathbb{R}^n$, we have

$$\begin{aligned}
&\mathbf{0} \preceq \liminf_{\substack{t\to 0+\\ u\to v}} \tfrac{1}{t} \odot \{(\boldsymbol{\Upsilon}_1(\bar{z}+tu) \ominus_{gH} \boldsymbol{\Upsilon}_1(\bar{z})) \ominus_{gH} (tu)^\top \odot \widehat{\mathbf{K}}_1\} \text{ and}\\
&\mathbf{0} \preceq \liminf_{\substack{t\to 0+\\ u\to v}} \tfrac{1}{t} \odot \{(\boldsymbol{\Upsilon}_2(\bar{z}+tu) \ominus_{gH} \boldsymbol{\Upsilon}_2(\bar{z})) \ominus_{gH} (tu)^\top \odot \widehat{\mathbf{K}}_2\}\\
\implies &\mathbf{0} \oplus \mathbf{0} \preceq \liminf_{\substack{t\to 0+\\ u\to v}} \tfrac{1}{t} \odot \{(\boldsymbol{\Upsilon}_1(\bar{z}+tu) \ominus_{gH} \boldsymbol{\Upsilon}_1(\bar{z})) \ominus_{gH} (tu)^\top \odot \widehat{\mathbf{K}}_1\}\\
&\qquad \oplus \liminf_{\substack{t\to 0+\\ u\to v}} \tfrac{1}{t} \odot \{(\boldsymbol{\Upsilon}_2(\bar{z}+tu) \ominus_{gH} \boldsymbol{\Upsilon}_2(\bar{z})) \ominus_{gH} (tu)^\top \odot \widehat{\mathbf{K}}_2\}\\
\implies &\mathbf{0} \preceq \liminf_{\substack{t\to 0+\\ u\to v}} \Big\{ \tfrac{1}{t} \odot \{(\boldsymbol{\Upsilon}_1(\bar{z}+tu) \ominus_{gH} \boldsymbol{\Upsilon}_1(\bar{z})) \ominus_{gH} (tu)^\top \odot \widehat{\mathbf{K}}_1\}\\
&\qquad \oplus \tfrac{1}{t} \odot \{(\boldsymbol{\Upsilon}_2(\bar{z}+tu) \ominus_{gH} \boldsymbol{\Upsilon}_2(\bar{z})) \ominus_{gH} (tu)^\top \odot \widehat{\mathbf{K}}_2\}\Big\}\\
&\text{(by Theorem 3.12 of [123])}\\
\implies &\mathbf{0} \preceq \liminf_{\substack{t\to 0+\\ u\to v}} \tfrac{1}{t} \odot \Big\{\{(\boldsymbol{\Upsilon}_1(\bar{z}+tu) \ominus_{gH} \boldsymbol{\Upsilon}_1(\bar{z})) \ominus_{gH} (tu)^\top \odot \widehat{\mathbf{K}}_1\}\\
&\qquad \oplus \{(\boldsymbol{\Upsilon}_2(\bar{z}+tu) \ominus_{gH} \boldsymbol{\Upsilon}_2(\bar{z})) \ominus_{gH} (tu)^\top \odot \widehat{\mathbf{K}}_2\}\Big\}\\
\implies &\mathbf{0} \preceq \liminf_{\substack{t\to 0+\\ u\to v}} \tfrac{1}{t} \odot \{((\boldsymbol{\Upsilon}_1 \oplus \boldsymbol{\Upsilon}_2)(\bar{z}+tu) \ominus_{gH} (\boldsymbol{\Upsilon}_1 \oplus \boldsymbol{\Upsilon}_2)(\bar{z})) \ominus_{gH} (tu)^\top \odot (\widehat{\mathbf{K}}_1 \oplus \widehat{\mathbf{K}}_2)\}\\
&\text{(by Lemma 1.3).}
\end{aligned}$$

Thus, $\widehat{\mathbf{K}} = \widehat{\mathbf{K}}_1 \oplus \widehat{\mathbf{K}}_2 \in \partial^-_{\mathscr{H}}(\boldsymbol{\Upsilon}_1 \oplus \boldsymbol{\Upsilon}_2)(\bar{z})$. □

The following result demonstrates a monotonic property of the gH-Dini Hadamard subdifferential set.

Theorem 4.6 (Monotonicity property of gH-Dini Hadamard subdifferentials). *Consider $\boldsymbol{\Upsilon}, \boldsymbol{S}$ be two IVFs on $\mathcal{Z}$ so that $\boldsymbol{\Upsilon}(\bar{z}) = \boldsymbol{S}(\bar{z})$ and $\boldsymbol{\Upsilon}(w) \preceq \boldsymbol{S}(w)$ in a neighbourhood of $\bar{z} \in \mathcal{Z}$. Then, $\partial^-_{\mathscr{H}} \boldsymbol{\Upsilon}(\bar{z}) \subseteq \partial^-_{\mathscr{H}} \boldsymbol{S}(\bar{z})$.*

Proof: Take an IVF $\boldsymbol{\Phi} : \mathcal{Z} \to I(\mathbb{R})$. Since we have $\boldsymbol{\Upsilon}(\bar{z}) = \mathbf{S}(\bar{z})$ and $\boldsymbol{\Upsilon}(w) \preceq \mathbf{S}(w)$ in a neighbourhood of $\bar{z} \in \mathcal{Z}$, we can select $\mathbf{S}(w) = \boldsymbol{\Upsilon}(w) \oplus \boldsymbol{\Phi}(w)$, where $\bar{z}$ is a weak efficient point of an IVF $\mathbf{\Phi}$. So, $\mathbf{0} \in \partial^-_{\mathscr{H}}\mathbf{\Phi}(\bar{z})$ as per Theorem 4.9. Thus, $\partial^-_{\mathscr{H}}\mathbf{\Upsilon}(\bar{z}) \subseteq \partial^-_{\mathscr{H}}\mathbf{\Upsilon}(\bar{z}) \oplus \partial^-_{\mathscr{H}}\mathbf{\Phi}(\bar{z}) \subseteq \partial^-_{\mathscr{H}}\mathbf{S}(\bar{z})$ following Theorem 4.5. $\square$

Theorem 4.7 *For an IVF $\mathbf{\Upsilon} : \mathcal{Z} \to I(\mathbb{R})$, if $\mathbf{\Upsilon}$ is gH-Hadamard differentiable at $\varsigma \in \mathcal{Z}$ with gH-Hadamard derivative $\mathbf{\Upsilon}_{\mathscr{H}}(\varsigma)$, then $\mathbf{\Upsilon}$ is also gH-Dini Hadamard subdifferential at $\bar{z}$.*

Proof: Since $\mathbf{\Upsilon}$ is gH-Hadamard differentiable at ς with given gH-Hadamard derivative $\mathbf{\Upsilon}_{\mathscr{H}}(\varsigma)$, we get for $t > 0$, $v \in \mathbb{R}^n$ that

$$\mathbf{\Upsilon}_{\mathscr{H}}(\varsigma)(v) = \lim_{\substack{t\to 0+\\ h\to v}} \tfrac{1}{t} \odot \{\mathbf{\Upsilon}(\varsigma + th) \ominus_{gH} \mathbf{\Upsilon}(\varsigma)\}$$
$$\implies \lim_{\substack{t\to 0+\\ h\to v}} \tfrac{1}{t} \odot \{\mathbf{\Upsilon}(\varsigma + th) \ominus_{gH} \mathbf{\Upsilon}(\varsigma)\} \ominus_{gH} \mathbf{\Upsilon}_{\mathscr{H}}(\varsigma)(v) = \mathbf{0}.$$

Hence, $\mathbf{\Upsilon}$ follows gH-Fréchet differentiability at ς with gH-Fréchet derivative $\mathbf{\Upsilon}_{\mathscr{H}}(\varsigma)$, and $\mathbf{\Upsilon}_{\mathscr{H}}(\varsigma)$ is a linear and gH-continuous IVF. Observe that

$$\lim_{\substack{t\to 0+\\ h\to v}} \tfrac{1}{t} \odot \{(\mathbf{\Upsilon}(\varsigma + th) \ominus_{gH} \mathbf{\Upsilon}(\varsigma)) \ominus_{gH} t \odot \mathbf{\Upsilon}_{\mathscr{H}}(\varsigma)(h)\} = \mathbf{0}$$
$$\implies \liminf_{\substack{t\to 0+\\ h\to v}} \tfrac{1}{t} \odot \{(\mathbf{\Upsilon}(\varsigma + th) \ominus_{gH} \mathbf{\Upsilon}(\varsigma)) \ominus_{gH} (th)^\top \odot \mathbf{\Upsilon}_{\mathscr{H}}(\varsigma)(1)\} = \mathbf{0}.$$

Thus, $\mathbf{\Upsilon}_{\mathscr{H}}(\varsigma)(1) \in \partial^-_{\mathscr{H}}\mathbf{\Upsilon}(\varsigma)$, and therefore $\mathbf{\Upsilon}$ is also gH-Dini Hadamard subdifferential at ς. $\square$

Note 4.1 *The converse of Theorem 4.7 is does not hold. Let the IVF $\mathbf{\Upsilon} : \mathbb{R} \to I(\mathbb{R})$ is defined as*

$$\mathbf{\Upsilon}(w) = \begin{cases} |w| \sin^2 \frac{1}{w} \odot [1, 2], & w \neq 0 \\ 0, & w = 0. \end{cases}$$

To compute the gH-Dini Hadamard subdifferential set of $\mathbf{\Upsilon}$ at 0, take $\boldsymbol{K} = [a, b] \in \partial^-_{\mathscr{H}}\mathbf{\Upsilon}(0)$. Then, for $t > 0$, $v \in \mathbb{R}^n$,

$$\begin{aligned}
\boldsymbol{0} &\preceq \liminf_{\substack{t\to 0+\\ h\to v}} \tfrac{1}{t} \odot \{(\mathbf{\Upsilon}(th) \ominus_{gH} \mathbf{\Upsilon}(0)) \ominus_{gH} (th)^\top \odot \boldsymbol{K}\} \\
&= \liminf_{\substack{t\to 0+\\ h\to v}} \tfrac{1}{t} \odot \{(|th| \sin^2\left(\tfrac{1}{th}\right) \odot [1, 2] \ominus_{gH} \mathbf{\Upsilon}(0)) \ominus_{gH} (th)^\top \odot [a, b]\} \\
&= \liminf_{\substack{t\to 0+\\ h\to v}} \left\{(|h| \sin^2\left(\tfrac{1}{th}\right) \odot [1, 2] \ominus_{gH} \mathbf{\Upsilon}(0)) \ominus_{gH} h^\top \odot [a, b]\right\} \\
&= \liminf_{\substack{t\to 0+\\ h\to v}} \left\{|h| \sin^2\left(\tfrac{1}{th}\right) \odot [1, 2] \ominus_{gH} h^\top \odot [a, b]\right\}
\end{aligned}$$

$$= \liminf_{\substack{t \to 0+ \\ h \to v}} \left\{ |h| \sin^2\left(\tfrac{1}{th}\right) \odot [1,2] \right\} \ominus_{gH} h^\top \odot [a,b] \text{ using ((ii)) of Lemma 1.1}$$

$$\text{or, } \quad \boldsymbol{0} \preceq \boldsymbol{0} \ominus_{gH} v^\top \odot [a,b].$$

Thus, $\boldsymbol{\Upsilon}$ *is* gH*-Dini Hadamard subdifferentiable at 0,* $\partial^-_{\mathscr{H}} \boldsymbol{\Upsilon}(0)(0) = \boldsymbol{0}$ *and* $\boldsymbol{\Upsilon}_{\mathscr{H}}(0)(h) = |h| \sin^2(\frac{1}{th}) \odot [1,2], h \neq 0$. *Since* $\boldsymbol{\Upsilon}_{\mathscr{H}}(0)$ *is not a linear IVF,* $\boldsymbol{\Upsilon}$ *is not* gH*-Hadamard differentiable at* 0.

The theorem below provides a criterion for a class of interval-valued functions to be gH-Dini Hadamard subdifferentiable.

Theorem 4.8 (Criterion for gH-Dini-Hadamard subdifferentiability). *Consider* $\boldsymbol{\Upsilon}$ *to be a* gH*-lower semicontinuous IVF on* $\mathcal{Z}$ *and* $\bar{z} \in \mathcal{Z}$. *Then, the following assertions are equivalent:*

(a) $\widehat{\boldsymbol{K}} \in \partial^-_{\mathscr{H}} \boldsymbol{\Upsilon}(\bar{z})$.

(b) For every $\epsilon > 0$ *and any compact set* $R \subset \mathcal{Z}$ *not containing the origin, there exists* $\delta > 0$ *so that the IVF* $\boldsymbol{\Psi} : R \to I(\mathbb{R})$, *expressed as*

$$\boldsymbol{\Psi}(\varsigma) = (\boldsymbol{\Upsilon}(\bar{z} + \varsigma) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z})) \ominus_{gH} \varsigma^\top \odot \widehat{\boldsymbol{K}} \oplus \epsilon \|\varsigma\|$$

achieves a weak efficient value on R *at zero.*

Proof: *(a)* $\implies$ *(b)*. On contrary, consider a compact set R containing zero so that for any positive number ϵ, ((b)) does not hold. Since R is compact, there is a sequence $\{\varsigma_n\}$ in R converging to $\varsigma \in R$ and $\{\delta_n\}$ in $\mathbb{R}_+$ converging to 0 so that $\boldsymbol{\Psi}(\delta_n \varsigma_n) \prec \boldsymbol{\Psi}(0)$ for all n. This results in

$$\begin{aligned}
& \boldsymbol{\Upsilon}(\bar{z} + \delta_n \varsigma_n) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z}) \prec (\delta_n \varsigma_n)^\top \odot \widehat{\mathbf{K}} \ominus_{gH} \epsilon \|\delta_n \varsigma_n\| \\
\implies & \liminf_{n \to \infty} \tfrac{1}{\delta_n} \odot \{ \boldsymbol{\Upsilon}(\bar{z} + \delta_n \varsigma_n) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z}) \} \prec \liminf_{n \to \infty} \{ \varsigma_n^\top \odot \widehat{\mathbf{K}} \ominus_{gH} \epsilon \|\varsigma_n\| \} \\
\implies & \boldsymbol{\Upsilon}^-_{\mathscr{H}}(\bar{z})(\varsigma) \prec \varsigma^\top \odot \widehat{\mathbf{K}} \ominus_{gH} \epsilon \|\varsigma\| \\
\implies & \varsigma^\top \odot \widehat{\mathbf{K}} \preceq \varsigma^\top \odot \widehat{\mathbf{K}} \ominus_{gH} \epsilon \|\varsigma\|,
\end{aligned}$$

and this can not be possible. Thus, assertion (b) is true.

(b) $\implies$ *(a)*. Consider a sequence $\{h_n\}$ in R and $h_n \to h \neq 0$. Because R is a compact not containing zero, $\{h_n\}$ does not contain zero. Hence, for every $\epsilon > 0$ and any sequence $\{t_n\}$, $t_n > 0 \; \forall \; n$, converging to zero, we get for n, even when very large, that

$$\boldsymbol{\Psi}(0) = \mathbf{0} \preceq \tfrac{1}{t_n} \odot \inf_{h \in R} \{ (\boldsymbol{\Upsilon}(\bar{z} + t_n h_n) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z})) \ominus_{gH} (t_n h_n)^\top \odot \widehat{\mathbf{K}} \}.$$

For $\epsilon > 0$, there exist $\delta > 0$ so that for all $0 < h \leq \delta$,

$$\begin{aligned}
&\mathbf{0} \preceq \tfrac{1}{t_n} \odot \{(\boldsymbol{\Upsilon}(\bar{z} + t_n h_n) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z})) \ominus_{gH} h_n^\top \odot \widehat{\mathbf{K}} \oplus \epsilon \|h_n\|\} \\
\implies &h^\top \odot \widehat{\mathbf{K}} \ominus_{gH} \epsilon\|h\| \preceq \boldsymbol{\Upsilon}^-_{\mathscr{H}}(\bar{z})(h) \\
\implies &h^\top \odot \widehat{\mathbf{K}} \preceq \boldsymbol{\Upsilon}^-_{\mathscr{H}}(\bar{z})(h),
\end{aligned}$$

which results in $\widehat{\mathbf{K}} \in \partial^-_{\mathscr{H}} \boldsymbol{\Upsilon}(\bar{z})$. □

In the following example, we compute the gH-Dini-Hadamard subdifferential set for a composition of IVF and a linear operator.

Example 4.4.3 *Consider* $\boldsymbol{\Upsilon}$ *to be an IVF on* $\mathcal{Z}$*, expressed as*

$$\boldsymbol{\Upsilon}(z) = \boldsymbol{H}(Az) \text{ for all } z \in \mathcal{Z},$$

where $\boldsymbol{H} : \mathbb{R}^m \to I(\mathbb{R})$ *is an IVF and* $A : \mathcal{Z} \to \mathbb{R}^m$ *is a linear bounded operator mapping from* $\mathcal{Z}$ *onto* $\mathbb{R}^m$*. Then, we have*

$$\partial^-_{\mathscr{H}} \boldsymbol{\Upsilon}(z) = \{A^\top \odot \widehat{\boldsymbol{K}}_m : \widehat{\boldsymbol{K}}_m \in \partial^-_{\mathscr{H}} \boldsymbol{H}(Az), \text{ where } \widehat{\boldsymbol{K}}_m \in I(\mathbb{R})^m \text{ and } z \in \mathcal{Z}\}.$$

Proof: Assume $z \in \mathcal{Z}$. Then, for any $h \in \mathcal{Z}$, we get

$$\mathbf{H}^-_{\mathscr{H}}(Az)(Ah) \preceq \liminf_{\substack{u \to h \\ t \to 0+}} \tfrac{1}{t} \odot \{\mathbf{H}(Az + tAu) \ominus_{gH} \mathbf{H}(Az)\} = \boldsymbol{\Upsilon}^-_{\mathscr{H}}(z)(h).$$

Taking $\widehat{\mathbf{K}}_m \in \partial^-_{\mathscr{H}} \mathbf{H}(Az)$, we have

$$h^\top \odot (A^\top \odot \widehat{\mathbf{K}}_m) = (Ah)^\top \odot \widehat{\mathbf{K}}_m \preceq \boldsymbol{\Upsilon}^-_{\mathscr{H}}(z)(h),$$

which means that $A^\top \odot \widehat{\mathbf{K}}_m \in \partial^-_{\mathscr{H}} \boldsymbol{\Upsilon}(z)$.

Now, since A is onto, there exists $L > 0$ so that for each $v \in \mathbb{R}^n$, there is an $h \in \mathcal{Z}$ satisfying $Ah = v$ with $\|h\| \leq L\|v\|$. Here, for a given $v \in \mathbb{R}^m$ choose the sequence $\{(v_n, t_n)\}$ that satisfies

$$\mathbf{H}^-_{\mathscr{H}}(Az)(v) = \lim_{n \to \infty} \tfrac{1}{t_n} \odot \{\mathbf{H}(Az + t_n v_n) \ominus_{gH} \mathbf{H}(Az)\}.$$

Select an $h \in \mathcal{Z}$ where $Ah = v$ and h_n so that $\|h_n - h\| \leq L\|v_n - v\|$. Hence,

$$\boldsymbol{\Upsilon}^-_{\mathscr{H}}(z)(h) \preceq \lim_{n \to \infty} \tfrac{1}{t_n} \odot \{\boldsymbol{\Upsilon}(z + t_n h_n) \ominus_{gH} \boldsymbol{\Upsilon}(z)\} = \mathbf{H}^-_{\mathscr{H}}(Az)(v).$$

Now consider $\widehat{\mathbf{G}}_m \in \partial^-_{\mathscr{H}}\boldsymbol{\Upsilon}(z)$. Then, we have $h^\top \odot \widehat{\mathbf{G}}_m \preceq \mathbf{H}^-_{\mathscr{H}}(Az)(v)$ where $Ah = v$. This results in

$$h^\top \odot \widehat{\mathbf{G}}_m \preceq \mathbf{0} \text{ if } h \in \ \ker A. \tag{4.16}$$

Here, we claim that $\widehat{\mathbf{G}}_m = A^\top \odot \widehat{\mathbf{K}}_m$. Because A is onto, $\mathrm{Im}A = \mathbb{R}^m$. Using rank-nullity theorem, we have the dimension of the null space of A as $n - m$. Thus, $\ker A$ is nonempty. Now take $z \in \ker A$ and we get

$$Az = 0 \implies (Az)^\top \odot \widehat{\mathbf{K}}_m = \mathbf{0} \implies z^\top \odot (A^\top \odot \widehat{\mathbf{K}}_m) = \mathbf{0}.$$

Thus, there is $\widehat{\mathbf{G}}_m = A^\top \odot \widehat{\mathbf{K}}_m \in \partial^-_{\mathscr{H}}\boldsymbol{\Upsilon}(z)$ for some $\widehat{\mathbf{K}}_m \in \partial^-_{\mathscr{H}}\mathbf{H}(Az)$. $\square$

4.5 Optimality conditions by gH-Dini Hadamard subdifferential

In this section, we introduce a Fritz-John-type necessary optimality condition and a Karush-Kuhn-Tucker-type necessary and sufficient optimality condition for IOPs involving IVF as objective that is gH-Dini Hadamard subdifferentiable.

Theorem 4.9 (Fermat-type necessary optimality condition). *Let $\boldsymbol{\Upsilon} : \mathcal{Z} \to \overline{I(\mathbb{R})}$ be an extended IVF and $\bar{z}$ be a weak efficient point of $\min\limits_{w \in \mathcal{Z}} \boldsymbol{\Upsilon}(w)$, then we have $\widehat{\boldsymbol{0}} \in \partial^-_{\mathscr{H}}\boldsymbol{\Upsilon}(\bar{z})$.*

Proof: Consider $\bar{z}$ to be a weak efficient solution. Then, for all $w \in \mathcal{Z}$, we get

$$\begin{aligned}
&\boldsymbol{\Upsilon}(\bar{z}) \preceq \boldsymbol{\Upsilon}(w)\\
\implies &\mathbf{0} \preceq \boldsymbol{\Upsilon}(w) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z})\\
\implies &\mathbf{0} \preceq \boldsymbol{\Upsilon}(\bar{z} + tu) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z}), \text{ for } t > 0, u \in \mathbb{R}^n\\
\implies &\mathbf{0} \preceq \liminf_{\substack{t \to 0+\\ u \to v}} \tfrac{1}{t} \odot \left\{ (\boldsymbol{\Upsilon}(\bar{z} + tu) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z})) \ominus_{gH} (tu)^\top \odot \widehat{\mathbf{0}} \right\}.
\end{aligned}$$

Thus, $\widehat{\mathbf{0}} \in \partial^-_{\mathscr{H}}\boldsymbol{\Upsilon}(\bar{z})$. $\square$

Note 4.2 *The converse of the above result (Theorem 4.9) does not hold. Let $\boldsymbol{\Upsilon} : \mathbb{R} \to I(\mathbb{R})$ by $\boldsymbol{\Upsilon}(w) = w^3 \cos w \odot [1, 2]$. Then, we have*

$$\liminf_{\substack{t \to 0+\\ u \to v}} \tfrac{1}{t} \odot \{(tu)^3 \cos(tu) \odot [1, 2]\} = \liminf_{t \to 0^+} \{t^2 v^3 \cos(tv) \odot [1, 2]\} = \boldsymbol{0}.$$

Thus, $\boldsymbol{0} \in \partial^-_{\mathscr{H}}\boldsymbol{\Upsilon}(0)$, but 0 is not an efficient point of $\boldsymbol{\Upsilon}$ as $\boldsymbol{\Upsilon}(\pi) = [-2\pi^3, -\pi^3] \prec \boldsymbol{\Upsilon}(0) = [0, 0]$.

To establish a Fritz-John-type optimality condition, we rely on several auxiliary concepts and results (Lemma 5, Definitions 11 and 12, Lemmas 6 and 7, and Theorem 11) from [25] which are crucial for the subsequent derivations in this section.

Theorem 4.10 (Fritz-John-type necessary optimality condition). *Consider $\mathscr{Z}$ to be a nonempty open set in $\mathbb{R}^n$ and $\boldsymbol{\Upsilon} : \mathbb{R}^n \to I(\mathbb{R})$ and $\boldsymbol{G}_j : \mathbb{R}^n \to I(\mathbb{R})$ for $j = 1, 2, \ldots, m$ to be IVFs. Take the constrained IOP*

$$\left.\begin{array}{ll} \min & \boldsymbol{\Upsilon}(w) \\ \textit{subject to} & \boldsymbol{G}_j(w) \preceq \boldsymbol{0}, j = 1, 2, \ldots, m \\ & w \in \mathscr{Z}. \end{array}\right\} \tag{4.17}$$

For any feasible $\bar{z} \in \mathscr{Z}$, let us define $J(\bar{z}) = \{j : \boldsymbol{G}_j(\bar{z}) = \boldsymbol{0}\}$. Consider at $\bar{z} \in \mathscr{Z}$, $\partial^+_{\mathscr{H}} \boldsymbol{\Upsilon}(\bar{z})$ and $\partial^-_{\mathscr{H}} \boldsymbol{G}_j(\bar{z})$ to be nonempty for all $j = 1, 2, \ldots m$. If $\bar{z}$ is a weak efficient point of (4.17), then there are constants $u_0, u_1, u_2, \ldots, u_m$ such that for any $d \in \mathbb{R}^n$,

$$\left\{\begin{array}{l} 0 \in u_0 \odot \partial^+_{\mathscr{H}} \boldsymbol{\Upsilon}(\bar{z})(d) \oplus \bigoplus_{j=1}^{m} u_j \odot \partial^-_{\mathscr{H}} \boldsymbol{G}_j(\bar{z})(d), \\ u_j \odot \boldsymbol{G}_j(\bar{z}) = \boldsymbol{0}, j = 1, 2, \ldots, m, \\ u_0 \geq 0, u_j \geq 0, j = 1, 2, \ldots, m, \\ (u_0, u) \neq (0, \boldsymbol{0}^m_v), \end{array}\right.$$

where u is the vector $(u_1, u_2, \ldots, u_m)$.

Proof: Consider $\widehat{\mathbf{K}}_j \in \partial^-_{\mathscr{H}} \mathbf{G}_j(\bar{z}), j = 1, 2, \ldots, m$. Utilizing Theorem 4.3, we can get an IVF $\mathbf{H}_j : \mathscr{Z} \to I(\mathbb{R})$ such that $\mathbf{H}_j$ is gH-Hadamard differentiable at $\bar{z}$ and

$$\mathbf{H}_j(\bar{z}) = \mathbf{G}_j(\bar{z}) \text{ and } \mathbf{H}_{j\mathscr{H}}(\bar{z}) = \widehat{\mathbf{K}}_j \text{ and } \mathbf{H}_j(w) \preceq \mathbf{G}_j(w)\ \forall\ w \in \mathscr{Z} \text{ and } j = 1, 2, \ldots, m. \tag{4.18}$$

From the above relation we can state that

$$\begin{aligned} & \mathbf{H}_{j\mathscr{H}}(\bar{z}) = \widehat{\mathbf{K}}_j \in \partial^-_{\mathscr{H}} \mathbf{G}_j(\bar{z}), j = 1, 2, \ldots, m, \\ & \bar{z} \in \{w \in \mathscr{Z} : \mathbf{H}_j(w) \preceq \mathbf{0}, j = 1, 2, \ldots, m\} \subseteq \mathbb{R}^n. \end{aligned} \tag{4.19}$$

Clearly for $j = 1, 2, \ldots, m$, $\mathbf{H}_j$ are gH-continuous at $\bar{z}$ since $\mathbf{H}_j$ is gH-Hadamard differentiable at $\bar{z}$.

Consider $\widehat{\mathbf{K}} \in \partial^+_{\mathscr{H}} \boldsymbol{\Upsilon}(\bar{z})$. From Remark 4.4.1, we have $-1 \odot \widehat{\mathbf{K}} \in \partial^-_{\mathscr{H}}(-1 \odot \boldsymbol{\Upsilon})(\bar{z})$. Using the variational description on $-1 \odot \boldsymbol{\Upsilon}$ from Theorem 4.3, an IVF $\mathbf{H} : \mathscr{Z} \to I(\mathbb{R})$

can be identified that is gH-Hadamard differentiable at $\bar{z}$ and

$$\mathbf{H}(\bar{z}) = -1 \odot \mathbf{\Upsilon}(\bar{z}) \text{ and } \mathbf{H}_{\mathscr{H}}(\bar{z}) = -1 \odot \widehat{\mathbf{K}} \text{ and } \mathbf{H}(w) \preceq -1 \odot \mathbf{\Upsilon}(w) \text{ for every } w \in \mathcal{Z}. \tag{4.20}$$

Consider an IVF $\mathbf{\Phi} : \mathcal{Z} \to I(\mathbb{R})$ expressed as $\mathbf{\Phi}(w) = -1 \odot \mathbf{H}(w)$. Then, using (4.20), we get $\mathbf{\Phi}(\bar{z}) = -1 \odot \mathbf{H}(\bar{z})$ and $\mathbf{\Phi}_{\mathscr{H}}(\bar{z}) = \widehat{\mathbf{K}} \in \partial^{+}_{\mathscr{H}} \mathbf{\Upsilon}(\bar{z})$. For $\bar{z}$ as a weak efficient point of (4.17), we get that

$$\mathbf{\Phi}(\bar{z}) = \mathbf{\Upsilon}(\bar{z}) \preceq \mathbf{\Upsilon}(w) \preceq \mathbf{\Phi}(w) \text{ for every } w \in \mathcal{Z}. \tag{4.21}$$

From(4.19) and (4.21), it is seen that $\bar{z}$ is weak efficient solution to the IOP

$$\min\{\mathbf{\Phi}(w) \ : \ \mathbf{H}_j(w) \preceq \mathbf{0}, \ j = 1, 2, \ldots, m\}.$$

Because $\mathbf{H}_j$ (for all $j = 1, 2, \ldots, m$) and $\mathbf{\Phi}$ are gH-Hadamard differentiable at $\bar{z}$ (from latter half of Theorem 12 in [25]), there are constants $u_0, u_1, u_2, \ldots, u_m$ so that for any $d \in \mathbb{R}^n$,

$$0 \in \left(u_0 \odot \mathbf{\Phi}_{\mathscr{H}}(\bar{z})(d) \oplus \bigoplus_{j=1}^{m} u_j \odot \mathbf{H}_{j\mathscr{H}}(\bar{z})(d) \right). \tag{4.22}$$

For $j \in J(\bar{z})$, we have $\mathbf{H}_j(\bar{z}) = \mathbf{0}$. Thus, $u_j \odot \mathbf{H}_j(\bar{z}) = \mathbf{0}$. For $\mathbf{H}_j(\bar{z}) \prec \mathbf{0}$ for all $j \notin J(\bar{z})$, setting $u_j = 0$ for $j \notin J(\bar{z})$ and using (4.18), (4.20), and (4.22), we get the result. $\square$

Definition 4.3 *Consider $\beta_1, \beta_2, \ldots, \beta_m$ to be m real numbers satisfying*

$$0 \in \beta_1 \odot \boldsymbol{Y}_1 \oplus \beta_2 \odot \boldsymbol{Y}_2 \oplus \ldots \beta_m \odot \boldsymbol{Y}_m \text{ if and only if } \beta_1 = 0, \beta_2 = 0, \ldots, \beta_m = 0.$$

Then, the set of m intervals $\{\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_m\}$ are called linearly independent.

Theorem 4.11 (Karush-Kuhn-Tucker-type necessary optimality criterion). *Consider $\mathcal{Z}$ to be a nonempty open set in $\mathbb{R}^n$ and $\mathbf{\Upsilon} : \mathbb{R}^n \to I(\mathbb{R})$ and $\boldsymbol{G}_j : \mathbb{R}^n \to I(\mathbb{R}), j = 1, 2, \ldots, m$, to be IVFs. Let $\bar{z}$ be a feasible point of the constrained IOP:*

$$\begin{aligned} &\min && \mathbf{\Upsilon}(w) \\ &\textit{subject to} && \boldsymbol{G}_j(w) \preceq \boldsymbol{0}, j = 1, 2, \ldots, m \\ & && w \in \mathcal{Z}. \end{aligned} \tag{4.23}$$

Take $J(\bar{z}) = \{j : \boldsymbol{G}_j(\bar{z}) = \boldsymbol{0}\}$. Consider *(i)* $\partial^+_{\mathscr{H}} \boldsymbol{\Upsilon}(\bar{z})$ *and* $\partial^-_{\mathscr{H}} \boldsymbol{G}_j(\bar{z})$ *to be nonempty for* $j = 1, 2, \ldots, m,$ *and*

(ii) the set of intervals $\{\partial^-_{\mathscr{H}} \boldsymbol{G}_j(\bar{z})(d) : j \in 1, 2, \ldots, m, d \in \mathbb{R}^n\}$ *to be linearly independent.*

For $\bar{z}$ *to be a weak efficient solution, there are constants* $u_1, u_2, \ldots, u_m$ *such that for any* $d \in \mathbb{R}^n$*,*

$$\begin{cases} 0 \in \left(u_0 \odot \partial^+_{\mathscr{H}} \boldsymbol{\Upsilon}(\bar{z})(d) \oplus \bigoplus_{j=1}^{m} u_j \odot \partial^-_{\mathscr{H}} \boldsymbol{G}_j(\bar{z})(d) \right), \\ u_j \odot \boldsymbol{G}_j(\bar{z}) = \boldsymbol{0}, j = 1, 2, \ldots, m, \\ u_j \geq 0, j = 1, 2, \ldots, m. \end{cases}$$

Proof: Using Theorem 4.10 and from (i) and (ii) we easily get (4.22). Now using (4.19), we get $\mathbf{H}_{j\mathscr{H}}(\bar{z}) \in \partial^-_{\mathscr{H}} \mathbf{G}_j(\bar{z}), j = 1, 2, \ldots, m$. This means that

$$\{\mathbf{H}_{j\mathscr{H}}(\bar{z})(d) : j = 1, 2, \ldots, m, d \in \mathbb{R}^n\} \subseteq \{\partial^-_{\mathscr{H}} \mathbf{G}_j(\bar{z})(d) : j = 1, 2, \ldots, m, d \in \mathbb{R}^n\}.$$

From the fact that any nonempty subset of linearly independent set of intervals is linearly independent, using (ii) we get that $\{\mathbf{H}_{j\mathscr{H}}(\bar{z})(d) : j = 1, 2, \ldots, m, d \in \mathbb{R}^n\}$ is linearly independent. Because $\mathbf{H}_j$ (for all $j = 1, 2, \ldots, m$) and $\boldsymbol{\Phi}$ are gH-Hadamard differentiable at $\bar{z}$, from Theorem 13 in [25] there are constants $u_0 > 0$ and $u'_j \geq 0$ for each $j = 1, 2, \ldots, m$, satisfying

$$0 \in \left(u_0 \odot \boldsymbol{\Phi}_{\mathscr{H}}(\bar{z})(d) \oplus \bigoplus_{j=1}^{m} u'_j \odot \mathbf{H}_{j\mathscr{H}}(\bar{z})(d) \right). \tag{4.24}$$

Consider $u_j = u'_j / u_0, j = 1, 2, \ldots, m$. Then, we have $u_j \geq 0$ for all $j = 1, 2, \ldots, m$ and

$$0 \in \left(u_0 \odot \partial^+_{\mathscr{H}} \boldsymbol{\Upsilon}(\bar{z})(d) \oplus \bigoplus_{j=1}^{m} u_j \odot \partial^-_{\mathscr{H}} \mathbf{G}_j(\bar{z})(d) \right).$$

With $j \in J(\bar{z})$, $\mathbf{G}_j(\bar{z}) = \mathbf{0}$. Thus, $u_j \odot \mathbf{G}_j(\bar{z}) = \mathbf{0}$. If $j \notin J(\bar{z})$, $\mathbf{G}_j(\bar{z}) \prec \mathbf{0}$. Thus, setting $u_j = 0$ for $j \notin J(\bar{z})$, we prove the theorem. □

Theorem 4.12 (Karush-Kuhn-Tucker-type sufficient optimality condition). *Consider* $\mathcal{Z}$ *to be a convex subset of* $\mathbb{R}^n$ *and* $\boldsymbol{\Upsilon} : \mathcal{Z} \to I(\mathbb{R})$ *and* $\boldsymbol{G}_j : \mathcal{Z} \to I(\mathbb{R}), j = 1, 2, \ldots, m$ *to be gH-Dini-Hadamard subdifferentiable IVFs. Let* $\bar{z} \in \mathcal{Z}$ *be a feasible point of the*

IOP

$$\left.\begin{array}{ll} \min & \boldsymbol{\Upsilon}(w) \\ \textit{subject to} & \boldsymbol{G}_j(w) \preceq \boldsymbol{0}, j = 1, 2, \ldots, m \\ & w \in \mathcal{Z}. \end{array}\right\} \tag{4.25}$$

If there are real constants $u_1, u_2, \ldots, u_m$ such that

$$\begin{cases} \partial^-_{\mathscr{H}} \boldsymbol{\Upsilon}(\bar{z})(v) \oplus \bigoplus_{j=1}^{m} u_j \odot \partial^-_{\mathscr{H}} \boldsymbol{G}_j(\bar{z})(v) \nprec \boldsymbol{0} \textit{ for all } v \in \mathbb{R}^n, \\ u_j \odot \boldsymbol{G}_j(\bar{z}) = \boldsymbol{0}, j = 1, 2, \ldots, m, \\ u_j \geq 0, j = 1, 2, \ldots, m, \end{cases}$$

then $\bar{z}$ is an efficient point of the IOP (4.25).

Proof: Because $\mathcal{Z}$ is a convex subset of $\mathbb{R}^n$, an element w in $\mathcal{Z}$ of the form $\bar{z} + tv \in \mathcal{Z}$ for $0 \leq t < 1$, where $v = w - \bar{z}$. From the hypothesis, for every $v \in \mathbb{R}^n$ and fix $\bar{z}$ such that $\mathbf{G}_j(\bar{z} + tv) \preceq \mathbf{0}$ for all $j = 1, 2, \ldots, m$, we have

$$\partial^-_{\mathscr{H}} \boldsymbol{\Upsilon}(\bar{z})(v) \oplus \bigoplus_{j=1}^{m} u_j \odot \partial^-_{\mathscr{H}} \mathbf{G}_j(\bar{z})(v) \nprec \mathbf{0}$$

With $t > 0$, we get

$$\begin{aligned} & \partial^-_{\mathscr{H}} \boldsymbol{\Upsilon}(\bar{z})(tv) \oplus \bigoplus_{j=1}^{m} u_j \odot \partial^-_{\mathscr{H}} \mathbf{G}_j(\bar{z})(tv) \nprec \mathbf{0} \\ \Longrightarrow\ & (tv)^\top \odot \partial^-_{\mathscr{H}} \boldsymbol{\Upsilon}(\bar{z}) \oplus \bigoplus_{j=1}^{m} u_j \odot (tv)^\top \odot \partial^-_{\mathscr{H}} \mathbf{G}_j(\bar{z}) \nprec \mathbf{0}. \end{aligned}$$

For every ϵ and $\epsilon_j > 0$ $(j = 1, 2, \ldots, m)$ there is $\delta > 0$ such that for all $t > 0$,

$$((\boldsymbol{\Upsilon}(\bar{z} + tv) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z})) \oplus [\epsilon, \epsilon]) \ \oplus \ \bigoplus_{j=1}^{m} u_j \odot ((\mathbf{G}_j(\bar{z} + tv) \ominus_{gH} \mathbf{G}_j(\bar{z})) \oplus [\epsilon_j, \epsilon_j]) \nprec \mathbf{0}$$

(using (4.8) of Theorem 4.2 and (i) of Lemma 1 in [25]).

Taking ϵ and ϵ_j to tend to zero, for each $j = 1, 2, \ldots, m$, we have

$$\begin{aligned} & (\boldsymbol{\Upsilon}(\bar{z} + tv) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z})) \oplus \bigoplus_{j=1}^{m} u_j \odot \mathbf{G}_j(\bar{z} + tv)) \nprec \mathbf{0} \\ \Longrightarrow\ & (\boldsymbol{\Upsilon}(\bar{z} + tv) \ominus_{gH} \boldsymbol{\Upsilon}(\bar{z})) \nprec \mathbf{0}, \text{ provided that } u_j \odot \mathbf{G}_j(\bar{z} + tv) \preceq \mathbf{0}, j = 1, 2, \ldots, m \end{aligned}$$

(by (ii) Lemma 1 in [25])

$\Longrightarrow \boldsymbol{\Upsilon}(\bar{z}+tv) \nprec \boldsymbol{\Upsilon}(\bar{z})$.

Therefore, because v is arbitrary, we have$\bar{z}$ as an efficient point of the IOP. $\square$

4.6 An application of gH-Dini Hadamard subdifferential

In this section, we aim to apply the concept of the gH-Dini Hadamard subdifferential to the analysis of control system. To adapt the study to the interval-valued framework, we consider the state variables in the control system to be interval-valued. Consequently, we first establish the concept of the gH-Dini Hadamard subdifferential for IVFs with interval variables. Additionally, we introduce a few concepts necessary for the analysis, such as compact sets of interval vectors, gH-absolutely continuous IVFs, and gH-upper semicontinuous IVFs of interval variables.

Definition 4.4 (gH-Dini Hadamard subdifferential for IVFs of interval variables). *Consider $\emptyset \neq \boldsymbol{\mathcal{Z}} \subseteq I(\mathbb{R})^n$. Also, consider $\boldsymbol{H} \in I(\mathbb{R})^n$ and $\boldsymbol{Z}_0 \in \boldsymbol{\mathcal{Z}}$. For an IVF $\boldsymbol{V}: \boldsymbol{\mathcal{Z}} \to I(\mathbb{R})$, we write*

$$\boldsymbol{V}^-_{\mathscr{H}}(\boldsymbol{Z}_0)(\boldsymbol{H}) = \liminf_{\substack{\tilde{\boldsymbol{H}} \to \boldsymbol{H} \\ \beta \to 0+}} \tfrac{1}{t} \odot \{\, \boldsymbol{V}(\boldsymbol{Z}_0 \oplus \beta \odot \tilde{\boldsymbol{H}}) \ominus_{gH} \boldsymbol{V}(\boldsymbol{Z}_0)\}, \text{ given that the limit exists.}$$

The gH-Dini Hadamard subdifferential of $\boldsymbol{V}$ at $\boldsymbol{Z}_0$ is the set

$$\partial^-_{\mathscr{H}} \boldsymbol{V}(\boldsymbol{Z}_0) = \{\widehat{\boldsymbol{K}} \in I(\mathbb{R})^n : \widehat{\boldsymbol{K}} \odot \boldsymbol{H} \preceq \boldsymbol{V}^-_{\mathscr{H}}(\boldsymbol{Z}_0)(\boldsymbol{H}) \ \forall\ \boldsymbol{H} \in I(\mathbb{R})^n\}.$$

Definition 4.5 (Compact set in $I(\mathbb{R})^n$). *A set $\boldsymbol{\mathcal{Z}} \subseteq I(\mathbb{R})^n$ is referred to as compact if any sequence $\{\widehat{\boldsymbol{Z}}_n\}_{n\in\mathbb{N}}$ in $\boldsymbol{\mathcal{Z}}$ includes a convergent subsequence $\{\widehat{\boldsymbol{Z}}_{n_i}\}_{i\in\mathbb{N}}$ with limit in $\boldsymbol{\mathcal{Z}}$.*

Lemma 4.1 *Consider $\boldsymbol{\mathcal{Z}} \neq \emptyset$, to be a compact subset of $I(\mathbb{R})^n$. If $\boldsymbol{\Upsilon} : \boldsymbol{\mathcal{Z}} \to I(\mathbb{R})$ is gH-continuous, $\boldsymbol{\Upsilon}(\boldsymbol{\mathcal{Z}})$ is a compact set in $I(\mathbb{R})$.*

Proof: We can prove this in the same manner as in the case of real-valued functions of real variables (given in p.89 of [159]). $\square$

Definition 4.6 (gH-absolutely continuous IVF). *An IVF $\boldsymbol{\Upsilon} : [a,b] \to I(\mathbb{R})$ is called gH-absolutely continuous on $[a,b]$ if for any given $\epsilon > 0$ there exists a $\delta > 0$ satisfying*

$$\sum_{\iota=1}^{m} \|\boldsymbol{\Upsilon}(d_\iota) \ominus_{gH} \boldsymbol{\Upsilon}(c_\iota)\| \leq \epsilon$$

for any finite collection of non-overlapping subintervals $\{[c_\iota, d_\iota] : 1 \leq \iota \leq m\}$ *of* $[a, b]$ *with* $\sum_{\iota=1}^{m}(d_\iota - c_\iota) < \delta$.

Definition 4.7 (gH-upper semicontinuity of extended IVFs of interval variables). *Consider* $\emptyset \neq \boldsymbol{\mathscr{Z}} \subseteq I(\mathbb{R})^n$. *We write the upper limit of an extended IVF* $\boldsymbol{\Upsilon}$ *at* $\widehat{\widehat{\boldsymbol{Z}}} \in \boldsymbol{\mathscr{Z}}$ *as* $\limsup\limits_{\widehat{\boldsymbol{Z}} \to \widehat{\widehat{\boldsymbol{Z}}}} \boldsymbol{\Upsilon}(\widehat{\boldsymbol{Z}})$. $\boldsymbol{\Upsilon}$ *is said to be* gH*-upper semicontinuous at* $\widehat{\widehat{\boldsymbol{Z}}} \in \boldsymbol{\mathscr{Z}} \subset I(\mathbb{R})^n$ *if*

$$\limsup_{\widehat{\boldsymbol{Z}} \to \widehat{\widehat{\boldsymbol{Z}}}} \boldsymbol{\Upsilon}(\widehat{\boldsymbol{Z}}) \preceq \boldsymbol{\Upsilon}(\widehat{\widehat{\boldsymbol{Z}}}).$$

Lemma 4.2 *Consider* $\boldsymbol{\mathscr{Z}}$ *to be a compact subset of* $I(\mathbb{R})^n$ *and* $\boldsymbol{\Upsilon} : \boldsymbol{\mathscr{Z}} \to I(\mathbb{R})$ *to be* gH*-upper semicontinuous. Then,* $\boldsymbol{\Upsilon}$ *achieves supremum on* $\boldsymbol{\mathscr{Z}}$.

Proof: We can show this in a manner similar to the case of real-valued functions of real variables (given in [156]). $\square$

For the study on control system, let a non-smooth interval control system be given by

$$\dot{\mathbf{Z}} = \boldsymbol{\Upsilon}(\mathbf{Z}(t), \boldsymbol{\zeta}(t)) \text{ almost everywhere for } 0 \leq t \leq T, \tag{4.26}$$

where the state $\mathbf{Z}(t)$ in $I(\mathbb{R})^n$ is gH-absolutely continuous and the control $\boldsymbol{\zeta} : [0, \infty) \to U$ is considered to be a subset of $\mathbb{R}^m$. For the IVF $\boldsymbol{\Upsilon} : I(\mathbb{R})^n \times U \to I(\mathbb{R})^n$, let $\boldsymbol{\Upsilon}$ be gH-continuous and measurable and assume that it admits an equilibrium point at the origin, i.e., $\boldsymbol{\Upsilon}(\mathbf{0}, 0) = \mathbf{0}$. For the state variable, we consider that $\mathbf{Z}$ is a gH-absolutely continuous IVF on $[a, b]$ and $\dot{\mathbf{Z}}(t)$ exists almost everywhere in $[a, b]$. Further, for an initial condition $\mathbf{Z}(0) = \mathbf{Z}_0$ and the control $\boldsymbol{\zeta}$, we write any solution of system (4.26) at time t as $\mathbf{Z}(t; \mathbf{Z}_0, \boldsymbol{\zeta})$.

For the remainder of this paper, we use the integration of interval-valued functions following Stefanini et al. [167]. To establish the main result (Theorem 4.13 on the asymptotic controllability of (4.26)), we need the following lemmas.

Lemma 4.3 *Consider* $\boldsymbol{V}$ *to be a* gH*-continuous IVF from* $[0, T]$ *to* $I(\mathbb{R})$*, and* $\boldsymbol{\Psi}$ *to be an* gH*-upper semicontinuous IVF from* $[0, T]$ *to* $I(\mathbb{R})$*, which is bounded from above. Let*

$$\boldsymbol{V}^{-}_{\mathscr{H}}(\boldsymbol{Z}(t))(\boldsymbol{H}) \oplus \boldsymbol{\Psi}(\boldsymbol{Z}(t)) \preceq \boldsymbol{0} \;\;\forall\;\; t \in [0, T). \tag{4.27}$$

Then,

$$\boldsymbol{V}(\boldsymbol{Z}(s)) \ominus_{gH} \boldsymbol{V}(\boldsymbol{Z}(r)) \oplus \int_r^s \boldsymbol{\Psi}(\boldsymbol{Z}(\tau))d\tau \preceq \boldsymbol{0} \textit{ for any } r, s \in (0, \tau) \textit{ with } r < s.$$

Proof: Possibly, let $\epsilon > 0$, such that

$$\epsilon(s-r) \preceq \mathbf{V}(\mathbf{Z}(s)) \ominus_{gH} \mathbf{V}(\mathbf{Z}(r)) \oplus \int_r^s \mathbf{\Psi}(\mathbf{Z}(\tau))d\tau.$$

Generate an IVF $\mathbf{G}$ as

$$\mathbf{G}(\mathbf{Z}(t)) = \mathbf{V}(\mathbf{Z}(t)) \ominus_{gH} \mathbf{V}(\mathbf{Z}(r)) \oplus \int_r^t \mathbf{\Psi}(\mathbf{Z}(\tau))d\tau \ominus_{gH} \epsilon(t-r).$$

Then,

$$\begin{aligned} \mathbf{G}^-_{\mathscr{H}}(\mathbf{Z}(r))(\mathbf{H}) &= \liminf_{\substack{\tilde{\mathbf{H}}\to\mathbf{H}\\ t\to r+}} \Big\{ \tfrac{1}{t-r} \odot \{\mathbf{V}(\mathbf{Z}(r) \oplus t \odot \tilde{\mathbf{H}}) \ominus_{gH} \mathbf{V}(\mathbf{Z}(r))\} \\ &\qquad \oplus \tfrac{1}{t-r} \odot \int_r^t \mathbf{\Psi}(\mathbf{Z}(\tau))d\tau \ominus_{gH} \epsilon \Big\}. \end{aligned}$$

Let an $\eta > 0$. Because $\mathbf{\Psi}$ is gH-upper semicontinuous on $[0, T]$, there is $\delta > 0$ such that $(\tau - r) \in (0, \delta)$ means that $\mathbf{\Psi}(\mathbf{Z}(\tau)) \preceq \mathbf{\Psi}(\mathbf{Z}(r)) \oplus \eta$. Thus,

$$\begin{aligned} \mathbf{G}^-_{\mathscr{H}}(\mathbf{Z}(r))(\mathbf{H}) &\preceq \liminf_{\substack{\tilde{\mathbf{H}}\to\mathbf{H}\\ t\to r+}} \Big\{ \tfrac{1}{t-r} \odot \{\mathbf{V}(\mathbf{Z}(r) \oplus t \odot \tilde{\mathbf{H}}) \ominus_{gH} \mathbf{V}(\mathbf{Z}(r))\} \oplus \mathbf{\Psi}(\mathbf{Z}(r)) \oplus \eta \ominus_{gH} \epsilon \Big\} \\ &= \mathbf{V}^-_{\mathscr{H}}(\mathbf{Z}(r))(\mathbf{H}) \oplus \mathbf{\Psi}(\mathbf{Z}(r)) \ominus_{gH} \epsilon \oplus \eta. \end{aligned}$$

Because η is arbitrary, from (4.27), we have

$$\mathbf{G}^-_{\mathscr{H}}(\mathbf{Z}(r))(\mathbf{H}) = \mathbf{V}^-_{\mathscr{H}}(\mathbf{Z}(r))(\mathbf{H}) \oplus \mathbf{\Psi}(\mathbf{Z}(r)) \ominus_{gH} \epsilon \preceq \mathbf{0}. \tag{4.28}$$

As $\mathbf{G}(\mathbf{Z}(r)) = \mathbf{0}$ for some $t^0 > r$, we get using (4.28) that $\mathbf{G}(\mathbf{Z}(t^0)) \prec \mathbf{0}$. Let $t^1 = \sup\{t < T : \mathbf{G}(\mathbf{Z}(t)) = \mathbf{0}\}$. Since the IVF $\mathbf{G}$ is gH-continuous and $\mathbf{0} \prec \mathbf{G}(\mathbf{Z}(s))$, we have that $0 < t < t^1 < s$. Because at the right of $t^1, \mathbf{0} \preceq \mathbf{G}(\mathbf{Z}(t))$, we have

$$\mathbf{0} \preceq \mathbf{G}^-_{\mathscr{H}}(\mathbf{Z}(t^1))(\mathbf{H}),$$

which is in contradiction with (4.28). Therefore, we have the desired result. $\square$

Lemma 4.4 *Consider a subset $\boldsymbol{\mathcal{M}} \neq \emptyset$ of $I(\mathbb{R})$ to be compact. Let $\boldsymbol{Z}(t) \in \boldsymbol{\mathcal{M}}$ for $p \leq t \leq q$, then*

$$\boldsymbol{Z}_{mean} = \tfrac{1}{q-p} \int_p^q \boldsymbol{Z}(t)dt \in co(\boldsymbol{\mathcal{M}}),$$

where $co(\boldsymbol{\mathcal{M}})$ is the convex hull of the set $\boldsymbol{\mathcal{M}}$.

Proof: We can prove this in a manner similar to that of real-valued functions of real variables (as given in p. 63 of [53]). ☐

Lemma 4.5 *Consider* $\boldsymbol{V}$ *to be a* gH*-continuous function from* $[0, T]$ *to* $I(\mathbb{R})$ *such that*

$$\boldsymbol{V}^{-}_{\mathscr{H}}(\boldsymbol{Z}_0)(\boldsymbol{H}) \preceq \boldsymbol{0}\ \forall\ t \in [0, T).$$

Then, for all $0 < t < s < T$, $\boldsymbol{V}$ *is a decreasing IVF with respect to* t.

Proof: Consider that for some $\epsilon > 0$, $\epsilon t \preceq \mathbf{V}(\mathbf{Z}(t)) \ominus_{gH} \mathbf{V}(\mathbf{Z}(0))$. Let $\mathbf{Z}(t) = \mathbf{Z}_0 \oplus t \odot \tilde{\mathbf{H}}$. Take $\mathbf{G}$ as $\mathbf{G}(\mathbf{Z}(t)) = \ (\mathbf{V}(\mathbf{Z}(t)) \ominus_{gH} \mathbf{V}(\mathbf{Z}(0)) \ominus_{gH} \epsilon t$. Then,

$$\begin{aligned}\mathbf{G}^{-}_{\mathscr{H}}(\mathbf{Z}_0)(\mathbf{H}) &= \liminf_{\substack{\tilde{\mathbf{H}} \to \mathbf{H} \\ t \to 0+}} \left\{ \tfrac{1}{t} \odot \{\mathbf{V}(\mathbf{Z}(t)) \ominus_{gH} \mathbf{V}(\mathbf{Z}(0))\} \ominus_{gH} \epsilon t \right\} \\ &= \mathbf{V}^{-}_{\mathscr{H}}(\mathbf{Z}_0)(\mathbf{H}) \ominus_{gH} \epsilon \preceq \mathbf{0}. \end{aligned} \tag{4.29}$$

Specifically, the above is true at 0. Thus, because $\mathbf{G}(\mathbf{Z}(0)) = \mathbf{0}$ for some $t^0 > 0$, we get $\mathbf{G}(\mathbf{Z}(t^0)) \prec \mathbf{0}$. Let $t^1 = \sup\{t < \infty : \mathbf{G}(\mathbf{Z}(t)) = \mathbf{0}\}$. Because at the right of t^1, $\mathbf{0} \prec \mathbf{G}(\mathbf{Z}(s))$, we have

$$\mathbf{0} \prec \mathbf{G}^{-}_{\mathscr{H}}(\mathbf{Z}_0)(\mathbf{H}),$$

which is in contradiction with (4.29). As a result, we have the proof. ☐

In the following, we examine the notion of asymptotic controllability and the control-Lyapunov pair for the nonsmooth dynamical control system (4.26). Subsequently, we study an equivalence between asymptotic controllability and the existence of a control-Lyapunov pair in terms of the proposed gH-Dini Hadamard subdifferential.

Definition 4.8 (Asymptotically controllable). *The system (4.26) is said to be asymptotically controllable if there is nondecreasing functions* $\theta, \tilde{\theta} : \mathbb{R}_+ \to \mathbb{R}_+$ *so that* $\lim_{r \to 0+} \tilde{\theta}(r) = 0$, *with any one of the three properties below:*

(i) For each $\boldsymbol{Z}_0 \in I(\mathbb{R})$, *there exists a control* $\boldsymbol{\zeta} : \mathbb{R}_+ \to U$ *and corresponding trajectory* $\boldsymbol{Z} : \mathbb{R}_+ \to I(\mathbb{R})$ *starting from the initial condition* $\boldsymbol{Z}(0) = \boldsymbol{Z}_0$ *with* $\boldsymbol{Z}(t) \to \boldsymbol{0}$ *as* $t \to \infty$, $\|\boldsymbol{\zeta}\| \leq \theta(\|\boldsymbol{Z}_0\|)$, *and* $\sup\{\|\boldsymbol{Z}(t)\| : 0 \leq t \leq \infty\} \leq \tilde{\theta}(\|\boldsymbol{Z}_0\|)$.

(ii) For each $\boldsymbol{Z}_0 \in \mathbb{R}^n$, *there exists an ordinary control* $\boldsymbol{\zeta}$ *such that the trajectory for each* $\boldsymbol{Z}(t) = \boldsymbol{Z}(t; \boldsymbol{Z}_0, \boldsymbol{\zeta})$ *is defined for any* $t \geq 0$ *and*

$$\lim_{t \to \infty} \boldsymbol{Z}(t) = \boldsymbol{0}. \tag{4.30}$$

(iii) For each $\epsilon > 0$, there is a $\delta > 0$ such that for any state with $\|\boldsymbol{Z}_0\| < \delta$ there is a control $\boldsymbol{\zeta}$ as in ((ii)) such that $\|\boldsymbol{Z}(t)\| < \epsilon$ for all $t \geq 0$.

Definition 4.9 (Control-Lyapunov pair). *A control-Lyapunov pair for the system (4.26) is defined as a pair $(\boldsymbol{V}, \boldsymbol{\Psi})$, where*

(i) $\boldsymbol{V} : I(\mathbb{R})^n \to I(\mathbb{R})$ is a gH-continuous, positive definite, proper IVF, and

(ii) $\boldsymbol{\Psi} : I(\mathbb{R})^n \to I(\mathbb{R})$ is gH-upper semicontinuous, positive definite, bounded above;

for this, there is a non-decreasing function $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ with the weak decreasing property that for each $\boldsymbol{Z}_0$ there is a $\boldsymbol{H} \in \overline{co}(\boldsymbol{\Upsilon}(\boldsymbol{Z}_0, U_{\nu(\|\boldsymbol{Z}_0\|)}))$ satisfying

$$\widehat{\boldsymbol{K}} \odot \boldsymbol{H} \preceq -1 \odot \boldsymbol{\Psi}(\boldsymbol{Z}_0) \text{ for all } \widehat{\boldsymbol{K}} \in \partial^-_{\mathscr{H}} \boldsymbol{V}(\boldsymbol{Z}_0). \tag{4.31}$$

If $\boldsymbol{V}$ is component of a control-Lyapunov pair $(\boldsymbol{V}, \boldsymbol{\Psi})$, it is saide to be a control-Lyapunov IVF for (4.26).

Theorem 4.13 *The system (4.26) is asymptotically controllable if and only if it admits a control-Lyapunov IVF $\boldsymbol{V}$ with the weak infintesimal decreasing property in the gH-Dini Hadamard subdifferential sense.*

Proof: Consider the system (4.26) to be asymptotically controllable. To make the system (4.26) stable by minimizing an interval-valued energy or cost function, we select $\mathbf{V}$ and $\mathbf{\Psi}$ satisfying

$$\mathbf{V}(\mathbf{Z}_0) = \inf\left\{\int_0^\infty \mathbf{\Psi}(\mathbf{Z}(\tau, \mathbf{Z}_0, \wp))\, d\tau \ \oplus \max\{\|\wp\| - \kappa, 0\}\right\},$$

where infimum varies over the collection of all relaxed controls $\wp : \mathbb{R}_+ \to U_{\nu(\|\mathbf{Z}_0\|)}$, κ is a constant which is due to the function θ in Definition 4.8, $U_{\nu(\|\mathbf{Z}_0\|)}$ is a ball $\{\boldsymbol{\zeta} : d(\boldsymbol{\zeta}, 0) \leq \nu(\|\mathbf{Z}_0\|)\}$, and $\|\wp\|$ is the essential supremum norm of $\wp$ over $[0, \infty)$. Select a $\mathbf{Z}_0$ and a relaxed control $\wp$. Take $\mathbf{Z}(t) = \mathbf{Z}(t, \mathbf{Z}_0, \wp)$. We consider that the new initial state is $\mathbf{Z}(t)$ on $[t, \infty)$, and the control $\tilde{\wp}$ is the restriction of $\wp$ on the interval $[t, \infty)$. Then, $\mathbf{V}(\mathbf{Z}(t))$ is a bounded from above by the cost if we use $\tilde{\wp}$, i.e.,

$$\begin{aligned}\mathbf{V}(\mathbf{Z}(t)) &\preceq \int_t^\infty \mathbf{\Psi}(\mathbf{Z}(\tau))\, d\tau \oplus \max\{\|\tilde{\wp}\| - \kappa, 0\} \\ &\preceq \int_t^\infty \mathbf{\Psi}(\mathbf{Z}(\tau))\, d\tau \oplus \max\{\|\wp\| - \kappa, 0\} \\ &= \mathbf{V}(\mathbf{Z}_0) \ominus_{gH} \int_0^t \mathbf{\Psi}(\mathbf{Z}(\tau))\, d\tau. \end{aligned} \tag{4.32}$$

Because $\mathbf{Z}$ is gH-absolutely continuous, we have

$$\begin{aligned}\mathbf{Z}(t) \ominus_{gH} \mathbf{Z}_0 = \int_0^t \boldsymbol{\Upsilon}(\mathbf{Z}(s), \wp(s))ds &= \int_0^t \boldsymbol{\Upsilon}(\mathbf{Z}_0, \wp(s))ds + o(t) \\ &\in t \odot \overline{\text{co}}(\boldsymbol{\Upsilon}(\mathbf{Z}_0, U_{\nu(\|\mathbf{Z}_0\|)}) + o(t).\end{aligned}$$

Thus, we get a sequence $\{t_j\}$ such that $t_j > 0$ and $t_j \to 0$ having the property that if $\mathbf{Z}(t_j) = \mathbf{Z}_0 \oplus t_j \odot \mathbf{H}_j$, then $\mathbf{H}_j \to \mathbf{H}$ for some $\mathbf{H} \in \overline{\text{co}}(\boldsymbol{\Upsilon}(\mathbf{Z}_0, U_{\nu(\|\mathbf{Z}_0\|)})$. Using (4.32), we have

$$\begin{aligned}&\mathbf{V}(\mathbf{Z}(t_j)) \ominus_{gH} \mathbf{V}(\mathbf{Z}_0) \preceq \int_0^{t_j} -1 \odot \boldsymbol{\Psi}(\mathbf{Z}(\tau))\ d\tau \text{ for all } t_j \in [0, \infty) \\ \implies &\liminf_{\substack{\mathbf{H}_j \to \mathbf{H} \\ t \to 0+}} t_j^{-1} \odot \{\mathbf{V}(\mathbf{Z}_0 \oplus t_j \odot \mathbf{H}_j) \ominus_{gH} \mathbf{V}(\mathbf{Z}_0)\} \preceq -1 \odot \boldsymbol{\Psi}(\mathbf{Z}_0) \\ \implies &\mathbf{V}_{\mathscr{H}}^{-}(\mathbf{Z}_0) \odot \mathbf{H} \preceq -1 \odot \boldsymbol{\Psi}(\mathbf{Z}_0) \\ \implies &\widehat{\mathbf{K}} \odot \mathbf{H} \preceq -1 \odot \boldsymbol{\Psi}(\mathbf{Z}_0) \text{ for all } \widehat{\mathbf{K}} \in \partial_{\mathscr{H}}^{-} \mathbf{V}(\mathbf{Z}_0).\end{aligned}$$

Thus, $(\mathbf{V}, \boldsymbol{\Psi})$ is a Lyapunov pair.

In the converse sense, assume that $(\mathbf{V}, \boldsymbol{\Psi})$ is a Lyapunov pair. For any given $\epsilon > 0$, select $r \in (0, \epsilon]$ so that we have

$$\mathcal{B}_r = \{\mathbf{Z} \in I(\mathbb{R})^n : \|\mathbf{Z}\| \leq r\} \subset \boldsymbol{\mathcal{Z}}.$$

Take $\boldsymbol{\alpha} = \min\limits_{\|\mathbf{Z}\|=r} \mathbf{V}(\mathbf{Z})$. Then, we have $\boldsymbol{\alpha} \in I(\mathbb{R}_+)$. Let $\mathbf{0} \prec \boldsymbol{\beta} \prec \boldsymbol{\alpha}$ and

$$\Lambda_{\boldsymbol{\beta}} = \{\mathbf{Z} \in \mathcal{B}_r : \mathbf{V}(\mathbf{Z}) \preceq \boldsymbol{\beta}\}.$$

Then, $\Lambda_{\boldsymbol{\beta}}$ lies in the interior of $\mathcal{B}_r$. The set $\Lambda_{\boldsymbol{\beta}}$ has the characteristic that any trajectory that starts in $\Lambda_{\boldsymbol{\beta}}$ at $t = 0$ remains in $\Lambda_{\boldsymbol{\beta}}$ for all $t \geq 0$.

It is true that if $\mathbf{Z}_0 \in \Lambda_\beta$, then $\mathbf{Z}(t) \in \Lambda_\beta$ $\forall t$. Select $\mathbf{Z}_0$ with $\mathbf{V}(\mathbf{Z}_0) \preceq \frac{\beta}{2}$. As $\mathbf{V}$ is gH-continuous and $\mathbf{V}_{\mathscr{H}}^{-}(\mathbf{Z}_0)(\mathbf{H}) \preceq \mathbf{0}$, for every $[\underline{\epsilon}_1, \overline{\epsilon}_1]$, we get

$$\mathbf{V}(\mathbf{Z}_0 \oplus t_n \odot \mathbf{H}_n) \preceq \mathbf{V}(\mathbf{Z}_0) \oplus [\underline{\epsilon}_1, \overline{\epsilon}_1] \tag{4.33}$$

for infinitely many values of n. Take $[\underline{\epsilon}_1, \overline{\epsilon}_1] = \frac{1}{2} \odot \mathbf{V}(\mathbf{Z}_0)$. Because $\mathbf{V}$ is positive definite, $\mathbf{0} \prec [\underline{\epsilon}_1, \overline{\epsilon}_1]$. This means that

$$\mathbf{V}(\mathbf{Z}(t_n)) = \mathbf{V}(\mathbf{Z}_0 \oplus t_n \odot \mathbf{H}_n) \preceq \tfrac{3}{2} \odot \mathbf{V}(\mathbf{Z}_0) \preceq \tfrac{3}{4} \odot \boldsymbol{\beta} \preceq \boldsymbol{\beta}.$$

Moreover, there is $\delta > 0$ satisfying $\|\mathbf{Z}(t)\| \leq \delta \implies \mathbf{V}(\mathbf{Z}(t)) \preceq \boldsymbol{\beta}$. Thus, we have

$$\mathcal{B}_\delta \subset \Lambda_\beta \subset \mathcal{B}_r,$$

and further

$$\mathbf{Z}_0 \in \mathcal{B}_\delta \implies \mathbf{Z}_0 \in \Lambda_\beta \implies \mathbf{Z}(t) \in \Lambda_\beta \implies \mathbf{Z}(t) \in \mathcal{B}_r.$$

As a consequence, it must be true that

$$\|\mathbf{Z}(0)\| \leq \delta \implies \|\mathbf{Z}(t)\| \leq r \leq \epsilon \; \forall \; t > 0.$$

For establishing the asymptotic stability of (4.26), we have to show that $\mathbf{Z}(t) \to \mathbf{0}$ as $t \to \infty$. In other words, for every $\nu > 0$, there exists $T > 0$ satisfying $\|\mathbf{Z}(t)\| < \nu$ for all $t > T$. Again from the earlier argument, we have that for every $\nu > 0$ we can select $\mu > 0$ so that $\Lambda_\mu \subset B_\nu$. Thus, it suffices to prove that $\mathbf{V}(\mathbf{Z}(t)) \to 0$ as $t \to \infty$.

Because $\mathbf{V}$ is a gH-continuous function with $\mathbf{K} \odot \mathbf{H} \preceq \mathbf{0}$ for all $\mathbf{K} \in \partial^-_{\mathscr{H}} \mathbf{V}(\mathbf{Z}_0)$, $\mathbf{V}^-_{\mathscr{H}}(\mathbf{Z}_0)(\mathbf{H})$ exists satisfying $\mathbf{V}^-_{\mathscr{H}}(\mathbf{Z}_0)(\mathbf{H}) \preceq \mathbf{0}$ for all $t \in (0, T)$, meaning that $\mathbf{V}(\mathbf{Z}(t))$ is a decreasing IVF and is bounded below by $\mathbf{0}$. Moreover, this means that

$$\mathbf{V}(\mathbf{Z}(t)) \to \mathbf{C} \text{ as } t \to \infty.$$

We prove $\mathbf{C} = \mathbf{0}$ using contradiction. Because $\mathbf{0} \prec \mathbf{V}(\mathbf{Z}(t))$ for all $t > 0$, $\mathbf{C} \not\prec \mathbf{0}$. From gH-continuity of $\mathbf{V}(\mathbf{Z})$, there exists $\varrho > 0$ such that $\mathcal{B}_\varrho \subset \Lambda_{\mathbf{C}}$. The limit $\mathbf{V}(\mathbf{Z}(t)) \to \mathbf{C}$ means that the trajectory $\{\mathbf{Z}(t)\}$ resides outside the ball $\mathcal{B}_\varrho$ for all $t \geq 0$.

Consider $\max\limits_{\varrho \leq \|\mathbf{Z}\| \leq r} \boldsymbol{\Psi}(\mathbf{Z}) = -1 \odot \Gamma$, which exists since the gH-upper semi-continuous IVF $\boldsymbol{\Psi}(\mathbf{Z})$ admits a maximum over the compact set $\{\mathbf{Z} : \varrho \leq \|\mathbf{Z}\| \leq r\}$. As $\mathbf{V}$ is a gH-continuous function from $[0, t]$ to $I(\mathbb{R})$,

$$\mathbf{V}(\mathbf{Z}(t)) = \mathbf{V}(\mathbf{Z}_0) \oplus \int_0^t \boldsymbol{\Psi}(\mathbf{Z}(\tau)) \, d\tau \preceq \mathbf{V}(\mathbf{0}) \ominus_{gH} t \odot \Gamma. \tag{4.34}$$

Because the right-hand side ultimately becomes negative when t is large, (4.34) is in contradiction with the assumption that $\mathbf{0} \prec \mathbf{C}$. Thus, $\mathbf{C} = \mathbf{0}$ and we get the result. □

- *An Example of Unstable System*

Let us consider an interval spring-mass system influenced by Coulomb friction, represented by the interval differential equation

$$m \odot [\min\{\dot{\underline{z}}, \dot{\overline{z}}\}, \max\{\dot{\underline{z}}, \dot{\overline{z}}\}] = \mathbf{0} \ominus_{gH} (k \odot [\underline{y}(t), \overline{y}(t)] \oplus b \odot \operatorname{sgn}[\underline{z}, \overline{z}]) \oplus \boldsymbol{\zeta}, \tag{4.35}$$

where $[\underline{y}(t), \overline{y}(t)]$ refers to the interval state variable of the system (4.35) at time t, $[\underline{z}, \overline{z}] = [\min\{\dot{\underline{y}}(t), \dot{\overline{y}}(t)\}, \max\{\dot{\underline{y}}(t), \dot{\overline{y}}(t)\}]$ represents interval velocity, m is the mass of the system, $b > 0$ is the Coulomb friction, $k > 0$ is the spring constant, and $\boldsymbol{\zeta}$ is the Dirac control input given by

$$\boldsymbol{\zeta} = \sum_{i=1}^{\infty} \boldsymbol{\xi}(\underline{y}) \delta(t - t_i).$$

where $\boldsymbol{\xi}(\underline{y})$ is the wrong feedback control which is applied at some state-dependent impulsive time sequence $\{t_i\}_{i \in \mathbb{N}}$ satisfying $0 \leq t_0 < t_1 < t_2 < \cdots < t_i < t_{i+1} < \cdots$ with $\lim\limits_{i \to \infty} t_i = +\infty$. The closed-loop system seems to exhibit discrete-continuous interval dynamics (4.35) with

$$\underline{y}(t_i+) = \underline{y}(t_i-), \overline{y}(t_i+) = \overline{y}(t_i-), [\underline{z}_{t_i+}, \overline{z}_{t_i+}] = [\underline{z}_{t_i-}, \overline{z}_{t_i-}] \oplus \boldsymbol{\xi}(\underline{z}(t_i)), i = 1, 2, \ldots,$$

where $\underline{y}(t_i+) = \lim\limits_{t \to t_i + 0} \underline{y}(t)$, $\underline{y}(t_i-) = \lim\limits_{t \to t_i - 0} \underline{y}(t)$, $\boldsymbol{\xi}(\underline{y}) = -\sqrt{\frac{2\alpha|\underline{y}| - k y^2}{m}} \operatorname{sgn}(\underline{y})$ and applied to the system (4.35) at the time sequence $\{t_i\}_{i \in \mathbb{N}}$ so that

$$\|\underline{y}(t_i)\| \leq \frac{2\alpha}{k}, \underline{z}(t_i-) = \overline{z}(t_i-) = 0.$$

We see that (4.35) results in the following four cases.

- Case 1.

$$\left.\begin{aligned} &\dot{\underline{y}} = \underline{z} \\ &\dot{\underline{z}} = -\tfrac{b}{m}\operatorname{sgn}\underline{z} - \tfrac{k}{m}\underline{y} + u \\ &\dot{\overline{y}} = \overline{z} \\ &\dot{\overline{z}} = -\tfrac{b}{m}\operatorname{sgn}\overline{z} - \tfrac{k}{m}\overline{y} + u. \end{aligned}\right\} \tag{4.36}$$

The state trajectory for • Case 1. is depicted in Figure 4.3(a) with the initial condition $(\underline{y}(0), \underline{z}(0), \overline{y}(0), \overline{z}(0)) = (3, 0, 6, 2)^\top$, using parameters set as $b = 1, m = 1, k = 1$, and time $T \approx 30$. The applied stabilizing impulse has proven beneficial for system stability, effectively controlling the state of the system at certain impulsive points, leading the system to converge rapidly to the equilibrium position of 0. It is important to note that this trajectory is not feasible because the $\underline{y}$ curve ($\underline{z}$ curve) surpasses $\overline{y}$ curve ($\overline{z}$ curve).

- Case 2.

$$\left.\begin{aligned}
&\dot{\underline{y}} = \underline{z}\\
&\dot{\underline{z}} = -\frac{b}{m}\mathrm{sgn}\overline{z} - \frac{k}{m}\overline{y} + u\\
&\dot{\overline{y}} = \overline{z}\\
&\dot{\overline{z}} = -\frac{b}{m}\mathrm{sgn}\underline{z} - \frac{k}{m}\underline{y} + u
\end{aligned}\right\} \tag{4.37}$$

The state trajectory for • Case 2. is illustrated in Figure 4.3(b), with an initial condition of $(\underline{y}(0), \underline{z}(0), \overline{y}(0), \overline{z}(0)) = (3, 0, 6, 2)^\top$ and parameters set to $b = 1, m = 1, k = 1$, $T \approx 30$. In this figure, the applied destabilizing impulse introduces disturbances, such as kinks at the impulsive points $\{t_i\}_{i\in\mathbb{N}}$, causing the system to become unstable. It should be noted that this trajectory is not acceptable because at certain points the $\underline{y}$ curve ($\underline{z}$ curve) goes above $\overline{y}$ curve ($\overline{z}$ curve).

- Case 3.

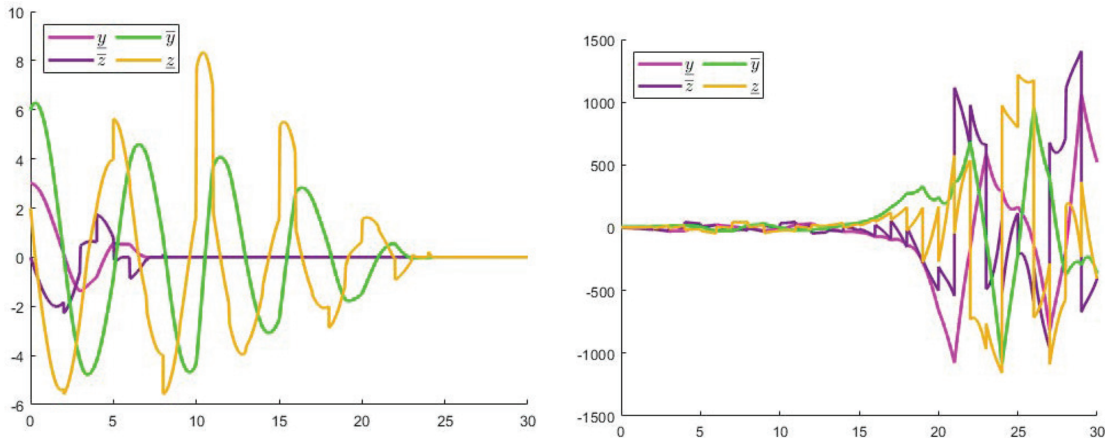
$$\left.\begin{aligned}
&\dot{\underline{y}} = \overline{z}\\
&\dot{\underline{z}} = -\frac{b}{m}\mathrm{sgn}\underline{z} - \frac{k}{m}\underline{y} + u\\
&\dot{\overline{y}} = \underline{z}\\
&\dot{\overline{z}} = -\frac{b}{m}\mathrm{sgn}\overline{z} - \frac{k}{m}\overline{y} + u
\end{aligned}\right\} \tag{4.38}$$

For the system (4.35) referring to • Case 3. with $b = 1, m = 1, k = 1$, the simulation curves of $(\underline{y}, \underline{z}, \overline{y}, \overline{z})^\top$ with the initial value $(3, 0, 6, 2)^\top$ are shown in Figure 4.4(a). This figure illustrates that the state finally becomes unbounded due to the destabilizing impulse within a settling time of $T \approx 30$. From the zoomed-in portion in Figure 4.4(b), located on the right side of Figure 4.4(a), it is evident that the $\underline{y}$ curve ($\underline{z}$ curve) always remain below $\overline{y}$ curve ($\overline{z}$ curve). Therefore, it represents a possible interval solution to the system (4.35).

- Case 4.

$$\left.\begin{aligned}
&\dot{\underline{y}} = \overline{z}\\
&\dot{\underline{z}} = -\frac{b}{m}\mathrm{sgn}\overline{z} - \frac{k}{m}\overline{y} + u\\
&\dot{\overline{y}} = \underline{z}\\
&\dot{\overline{z}} = -\frac{b}{m}\mathrm{sgn}\underline{z} - \frac{k}{m}\underline{y} + u
\end{aligned}\right\} \tag{4.39}$$

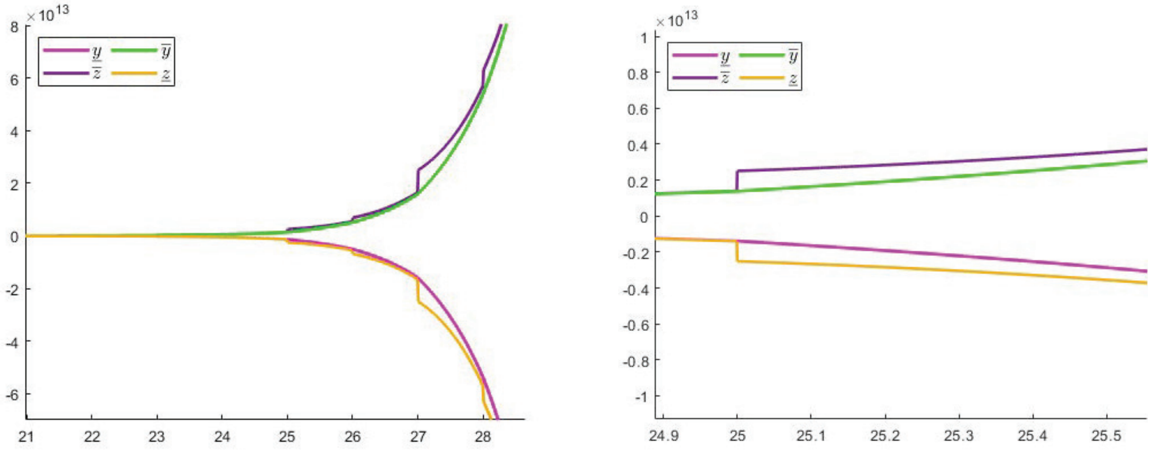
For the system (4.35) referring to • Case 4. with $b = 1, m = 1, k = 1$, the simulation curves of $(\underline{y}, \underline{z}, \overline{y}, \overline{z})^\top$ with the initial values $(3, 0, 6, 2)^\top$ are displayed in Figure 4.5. A valid trajectory cannot be derived from this figure because, at certain points, the $\underline{y}$ curve ($\underline{z}$ curve) goes above $\overline{y}$ curve ($\overline{z}$ curve).



(a) Trajectory for • Case 1.

(b) Trajectory for • Case 2.

Figure 4.3: Wrong trajectories obtained as per • Case 1. and • Case 2., respectively.



(a) Trajectory for • Case 3.

(b) Zoomed-in version of indicated portion

Figure 4.4: Valid trajectory and zoomed-in version of the rectangular portion in the left side figure for • Case 3.

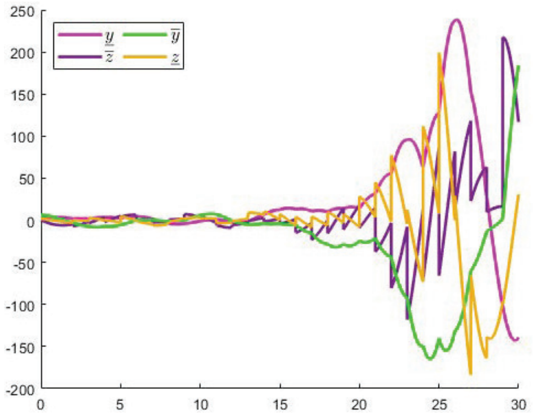


Figure 4.5: Wrong trajectory obtained as per • Case 4.

Summarizing all four possible cases, we observe that only • Case 3. for the system (4.35) provides a solution. From Figure • Case 3., it is evident that the solution curve in • Case 3. diverges as time progresses, indicating that the system (4.35) is not asymptotically stable.

4.7 Conclusion

In this chapter, we have introduced the concepts of gH-Dini Hadamard subdifferential and gH-Dini Hadamard superdifferential (Definitions 4.1 and 4.2) for IVFs, accompanied by illustrative examples. A connection between gH-Dini Hadamard subdifferential and gH-Dini Hadamard superdifferential has been established (Remark 4.4.1). The convexity (Theorem 4.1) and closedness (Theorem 4.2) of the gH-Dini Hadamard subdifferential set have been presented. It has been noted that the gH-Dini Hadamard subdifferential set adheres to various calculus rules, including the partial chain rule (Theorem 4.4) and the subadditive rule (Theorem 4.5).

We have also developed a smooth variational type description (Theorem 4.3) of gH-Dini Hadamard subgradients for IVFs. We have provided a criterion (Theorem 4.8) that determines when an IVF is gH-Dini Hadamard subdifferentiable. Then, we have presented two optimality condition—FJ type (Theorem 4.10) and KKT type (Theorem 4.11, Theorem 4.12)—for constrained IOPs. Finally, employing the proposed gH-Dini Hadamard subdifferential, we have investigated the asymptotic controllability of a nonsmooth dynamical control system with interval state variables, in the presence of a control-Lyapunov pair (Theorem 4.13). Towards the conclusion of the chapter, an example of an unstable nonsmooth control system is provided. This example could serve as inspiration for future researchers to explore this direction further.

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