Chapter 2

A high order numerical method for the variable- order time-fractional reaction-subdiffusion equation

In this chapter, we have designed a new high order numerical approximation for the variable-order Caputo derivative of order $(0 < \alpha(\mathtt{x}, \mathtt{t}) < 1)$. Later, this approximation is used to obtain the numerical solution of the variable-order time-fractional reaction subdiffusion equation (VO-TFRDE). The structure of this chapter is outlined as follows. In Sect. 2.1 we have given the formulation of the proposed problem and L-123 approximation of the variable-order Caputo derivative. A fully discrete scheme for the problem is constructed, and its matrix representation is given in Sect. 2.2. The stability of the proposed numerical scheme is discussed in Sect. 2.3. The numerical results and discussion with different cases are covered in Sect. 2.4, and Sect. 2.5 concludes the chapter.

2.1 Introduction

In this chapter, we present a high order new numerical approximation for variable order Caputo fractional derivative of order $0 < \alpha(\mathrm{x}, \mathtt{t}) < 1$, by using the idea of interpolation. Then, by using this approximation, a numerical scheme is presented by using finite difference approach for variable order time-fractional reaction-subdiffusion equation (VO-TFRSDE).

2.1.1 Formulation of the problem

Consider the following variable-order time-fractional reaction-subdiffusion equation [56, 184]:

$$\mathsf{y}_{\mathtt{t}}(\mathrm{x}, \mathtt{t}) = {}_{RL}D_{0,\mathtt{t}}^{1-\alpha(\mathrm{x},\mathtt{t})}\left(\kappa_1 \mathsf{y}_{\mathrm{xx}}(\mathrm{x}, \mathtt{t}) - \kappa_2 \mathsf{y}(\mathrm{x}, \mathtt{t})\right) + \mathsf{f}(\mathrm{x}, \mathtt{t}) \tag{2.1}$$

with initial and boundary conditions respectively

$$\mathsf{y}(\mathrm{x}, 0) = \varphi(\mathrm{x}), \tag{2.2}$$

$$\mathsf{y}(0, \mathtt{t}) = \psi_0(\mathtt{t}),\ \ \mathsf{y}(L, \mathtt{t}) = \psi_L(\mathtt{t}). \tag{2.3}$$

The Riemann-Liouville integration of order $\alpha(\mathrm{x}, \mathtt{t}) \geq 0$ for the function $\mathsf{f}(\mathrm{x}, \mathtt{t})$ is defined as [13]

$${}_{RL}D_{a,\mathtt{t}}^{-\alpha(\mathrm{x},\mathtt{t})}\mathsf{f}(\mathrm{x}, \mathtt{t}) = \frac{1}{\Gamma\alpha(\mathrm{x}, \mathtt{t})} \int_a^{\mathtt{t}} (\mathtt{t} - s)^{\alpha(\mathrm{x},\mathtt{t})-1}\mathsf{f}(\mathrm{x}, s)ds. \tag{2.4}$$

Performing Riemann-Liouville integration in time in (2.1), we get

$$ {}_{RL}D_{0,\mathtt{t}}^{-(1-\alpha(\mathrm{x},\mathtt{t}))}\mathsf{y}_{\mathtt{t}}(\mathrm{x},\mathtt{t}) = \kappa_1 \mathsf{y}_{\mathrm{xx}}(\mathrm{x},\mathtt{t}) - \kappa_2 \mathsf{y}(\mathrm{x},\mathtt{t}) + {}_{RL}D_{0,\mathtt{t}}^{-(1-\alpha(\mathrm{x},\mathtt{t}))}\mathsf{f}(\mathrm{x},\mathtt{t}). \tag{2.5} $$

Now, using the property [13]

$$ {}_{RL}D_{a,\mathtt{t}}^{-\alpha(\mathrm{x},\mathtt{t})}\left({}_{RL}D_{a,\mathtt{t}}^{\alpha(\mathrm{x},\mathtt{t})}\mathsf{f}(\mathrm{x},\mathtt{t})\right) = \mathsf{f}(\mathrm{x},\mathtt{t}) - \sum_{j=1}^{m}\left[{}_{RL}D_{a,\mathtt{t}}^{(\alpha(\mathrm{x},\mathtt{t})-j)}\mathsf{f}(\mathrm{x},\mathtt{t})\right]_{\mathtt{t}=a} \frac{(\mathtt{t}-a)^{\alpha(\mathrm{x},\mathtt{t})-j}}{\Gamma(\alpha(\mathrm{x},\mathtt{t})-j+1)}. \tag{2.6} $$

Here, $(m-1) < \alpha(\mathrm{x},\mathtt{t}) < m$. The relation between the Riemann-Liouville and Caputo derivative for the function $\mathsf{f}(\mathrm{x},\mathtt{t})$ is given by [13]

$$ {}_{RL}D_{a,\mathtt{t}}^{\alpha(\mathrm{x},\mathtt{t})}\mathsf{f}(\mathrm{x},\mathtt{t}) = {}_{C}D_{a,\mathtt{t}}^{\alpha(\mathrm{x},\mathtt{t})}\mathsf{f}(\mathrm{x},\mathtt{t}) + \sum_{k=0}^{m-1}\frac{1}{\Gamma(k+1-\alpha(\mathrm{x},\mathtt{t}))}\mathsf{f}^{k}(\mathrm{x},a)(\mathtt{t}-a)^{k-\alpha(\mathrm{x},\mathtt{t})}. \tag{2.7} $$

Taking $m=1$, the relationship takes the form

$$ {}_{RL}D_{a,\mathtt{t}}^{\alpha(\mathrm{x},\mathtt{t})}\mathsf{f}(\mathrm{x},\mathtt{t}) = {}_{C}D_{a,\mathtt{t}}^{\alpha(\mathrm{x},\mathtt{t})}\mathsf{f}(\mathrm{x},\mathtt{t}) + \frac{1}{\Gamma(1-\alpha(\mathrm{x},\mathtt{t}))}\mathsf{f}(\mathrm{x},a)(\mathtt{t}-a)^{-\alpha(\mathrm{x},\mathtt{t})}. \tag{2.8} $$

Using the relation (2.6) and (2.8) in eqn. (2.5), we get

$$ {}_{C}D_{0,\mathtt{t}}^{\alpha(\mathrm{x},\mathtt{t})}\mathsf{y}(\mathrm{x},\mathtt{t}) = \kappa_1 \mathsf{y}_{\mathrm{xx}}(\mathrm{x},\mathtt{t}) - \kappa_2 \mathsf{y}(\mathrm{x},\mathtt{t}) + {}_{RL}D_{0,\mathtt{t}}^{-(1-\alpha(\mathrm{x},\mathtt{t}))}\mathsf{f}(\mathrm{x},\mathtt{t}). \tag{2.9} $$

Thus, the problem (2.1) has been changed to the following

$$ {}_{C}D_{0,\mathtt{t}}^{\alpha(\mathrm{x},\mathtt{t})}\mathsf{y}(\mathrm{x},\mathtt{t}) = \kappa_1 \mathsf{y}_{\mathrm{xx}}(\mathrm{x},\mathtt{t}) - \kappa_2 \mathsf{y}(\mathrm{x},\mathtt{t}) + g(\mathrm{x},\mathtt{t}) \tag{2.10} $$

with boundaries

$$\mathtt{y}(\mathrm{x},0)=\varphi(\mathrm{x}), \tag{2.11}$$

$$\mathtt{y}(0,\mathtt{t})=\psi_0(\mathtt{t}),\ \mathtt{y}(L,\mathtt{t})=\psi_L(\mathtt{t}), \tag{2.12}$$

where

$$g(\mathrm{x},\mathtt{t})={}_{RL}D_{0,\mathtt{t}}^{-(1-\alpha(\mathrm{x},\mathtt{t}))}\mathtt{f}(\mathrm{x},\mathtt{t})=\frac{1}{\Gamma(1-\alpha(\mathrm{x},\mathtt{t}))}\int_0^{\mathtt{t}}\frac{\mathtt{f}(\mathrm{x},\eta)}{(\mathtt{t}-\eta)^{\alpha(\mathrm{x},\mathtt{t})}}d\eta.$$

Here ${}_CD_{0,\mathtt{t}}^{\alpha(\mathrm{x},\mathtt{t})}$ represent the variable order Caputo differential operator (VOCDO) with order $\alpha(\mathrm{x},\mathtt{t})$, where $(0<\alpha(\mathrm{x},\mathtt{t})<1)$. The VOCDO is defined as

$${}_CD_{0,\mathtt{t}}^{\alpha(\mathrm{x},\mathtt{t})}\mathtt{y}(\mathrm{x},\mathtt{t})=\begin{cases}\dfrac{1}{\Gamma(m-\alpha(\mathrm{x},\mathtt{t}))}\int_0^{\mathtt{t}}(\mathtt{t}-s)^{m-\alpha(\mathrm{x},\mathtt{t})-1}\mathtt{y}^m(\mathrm{x},s)ds, & m-1<\alpha(\mathrm{x},\mathtt{t})<m,\\ \mathtt{y}^m(\mathrm{x},\mathtt{t}), \quad \alpha(\mathrm{x},\eta)=m\in\mathcal{N}.\end{cases} \tag{2.13}$$

Moreover, the primary contributions of this chapter can be summarized in the following points:

- We have transformed our model (2.1) to (2.10) by using the properties of Riemann-Liouville derivative.
- We have extended the idea of Mokhtari and Mostajeran [185] to develope the numerical approximation of the variable-order Caputo derivative of order $0<\alpha(\mathrm{x},\mathtt{t})<1$.
- A difference scheme is proposed for solving variable-order time-fractional reaction-subdiffusion equation. The scheme is proved to be unconditionally stable.

- Computational algorithm is provided for better understanding of our numerical scheme.
- The proposed numerical scheme is validated on two numerical examples. The obtained numerical results are highly accurate and have higher order of convergence.

2.1.2 Formulation of L-123 approximation for the variable-order Caputo derivative

In this section, we present the L-123 approximation for the variable order Caputo fractional derivative of order $\alpha(\mathrm{x},\mathtt{t}) \in (0,1)$, which is motivated by the idea of Mokhtari and Mostajeran [185]. Let τ be the step size in time such that $\mathtt{t}_k = k\tau$ for any integer $k \geq 0$. Let $\mathtt{t}_{k+\frac{1}{2}} = (\mathtt{t}_{k+1} + \mathtt{t}_k)/2$,

$$\delta_{\mathtt{t}}\mathsf{f}(\mathrm{x}_\iota, \mathtt{t}_{k-\frac{1}{2}}) = \frac{1}{\tau}(\mathsf{f}(\mathrm{x}_\iota, \mathtt{t}_k) - \mathsf{f}(\mathrm{x}_\iota, \mathtt{t}_{k-1})), \tag{2.14}$$

$$\delta_{\mathtt{t}}^2\mathsf{f}(\mathrm{x}_\iota, \mathtt{t}_k) = \frac{1}{\tau}(\delta_{\mathtt{t}}\mathsf{f}(\mathrm{x}_\iota, \mathtt{t}_{k+\frac{1}{2}}) - \delta_{\mathtt{t}}\mathsf{f}(\mathrm{x}_\iota, \mathtt{t}_{k-\frac{1}{2}})), \tag{2.15}$$

$$\delta_{\mathtt{t}}^3\mathsf{f}(\mathrm{x}_\iota, \mathtt{t}_{k-\frac{1}{2}}) = \frac{1}{\tau^2}(\delta_{\mathtt{t}}\mathsf{f}(\mathrm{x}_\iota, \mathtt{t}_{k+\frac{1}{2}}) - 2(\delta_{\mathtt{t}}\mathsf{f}(\mathrm{x}_\iota, \mathtt{t}_{k-\frac{1}{2}})) + \delta_{\mathtt{t}}\mathsf{f}(\mathrm{x}_\iota, \mathtt{t}_{k-\frac{3}{2}})). \tag{2.16}$$

Let $\mathcal{P}_{1,j}\mathsf{f}(\mathrm{x},\mathtt{t})$ represent the linear interpolating polynomial of $\mathsf{f}(\mathrm{x},\mathtt{t})$ on every small interval $[\mathtt{t}_{j-1}, \mathtt{t}_j]$, $(1 \leq j \leq k)$, we get

$$\mathcal{P}_{1,j}\mathsf{f}(\mathrm{x}_\iota, \mathtt{t}_j) = \mathsf{f}(\mathrm{x}_\iota, \mathtt{t}_{j-1})\frac{(\mathtt{t}_j - \mathtt{t})}{\tau} + \mathsf{f}(\mathrm{x}_\iota, \mathtt{t}_j)\frac{(\mathtt{t} - \mathtt{t}_{j-1})}{\tau}. \tag{2.17}$$

The linear interpolation error is given by

$$\mathsf{f}(\mathrm{x},\mathtt{t}) - \mathcal{P}_{1,j}\mathsf{f}(\mathrm{x},\mathtt{t}) = \frac{\mathsf{f}''(\mathrm{x},\xi_j)}{2}(\mathtt{t} - \mathtt{t}_{j-1})(\mathtt{t} - \mathtt{t}_j), \tag{2.18}$$

where $\mathtt{t} \in [\mathtt{t}_{j-1}, \mathtt{t}_j]$, $\xi_j \in (\mathtt{t}_{j-1}, \mathtt{t}_j)$, $(1 \le j \le k)$.

Let $\mathcal{P}_{2,j}\mathsf{f}(\mathrm{x}, \mathtt{t})$ denote the quadratic interpolating polynomial for the function of $\mathsf{f}(\mathrm{x}, \mathtt{t})$ by using three points $(\mathtt{t}_{j-2}, \mathsf{f}(\mathrm{x}_\iota, \mathtt{t}_{j-2})), (\mathtt{t}_{j-1}, \mathsf{f}(\mathrm{x}_\iota, \mathtt{t}_{j-1}))$, $(\mathtt{t}_j, \mathsf{f}(\mathrm{x}_\iota, \mathtt{t}_j))$, and constraining the outcome onto a subinterval $[\mathtt{t}_{j-1}, \mathtt{t}_j]$ using the formula for $(2 \le j \le k)$, we have

$$\begin{aligned}\mathcal{P}_{2,j}\mathsf{f}(\mathrm{x}, \mathtt{t}) =& \mathsf{f}(\mathrm{x}_\iota, \mathtt{t}_{j-2})\frac{(\mathtt{t} - \mathtt{t}_{j-1})(\mathtt{t} - \mathtt{t}_j)}{2\tau^2} + \mathsf{f}(\mathrm{x}_\iota, \mathtt{t}_{j-1})\frac{(\mathtt{t} - \mathtt{t}_{j-2})(\mathtt{t}_j - \mathtt{t})}{\tau^2} \\ &+ \mathsf{f}(\mathrm{x}_\iota, \mathtt{t}_j)\frac{(\mathtt{t} - \mathtt{t}_{j-1})(\mathtt{t} - \mathtt{t}_{j-2})}{2\tau^2}.\end{aligned} \tag{2.19}$$

The quadratic interpolation error is given by

$$\mathsf{f}(\mathrm{x}, \mathtt{t}) - \mathcal{P}_{2,j}\mathsf{f}(\mathrm{x}, \mathtt{t}) = \frac{f'''(\mathrm{x}, \theta_j)}{6}(\mathtt{t} - \mathtt{t}_{j-2})(\mathtt{t} - \mathtt{t}_{j-1})(\mathtt{t} - \mathtt{t}_j), \tag{2.20}$$

$$\mathtt{t} \in [\mathtt{t}_{j-1}, \mathtt{t}_j], \theta_j \in (\mathtt{t}_{j-2}, \mathtt{t}_j), \ \ (2 \le j \le k).$$

Let $\mathcal{P}_{3,j}\mathsf{f}(\mathrm{x}, \mathtt{t})$ denote the Lagrange's cubic interpolation polynomial for the function $\mathsf{f}(\mathrm{x}, \mathtt{t})$ using four points $(\mathtt{t}_{j-n}, \mathsf{f}(\mathrm{x}_\iota, \mathtt{t}_{j-n}))$, $n = 3, 2, 1, 0$, $j \ge 3$ and taking a constraint of the result for the interval $[\mathtt{t}_{j-1}, \mathtt{t}_j]$, $(2 \le j \le k)$, we have

$$\begin{aligned}\mathcal{P}_{3,j}\mathsf{f}(\mathrm{x}, \mathtt{t}) =& \frac{(\mathtt{t}_j - \mathtt{t})(\mathtt{t} - \mathtt{t}_{j-1})(\mathtt{t} - \mathtt{t}_{j-2})}{6\tau^3}\mathsf{f}(\mathrm{x}_\iota, \mathtt{t}_{j-3}) + \frac{(\mathtt{t} - \mathtt{t}_j)(\mathtt{t} - \mathtt{t}_{j-1})(\mathtt{t} - \mathtt{t}_{j-3})}{2\tau^3} \\ & \mathsf{f}(\mathrm{x}_\iota, \mathtt{t}_{j-2}) + \frac{(\mathtt{t}_j - \mathtt{t})(\mathtt{t} - \mathtt{t}_{j-3})(\mathtt{t} - \mathtt{t}_{j-2})}{2\tau^3}\mathsf{f}(\mathrm{x}_\iota, \mathtt{t}_{j-1}) + \\ & \frac{(\mathtt{t} - \mathtt{t}_{j-3})(\mathtt{t} - \mathtt{t}_{j-1})(\mathtt{t} - \mathtt{t}_{j-2})}{6\tau^3}\mathsf{f}(\mathrm{x}_\iota, \mathtt{t}_j) \\ =& \mathcal{P}_{2,j}\mathsf{f}(\mathrm{x}, \mathtt{t}) + \frac{1}{6}(\mathtt{t} - \mathtt{t}_j)(\mathtt{t} - \mathtt{t}_{j-1})(\mathtt{t} - \mathtt{t}_{j-2})\delta_{\mathtt{t}}^3\mathsf{f}(\mathrm{x}_\iota, \mathtt{t}_{j-\frac{3}{2}}).\end{aligned} \tag{2.21}$$

Taking the first derivative, we get

$$(\mathcal{P}_{3,j}\mathtt{f}(\mathrm{x},\mathtt{t}))' = \delta_{\mathtt{t}}\mathtt{f}(\mathrm{x}_\iota,\mathtt{t}_{j-\frac{1}{2}}) + \delta_{\mathtt{t}}^2\mathtt{f}(\mathrm{x}_\iota,\mathtt{t}_{j-1})(\mathtt{t}-\mathtt{t}_{j-\frac{1}{2}}) + \frac{1}{6}\varphi_j(\mathrm{x},\mathtt{t})\delta_{\mathtt{t}}^3\mathtt{f}(\mathrm{x}_\iota,\mathtt{t}_{j-\frac{3}{2}}), \tag{2.22}$$

where $\varphi_j(\mathtt{t}) = 3\mathtt{t}^2 - 2\mathtt{t}(\mathtt{t}_{j-2}+\mathtt{t}_{j-1}+\mathtt{t}_j) + \mathtt{t}_j\mathtt{t}_{j-1} + \mathtt{t}_j\mathtt{t}_{j-2} + \mathtt{t}_{j-2}\mathtt{t}_{j-1}$. The error in the cubic interpolation is given by

$$\mathtt{f}(\mathrm{x},\mathtt{t}) - \mathcal{P}_{3,j}\mathtt{f}(\mathrm{x},\mathtt{t}) = \frac{\mathtt{f}^{(4)}(\mathrm{x},\xi_j)}{4!}\mathcal{P}_{\iota=0}^{3}(\mathtt{t}-\mathtt{t}_{j-\iota})$$
$$\mathtt{t}\in[\mathtt{t}_{j-1},\mathtt{t}_j],\ \xi_j\in(\mathtt{t}_{j-3},\mathtt{t}_j),\ (3\le j\le k). \tag{2.23}$$

Let $\mathtt{f}(\mathrm{x},\mathtt{t}) \in C^{2,3}_{\mathrm{x},\mathtt{t}}([0,L]\times[0,\mathtt{t}_k])$, the Caputo fractional derivative of order $\alpha(\mathrm{x}_\iota,\mathtt{t}_k)\in(0,1)$ is defined as

$$ {}_0^C D_{\mathtt{t}}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\mathtt{f}(\mathrm{x},\mathtt{t})|_{\mathtt{t}=\mathtt{t}_k} = \frac{1}{\Gamma(1-\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}\int_0^{\mathtt{t}_k}\frac{\mathtt{f}'(\mathrm{x}_\iota,s)}{(\mathtt{t}_k-s)^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}}ds, \tag{2.24}$$
$$= \frac{1}{\Gamma(1-\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}\sum_{j=1}^{k}\int_{\mathtt{t}_{j-1}}^{\mathtt{t}_j}\frac{\mathtt{f}'(\mathrm{x}_\iota,s)}{(\mathtt{t}_k-s)^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}}ds. \tag{2.25}$$

The basic idea behind L-123 approximation is that, we use the linear interpolation $\mathcal{P}_{1,j}\mathtt{f}(\mathrm{x},\mathtt{t})$ to approximate $\mathtt{f}(\mathrm{x},\mathtt{t})$ in the first sub-interval $[\mathtt{t}_0,\mathtt{t}_1]$, and quadratic interpolation $\mathcal{P}_{2,j}\mathtt{f}(\mathrm{x},\mathtt{t})$ in the second sub-interval $[\mathtt{t}_1,\mathtt{t}_2]$ and cubic interpolation $\mathcal{P}_{3,j}\mathtt{f}(\mathrm{x},\mathtt{t})$ in the intervals $[\mathtt{t}_{j-1},\mathtt{t}_j]$ for $j\ge 3$. Noticing,

$$\int_{\mathtt{t}_{j-1}}^{\mathtt{t}_j}\varphi_j(s)(\mathtt{t}_k-s)^{-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}ds = \frac{6\tau^{3-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}}{1-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\beta_{k-j}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}, \tag{2.26}$$

where

$$\beta_j^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} = -\Bigg(\frac{1}{6}((j+1)^{1-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} + 2j^{1-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}) + \frac{1}{2-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} j^{2-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}, \\ - \frac{1}{(2-\alpha(\mathrm{x}_\iota,\mathtt{t}_k))(3-\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}((j+1)^{3-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} - j^{3-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)})\Bigg),\ j \geq 0. \tag{2.27}$$

Then the numerical approach of the order $\alpha(\mathrm{x},\mathtt{t})$ derivative of the variable order Caputo fractional for the function $\mathsf{f}(\mathrm{x},\mathtt{t})$ is provided by:

$$ {}_0^C D_{\mathtt{t}}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} \mathsf{f}(\mathrm{x},\mathtt{t})|_{\mathtt{t}=\mathtt{t}_k} = \frac{1}{\Gamma(1-\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} \sum_{j=1}^{k} \int_{\mathtt{t}_{j-1}}^{\mathtt{t}_j} \frac{\mathsf{f}'(\mathrm{x}_\iota,s)}{(\mathtt{t}_k - s)^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}} ds \tag{2.28}$$

$$\approx \frac{1}{\Gamma(1-\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} \Bigg(\int_{\mathtt{t}_0}^{\mathtt{t}_1} (\mathtt{t}_k - s)^{-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} (\mathcal{P}_{1,1}\mathsf{f}(\mathrm{x},s))' ds \\ + \int_{\mathtt{t}_1}^{\mathtt{t}_2} (\mathtt{t}_k - s)^{-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} (\mathcal{P}_{2,2}\mathsf{f}(\mathrm{x},s))' ds \\ + \sum_{j=3}^{k} \int_{\mathtt{t}_{j-1}}^{\mathtt{t}_j} (\mathtt{t}_k - s)^{-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} (\mathcal{P}_{3,j}\mathsf{f}(\mathrm{x},s))' ds \Bigg) \tag{2.29}$$

$$= \frac{1}{\Gamma(1-\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} \Bigg(\delta_{\mathtt{t}} \mathsf{f}(\mathrm{x}_\iota,\mathtt{t}_{\frac{1}{2}}) \int_{\mathtt{t}_0}^{\mathtt{t}_1} (\mathtt{t}_k - s)^{-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} ds + \int_{\mathtt{t}_1}^{\mathtt{t}_2} (\mathtt{t}_k - s)^{-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} \\ (\delta_{\mathtt{t}} \mathsf{f}(\mathrm{x}_\iota,\mathtt{t}_{\frac{3}{2}}) + \delta_{\mathtt{t}}^2 \mathsf{f}_1 (s - \mathtt{t}_{\frac{3}{2}})) ds + \sum_{j=3}^{k} \int_{\mathtt{t}_{j-1}}^{\mathtt{t}_j} (\mathtt{t}_k - s)^{-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} (\delta_{\mathtt{t}} \mathsf{f}(\mathrm{x}_\iota,\mathtt{t}_{j-\frac{1}{2}}) \\ + \delta_{\mathtt{t}}^2 \mathsf{f}_{j-1}(s - \mathtt{t}_{j-\frac{1}{2}})) + \frac{1}{6}\varphi_j(s) \delta_{\mathtt{t}}^3 \mathsf{f}_{j-\frac{3}{2}} ds \Bigg) \tag{2.30}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(1-\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}\Bigg(\sum_{j=1}^{k}\delta_{\mathtt{t}}\mathsf{f}(\mathrm{x}_\iota,\mathtt{t}_{j-1/2})\int_{\mathtt{t}_{j-1}}^{\mathtt{t}_j}(\mathtt{t}_k-s)^{-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}ds \\
&+\sum_{j=2}^{k}\delta_{\mathtt{t}}^2\mathsf{f}(\mathrm{x}_\iota,\mathtt{t}_{j-1})\int_{\mathtt{t}_{j-1}}^{\mathtt{t}_j}(\mathtt{t}_k-s)^{-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}(s-\mathtt{t}_{j-\frac{1}{2}})ds \\
&+\sum_{j=3}^{k}\frac{1}{6}(\delta_{\mathtt{t}}^3\mathsf{f}(\mathrm{x}_\iota,\mathtt{t}_{j-\frac{3}{2}}))\int_{\mathtt{t}_{j-1}}^{\mathtt{t}_j}(\mathtt{t}_k-s)^{-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\varphi_j(s)ds\Bigg). \qquad (2.31)
\end{aligned}
$$

In accordance with [186], $D_{\mathtt{t}}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\mathsf{f}(\mathrm{x},\mathtt{t})$ is the traditional L1 operator and is generated from an estimate of $\mathsf{f}(\mathrm{x},\mathtt{t})$ using piecewise linear interpolation on each subinterval $[\mathtt{t}_{j-1},\mathtt{t}_j],(1\le j\le k)$ defined by

$$
\begin{aligned}
D_{\mathtt{t}}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\mathsf{f}(\mathrm{x},\mathtt{t})\Big|_{\mathtt{t}=\mathtt{t}_k} &= \frac{\tau^{1-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}}{\Gamma(2-\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}\sum_{j=1}^{k}a_{k-j}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\delta_{\mathtt{t}}\mathsf{f}(\mathrm{x}_\iota,\mathtt{t}_{j-\frac{1}{2}}), \qquad (2.32)\\
&= \frac{\tau^{-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}}{\Gamma(2-\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}\Bigg(\mathsf{f}(\mathrm{x}_\iota,\mathtt{t}_k)-\sum_{j=1}^{k-1}(a_{k-j-1}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}-a_{k-j}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}) \\
&\qquad \mathsf{f}(\mathrm{x}_\iota,\mathtt{t}_j)-a_{k-1}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\mathsf{f}(\mathrm{x}_\iota,\mathtt{t}_0)\Bigg), \qquad (2.33)
\end{aligned}
$$

with

$$
a_j^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} = (j+1)^{1-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} - j^{1-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}, \quad 0\le j\le k-1,
$$

and the Lagrange interpolation approximation of $\mathsf{f}(\mathrm{x},\mathtt{t})$ on each tiny interval $[\mathtt{t}_{j-1},\mathtt{t}_j]$, $(1\le j\le k)$ yields the L1-2 operator, which is indicated by $\mathbb{D}_{\mathtt{t}}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\mathsf{f}(\mathrm{x},\mathtt{t})$.

$$
\begin{aligned}
\mathbb{D}_{\mathtt{t}}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\mathsf{f}(\mathrm{x},\mathtt{t})\Big|_{\mathtt{t}=\mathtt{t}_k} &= D_{\mathtt{t}}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\mathsf{f}(\mathrm{x},\mathtt{t})\Big|_{\mathtt{t}=\mathtt{t}_k} + \frac{\tau^{2-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}}{\Gamma(2-\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}\sum_{j=2}^{k}b_{k-j}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} \\
&\quad \delta_{\mathtt{t}}^2\mathsf{f}(\mathrm{x}_\iota,\mathtt{t}_{j-1}) \qquad (2.34)
\end{aligned}
$$

with

$$
b_j^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} = \frac{(j+1)^{2-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} - j^{2-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}}{(2-\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} - \frac{(j+1)^{1-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} + j^{1-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}}{2}, \; j\ge 0.
$$

Now we define

$$\tilde{\mathbb{D}}_{\mathtt{t}}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\mathtt{f}(\mathrm{x},\mathtt{t})\Big|_{\mathtt{t}=\mathtt{t}_k} = \mathbb{D}_{\mathtt{t}}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\mathtt{f}(\mathrm{x},\mathtt{t})\Big|_{\mathtt{t}=\mathtt{t}_k} + \frac{\tau^{3-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}}{\Gamma(2-\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}\sum_{j=3}^{k}\beta_{k-j}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\delta_{\mathtt{t}}^3\mathtt{f}(\mathrm{x}_\iota,\mathtt{t}_{j-\frac{3}{2}}). \tag{2.35}$$

The operator $\tilde{\mathbb{D}}_{\mathtt{t}}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}$ is the L-123 operator for the variable-order Caputo derivative (VOCD) ${}_0^C D_{\mathtt{t}}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\mathtt{f}(\mathrm{x},\mathtt{t})$. The new L-123 formula (2.35) for VOCD can be rewritten as

$$\begin{aligned}
\tilde{\mathbb{D}}_{\mathtt{t}}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\mathtt{f}(\mathrm{x},\mathtt{t})\Big|_{\mathtt{t}=\mathtt{t}_k} &= \frac{\tau^{1-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}}{\Gamma(2-\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}\Bigg(\sum_{j=1}^{k} a_{k-j}^{(\mathrm{x}_\iota,\mathtt{t}_k)}\delta_{\mathtt{t}}\mathtt{f}(\mathrm{x}_\iota,\mathtt{t}_{j-\frac{1}{2}}) \\
&\quad + \sum_{j=2}^{k} b_{k-j}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}(\delta_{\mathtt{t}}\mathtt{f}(\mathrm{x}_\iota,\mathtt{t}_{j-\frac{1}{2}}) - \delta_{\mathtt{t}}\mathtt{f}(\mathrm{x}_\iota,\mathtt{t}_{j-\frac{3}{2}})) \\
&\quad + \sum_{j=3}^{k} \beta_{k-j}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}(\delta_{\mathtt{t}}\mathtt{f}(\mathrm{x}_\iota,\mathtt{t}_{j-\frac{1}{2}}) - 2\delta_{\mathtt{t}}\mathtt{f}(\mathrm{x}_\iota,\mathtt{t}_{j-\frac{3}{2}}) + \delta_{\mathtt{t}}\mathtt{f}(\mathrm{x}_\iota,\mathtt{t}_{j-\frac{5}{2}}))\Bigg)
\end{aligned} \tag{2.36}$$

$$= \frac{\tau^{1-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}}{\Gamma(2-\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}\sum_{j=1}^{k}\gamma_{k-j}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\delta_{\mathtt{t}}\mathtt{f}(\mathrm{x}_\iota,\mathtt{t}_{j-\frac{1}{2}}) \tag{2.37}$$

$$\begin{aligned}
&= \frac{\tau^{-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}}{\Gamma(2-\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}\Bigg(\gamma_0^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\mathtt{f}(\mathrm{x}_\iota,\mathtt{t}_k) - \sum_{j=1}^{k-1}(\gamma_{k-j-1}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} - \gamma_{k-j}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}) \\
&\quad \mathtt{f}(\mathrm{x}_\iota,\mathtt{t}_j) - \gamma_{k-1}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\mathtt{f}(\mathrm{x}_\iota,\mathtt{t}_0)\Bigg)
\end{aligned} \tag{2.38}$$

where, for $k=1$, $\gamma_0^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} = 1$;

for $k=2$, $\gamma_0^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} = a_0^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} + b_0^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} \in (1, 1.5)$, $\gamma_1^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} = a_1^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} - b_0^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} \in (-0.5, 1)$;

for $k = 3$,

$$\gamma_l^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} = \begin{cases} a_l^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} + b_l^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} + \beta_l^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}, & l = 0, \\ a_l^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} + b_l^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} - b_{j-1}^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} - 2\beta_{l-1}^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}, & l = 1, \\ a_l^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} - b_{l-1}^{(\alpha)} + \beta_{l-2}^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}, & l = 2. \end{cases} \tag{2.39}$$

Here, $\gamma_0^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} \in (1, 11/6)$, $\gamma_1^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} \in (-7/6, 1)$, and $\gamma_2^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} \in (0, 1)$.

For $k \geq 4$,

$$\gamma_l^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} = \begin{cases} a_l^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} + b_l^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} + \beta_l^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}, \quad l = 0, \\ a_l^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} + b_l^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} - b_{l-1}^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} + \beta_l^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} - 2\beta_{l-1}^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}, \quad l = 1, \\ a_l^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} + b_l^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} - b_{l-1}^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} + \beta_l^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} - 2\beta_{l-1}^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} + \beta_{l-2}^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}, \\ \hfill 2 \leq l \leq k-3, \\ a_l^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} + b_l^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} - b_{l-1}^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} - 2\beta_{l-1}^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} + \beta_{l-2}^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}, \quad l = k-2, \\ a_l^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} - b_{l-1}^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} + \beta_{l-2}^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}, \quad l = k-1. \end{cases} \tag{2.40}$$

The properties of coefficient $\beta_j^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}$ defined in (2.27) is given below.

Lemma 2.1.1 ([185]). For any $\alpha(\mathrm{x}_\iota,\mathtt{t}_k)$ $(0 < \alpha(\mathrm{x}_\iota,\mathtt{t}_k) < 1)$, we have

1. $\beta_j^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} \geq 0, \ j \leq 0;$
2. $\beta_j^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}$ is strictly monotonically decreasing for $j = 0, 1, 2, \ldots, k-1$.

Lemma 2.1.2 ([185]). For any $\alpha(\mathrm{x}_\iota,\mathtt{t}_k)$ $(0 < \alpha(\mathrm{x}_\iota,\mathtt{t}_k) < 1)$, and $\gamma_j^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}$ $(0 \leq j \leq k-1, k \geq 4)$ defined in (2.40), it holds

1. $\gamma_0^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} \geq |\gamma_1^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}|;$

2. $\gamma_j^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} > 0,\ j \neq 1;$

3. $\gamma_2^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} \geq \gamma_3^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} \geq \cdots \geq \gamma_{k-1}^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))};$

4. $\gamma_0^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} > \gamma_2^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))};$

5. $\sum_{j=0}^{k-1} \gamma_0^{(\alpha(\mathrm{x}_\iota,\mathtt{t}_k))} = k^{(1-\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}.$

Proof. Similar to as given in [185]. □

Theorem 2.1.1. Suppose $\mathsf{f}(\mathrm{x},\mathtt{t}) \in C^{2,4}_{\mathrm{x},\mathtt{t}}([0,L]\times[0,\mathtt{t}_k])$. For any α, $\tilde{\mathbb{D}}_{\mathtt{t}}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\mathsf{f}(\mathrm{x},\mathtt{t})|_{\mathtt{t}=\mathtt{t}_k}$ is defined in (2.36). Let $R(\mathsf{f}(\mathrm{x}_\iota,\mathtt{t}_k))$ denote the truncation error, then $|R(\mathsf{f}(\mathrm{x}_\iota,\mathtt{t}_k))| = {}_0^C D_{\mathtt{t}}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\mathsf{f}(\mathrm{x},\mathtt{t})|_{\mathtt{t}=\mathtt{t}_k} - \tilde{\mathbb{D}}_{\mathtt{t}}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\mathsf{f}(\mathrm{x},\mathtt{t})|_{\mathtt{t}=\mathtt{t}_k}$. Then we have

$$|R(\mathsf{f}(\mathrm{x}_\iota,\mathtt{t}_1))| \leq \frac{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}{2\Gamma(3-\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}\Delta\mathtt{t}^{2-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} \max_{\mathtt{t}_0\leq\mathtt{t}\leq\mathtt{t}_1} \mathsf{f}''(\mathrm{x},\mathtt{t}), \tag{2.41}$$

$$\begin{aligned}|R(\mathsf{f}(\mathrm{x}_\iota,\mathtt{t}_2))| \leq & \frac{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}{\Gamma(1-\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}\Bigg(\frac{1}{12}\max_{\mathtt{t}_0\leq\mathtt{t}\leq\mathtt{t}_1}|\mathsf{f}''(\mathrm{x},\mathtt{t})|(\mathtt{t}_2-\mathtt{t}_1)^{-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)-1}\Delta\mathtt{t}^3 \\ & + \frac{1}{3(1-\alpha(\mathrm{x}_\iota,\mathtt{t}_k))(2-\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}\left(\frac{1}{2}+\frac{1}{3-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\right) \\ & \max_{\mathtt{t}_0\leq\mathtt{t}\leq\mathtt{t}_2}|\mathsf{f}'''(\mathrm{x},\mathtt{t})|\Delta\mathtt{t}^{3-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\Bigg),\end{aligned} \tag{2.42}$$

$$\begin{aligned}|R(\mathsf{f}(\mathrm{x}_\iota,\mathtt{t}_k))| \leq & \frac{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}{\Gamma(1-\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}\Bigg(12(\mathtt{t}_k-\mathtt{t}_1)^{-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)-1}\max_{\mathtt{t}_0\leq\mathtt{t}\leq\mathtt{t}_1}|\mathsf{f}''(\mathrm{x},\mathtt{t})|\Delta\mathtt{t}^3 \\ & + \frac{1}{8}(\mathtt{t}_k-\mathtt{t}_2)^{-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)-1}\max_{\mathtt{t}_0\leq\mathtt{t}\leq\mathtt{t}_2}|\mathsf{f}'''(\mathrm{x},\mathtt{t})|\Delta\mathtt{t}^4 \\ & + \left(\frac{1}{2}+\frac{1}{12}\frac{27-10\alpha(\mathrm{x}_\iota,\mathtt{t}_k)+\alpha(\mathrm{x}_\iota,\mathtt{t}_k)^2}{\mathcal{P}_{\iota=1}^4(\alpha(\mathrm{x}_\iota,\mathtt{t}_k)-\iota)}\right) \\ & \max_{\mathtt{t}_0\leq\mathtt{t}\leq\mathtt{t}_k}|\mathsf{f}^4(\mathrm{x},\mathtt{t})|\Delta\mathtt{t}^{4-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\Bigg), \quad k\geq 3.\end{aligned} \tag{2.43}$$

Proof. Similar to given in [185]. □

2.2 Construction of numerical scheme for variable-order time-fractional reaction-subdiffusion equation

Consider the rectangular domain $\Omega(\mathrm{x},\mathtt{t}) = [0,L]\times[0,\mathcal{T}]$ is discretized by using uniform mesh with step length $h_{\mathrm{x}} = L/\mathcal{M}$ and $\tau = \mathcal{T}/\mathcal{N}$ in spatial and time direction respectively. $\mathrm{x}_\iota = \iota h, \iota = 0,1\dots\mathcal{M}$ stands for the space grid points, and $\mathtt{t}_k = k\tau, k = 0,1\dots\mathcal{N}$ for the time grid points. The numerical solution at the any grid point $(\mathrm{x}_\iota,\mathtt{t}_k)$ can be written as

$$
{}_0^C D_{\mathtt{t}}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\mathsf{f}(\mathrm{x},\mathtt{t})|_{\mathtt{t}=\mathtt{t}_k} = \kappa_1 \mathsf{y}_{\mathrm{xx}}(\mathrm{x}_\iota,\mathtt{t}_k) - \kappa_2\mathsf{y}(\mathrm{x}_\iota,\mathtt{t}_k) + g_\iota^k \tag{2.44}
$$

where $g_\iota^k = g(\mathrm{x}_\iota,\mathtt{t}_k)$ and the boundaries are $\mathsf{y}_\iota^0 = \varphi(\mathrm{x}_\iota), i = 0,1,2\dots\mathcal{M}$; $\mathsf{y}_0^k = \psi_0(\mathtt{t}_k)$; $\mathsf{y}_M^k = \psi_L(\mathtt{t}_k)$, $k = 1,2,\dots,\mathcal{N}$. The central difference method can be used to approximate the second-order spatial derivative as

$$
\mathsf{y}_{\mathrm{xx}}(\mathrm{x}_\iota,\mathtt{t}_k) = \frac{\mathsf{y}(\mathrm{x}_{i+1},\mathtt{t}_k) - 2\mathsf{y}(\mathrm{x}_\iota,\mathtt{t}_k) + \mathsf{y}(\mathrm{x}_{i-1},\mathtt{t}_k)}{h_{\mathrm{x}}^2} + O(h_{\mathrm{x}}^2). \tag{2.45}
$$

Now, using the L-123 approximation (2.37) to approximate the Caputo derivative of variable order in (2.44).

$$
\frac{\tau^{-\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}}{\Gamma(2-\alpha(\mathrm{x}_\iota,\mathtt{t}_k))}\sum_{j=0}^{k-1}\gamma_{i,k-j}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}[\mathsf{y}_\iota^{j+1} - \mathsf{y}_\iota^j] = \kappa_1\left[\frac{\mathsf{y}_{i+1}^k - 2\mathsf{y}_\iota^k + \mathsf{y}_{i-1}^k}{h_{\mathrm{x}}^2}\right] - \kappa_2\mathsf{y}_\iota^k + g_\iota^k \tag{2.46}
$$

$$\frac{\tau^{-\alpha(x_\iota,\mathtt{t}_k)}}{\Gamma(2-\alpha(x_\iota,\mathtt{t}_k))}\left[\gamma_0^{\alpha(x_\iota,\mathtt{t}_k)}\mathsf{y}_\iota^k-\sum_{j=1}^{k-1}(\gamma_{k-j-1}^{\alpha(x_\iota,\mathtt{t}_k)}-\gamma_{k-j}^{\alpha(x_\iota,\mathtt{t}_k)})\mathsf{y}_\iota^j-\gamma_{k-1}^{\alpha(x_\iota,\mathtt{t}_k)}\mathsf{y}_\iota^0\right]$$
$$=\kappa_1\frac{\mathsf{y}_{i+1}^k}{h_x^2}-(\frac{2}{h_x^2}\kappa_1+\kappa_2)\mathsf{y}_\iota^k+\kappa_1\mathsf{y}_{i-1}^k h_x^2+g_\iota^k \tag{2.47}$$

$$\mathsf{y}_\iota^k-\frac{1}{\gamma_0^{\alpha(x_\iota,\mathtt{t}_k)}}\left(\sum_{j=1}^{k-1}\left(\gamma_{k-j-1}^{\alpha(x_\iota,\mathtt{t}_k)}-\gamma_{k-j}^{\alpha(x_\iota,\mathtt{t}_k)}\right)\mathsf{y}_\iota^j+\gamma_{k-1}^{\alpha(x_\iota,\mathtt{t}_k)}\mathsf{y}_\iota^o\right)=\mu(x_\iota,\mathtt{t}_k)\left(\kappa_1\frac{\mathsf{y}_{i+1}^k}{h_x^2}\right.$$
$$\left.-(\frac{2}{h_x^2}\kappa_1+\kappa_2)\mathsf{y}_\iota^k+\kappa_1\mathsf{y}_{i-1}^k h_x^2\right)+\mu(x_\iota,\mathtt{t}_k)g_\iota^k, \tag{2.48}$$

here

$$\mu(x_\iota,\mathtt{t}_k)=\frac{\Gamma(2-\alpha(x_\iota,\mathtt{t}_k))}{\tau^{-\alpha(x_\iota,\mathtt{t}_k)}\gamma_0^{\alpha(x_\iota,\mathtt{t}_k)}}. \tag{2.49}$$

Then the designed scheme 2.48 can be written as,

$$\mathsf{y}_\iota^k-\mu(x_\iota,\mathtt{t}_k)\left(\kappa_1\frac{\mathsf{y}_{i+1}^k}{h_x^2}-(\frac{2}{h_x^2}\kappa_1+\kappa_2)\mathsf{y}_\iota^k+\kappa_1\frac{\mathsf{y}_{i-1}^k}{h_x^2}\right)$$
$$=\frac{1}{\gamma_0^{\alpha(x_\iota,\mathtt{t}_k)}}\left(\sum_{j=1}^{k-1}\left(\gamma_{k-j-1}^{\alpha(x_\iota,\mathtt{t}_k)}-\gamma_{k-j}^{\alpha(x_\iota,\mathtt{t}_k)}\right)\mathsf{y}_\iota^j+\gamma_{k-1}^{\alpha(x_\iota,\mathtt{t}_k)}\mathsf{y}_\iota^o\right)+\mu(x_\iota,\mathtt{t}_k)g_\iota^k. \tag{2.50}$$

Writing above scheme 2.50 in matrix form, we get

$$\mathsf{Y}^k+\mathbb{B}\mathbb{E}\mathsf{Y}^k=\frac{1}{\gamma_0^{\alpha(x_\iota,\mathtt{t}_k)}}\left[\sum_{j=1}^{k-1}\left(\gamma_{k-j-1}^{\alpha(x_\iota,\mathtt{t}_k)}-\gamma_{k-j}^{\alpha(x_\iota,\mathtt{t}_k)}\right)\mathsf{Y}^j+\gamma_{k-1}^{\alpha(x_\iota,\mathtt{t}_k)}\mathsf{Y}^0\right]+\mathbb{B}G^k. \tag{2.51}$$

Where $\mathsf{Y}^k = [u_1^k, u_2^k, \ldots, u_{\mathcal{M}-1}^k]^{\mathcal{T}}$ and $G^k = [g_1^k, g_2^k, \ldots, g_{\mathcal{M}-1}^k]^{\mathcal{T}}$.

$$\mathbb{B} = \begin{bmatrix} \mu(\mathrm{x}_1, \mathtt{t}_k) & 0 & \ldots & 0 \\ 0 & \mu(\mathrm{x}_2, \mathtt{t}_k) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mu(\mathrm{x}_{\mathcal{M}-1}, \mathtt{t}_k) \end{bmatrix}_{\{\mathcal{M}-1\times\mathcal{M}-1\}}. \tag{2.52}$$

$$\mathbb{E} = \begin{bmatrix} \frac{2\kappa_1}{h_{\mathrm{x}}^2} + \kappa_2 & -\frac{\kappa_1}{h_{\mathrm{x}}^2} & 0 & \ldots & 0 \\ -\frac{\kappa_1}{h_{\mathrm{x}}^2} & \frac{2\kappa_1}{h_{\mathrm{x}}^2} + \kappa_2 & -\frac{\kappa_1}{h_{\mathrm{x}}^2} & \ldots & 0 \\ 0 & -\frac{\kappa_1}{h_{\mathrm{x}}^2} & \frac{2\kappa_1}{h_{\mathrm{x}}^2} + \kappa_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \frac{2\kappa_1}{h_{\mathrm{x}}^2} + \kappa_2 \end{bmatrix}_{\{\mathcal{M}-1\times\mathcal{M}-1\}}. \tag{2.53}$$

2.3 Stability of the numerical scheme

Theorem 2.3.1. [187] The numerical scheme (2.50) is unconditionally stable. i.e.

$$\|\xi^k\|_2 \leq \|\xi^0\|_0. \tag{2.54}$$

Proof. Suppose ${\mathsf{Y}_\iota}^n$ be the approximate solution of numerical scheme and ${\bar{\mathsf{y}}_\iota}^n$ be the exact solution. The error in the numerical scheme is denoted by $\xi_\iota^k = {\bar{\mathsf{y}}_\iota}^k - {\mathsf{Y}_\iota}^k$. This error will also satisfy the difference scheme (2.51)

$$\xi^k + \mathbb{B}\mathbb{E}\xi^k = \frac{1}{\gamma_0^{\alpha(\mathrm{x}_\iota, \mathtt{t}_k)}} \left[\sum_{j=1}^{k-1} \left(\gamma_{k-j-1}^{\alpha(\mathrm{x}_\iota, \mathtt{t}_k)} - \gamma_{k-j}^{\alpha(\mathrm{x}_\iota, \mathtt{t}_k)} \right) \xi^j + \gamma_{k-1}^{\alpha(\mathrm{x}_\iota, \mathtt{t}_k)} \xi^0 \right]. \tag{2.55}$$

Since $\mathbb{E}$ is a symmetric matrix, thus there exist an orthogonal matrix Q such that

$$\mathbb{E} = Q^{\mathcal{T}}\Lambda Q, \tag{2.56}$$

where the transposed version of matrix Q is $Q^{\mathcal{T}}$, and $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_{\mathcal{M}-1})$ is a diagonal matrix with eigen values of $\mathbb{E}$. Using (2.56) and left multiplying both side of (2.55) by Q we get,

$$Q\xi^k + \mathbb{B}QQ^{\mathcal{T}}\Lambda Q\xi^k = \frac{1}{\gamma_0^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}}\left[\sum_{j=1}^{k-1}\left(\gamma_{k-j-1}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} - \gamma_{k-j}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\right)Q\xi^j + \gamma_{k-1}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}Q\xi^0\right]. \tag{2.57}$$

Let $Q\xi^k = \bar{\xi}^k$. As Q is unitary, it follows that $\|\xi^k\|_2 = \|\bar{\xi}^k\|_2$, thus the theorem will be proven if we state that

$$\|\bar{\xi}^k\|_2 \leq \|\bar{\xi}^0\|_0, \tag{2.58}$$

for each $i, i = 1, 2, \ldots, \mathcal{M}-1$ in (2.57), we obtain

$$(1 + \mu(\mathrm{x}_\iota, \mathtt{t}_k)\lambda_\iota)\,\bar{\xi}^k = \frac{1}{\gamma_0^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}}\left[\sum_{j=1}^{k-1}\left(\gamma_{k-j-1}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} - \gamma_{k-j}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\right)\bar{\xi}^j + \gamma_{k-1}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\bar{\xi}^0\right]. \tag{2.59}$$

From (2.59), we now demonstrate that the following inequality applies for each instance of $\iota, \iota = 1, 2, \ldots, \mathcal{M}-1$,

$$|\bar{\xi}_\iota^k| \leq |\bar{\xi}_\iota^0|, \tag{2.60}$$

for $k = 1$, we clearly have

$$(1 + \mu(\mathrm{x}_\iota, \mathtt{t}_1)\lambda_\iota)\,\bar{\xi}^1 = \bar{\xi}^0. \tag{2.61}$$

$$|\bar{\xi}^1| = \frac{1}{|1 + \mu(\mathrm{x}_\iota, \mathtt{t}_1)\lambda_\iota|}|\bar{\xi}^0|, \tag{2.62}$$

we have

$$\frac{1}{|1 + \mu(\mathrm{x}_\iota, \mathtt{t}_1)\lambda_\iota|} \leq 1, \tag{2.63}$$

$$\Rightarrow |\bar{\xi}_j^1| \leq |\bar{\xi}_j^0|. \tag{2.64}$$

Now, suppose the inequality as below is true for $j = 0, 1, 2, \ldots, k-1$

$$|\bar{\xi}_\iota^j| \leq |\bar{\xi}_\iota^0|. \tag{2.65}$$

Consequently, using mathematical induction, we arrive at the following conclusion for $j = k$.

$$|\bar{\xi^k}| = \frac{1}{(1+\mu(\mathrm{x}_\iota, \mathtt{t}_k)\lambda_\iota)\,\gamma_0^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}} \left| \sum_{j=1}^{k-1} \left(\gamma_{k-j-1}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} - \gamma_{k-j}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\right)\bar{\xi^j} + \gamma_{k-1}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\bar{\xi^0} \right|, \tag{2.66}$$

$$\leq \frac{1}{(1+\mu(\mathrm{x}_\iota, \mathtt{t}_k)\lambda_\iota)\,\gamma_0^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}} \left| \sum_{j=1}^{k-1} \left(\gamma_{k-j-1}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} - \gamma_{k-j}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\right) \right| \left|\bar{\xi^j}\right| + \left|\gamma_{k-1}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\right| \left|\bar{\xi^0}\right|, \tag{2.67}$$

$$= \frac{1}{(1+\mu(\mathrm{x}_\iota, \mathtt{t}_k)\lambda_\iota)\,\gamma_0^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}} \left| \sum_{j=1}^{k-1} \left(\gamma_{k-j-1}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)} - \gamma_{k-j}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\right) \right| \left|\bar{\xi^0}\right| + \left|\gamma_{k-1}^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}\right| \left|\bar{\xi^0}\right|, \tag{2.68}$$

$$= \frac{1}{(1+\mu(\mathrm{x}_\iota, \mathtt{t}_k)\lambda_\iota)\,\gamma_0^{\alpha(\mathrm{x}_\iota,\mathtt{t}_k)}} \left|\bar{\xi^0}\right|, \tag{2.69}$$

which means

$$\left|\bar{\xi^k}\right| \leq \left|\bar{\xi^0}\right|, \;\; k = 0, 1, \ldots, \mathcal{N}. \tag{2.70}$$

Result (2.70) implies that

$$\|\bar{\xi^k}\|_2 \leq \|\bar{\xi^0}\|_0, \;\; k = 0, 1, \ldots, \mathcal{N}. \tag{2.71}$$

The system is therefore unconditionally stable. □

The algorithm for solving by the proposed numerical scheme (2.51) is given below.

Algorithm 1: Assessment of the VO-TFRSDE numerical solution (2.1)-(2.3)

Input: The domain $\Omega = [0, L] \times [0, \mathcal{T}]$, $\mathcal{M}$, $h_{\mathrm{x}} = L/\mathcal{M}$, $\mathcal{N}$, $\tau = \mathcal{T}/\mathcal{N}$, initial condition $\varphi(\mathrm{x})$, boundary conditions $\psi_0(\mathtt{t})$, $\psi_L(\mathtt{t})$ and $\alpha(\mathrm{x}_\iota, \mathtt{t}_k) \in (0, 1)$

Output: The numerical solution at each discretization point $(\mathrm{x}_\iota, \mathtt{t}_k)$

for Numerical solution of by difference scheme **do**

- **Step-1.1** Consider the step length in time and space variables $\mathtt{t}_k = k\tau,\ k = 0, 1, \ldots, \mathcal{N},\ \mathrm{x}_\iota = \iota h_{\mathrm{x}},\ \iota = 0, 1, \ldots, \mathcal{M}$.
- **Step-1.2** Convert the problem (2.1)-(2.3) into the problem (2.10)-(2.12) by using the transformation (2.6)-(2.8).
- **Step-1.3** Apply L-123 approximation (2.36) to discretized the VOCD in time and the central difference scheme in spatial direction.
- **Step-1.4** Use the boundary conditions $\mathtt{y}(0, \mathtt{t}_k) = \psi_0(\mathtt{t}_k)$, and $\mathtt{y}(L, \mathtt{t}_k) = \psi_L(\mathtt{t}_k)$ to calculate the value of $\mathtt{y}_\iota^0$ and $\mathtt{y}_{\mathcal{M}}^{\mathcal{N}}$, respectively.
- **Step-1.5** Calculate the numerical solution $\mathtt{y}_\iota^k,\ k \geq 1$, by using the difference scheme (2.50).
- **Step-1.6** Repeat step (1.4) and (1.5) and use all the values of $\mathtt{y}_\iota^k$ at each previous time levels till we get the discrete solution of at each discretised points $(\mathrm{x}_\iota, \mathtt{t}_k)$.

end

2.4 Numerical results

In this section, we give the numerical example of the VO-TFRSDE to demonstrate the effectiveness of the numerical scheme. The accuracy of the proposed scheme for both cases is demonstrated by the following error norms

$$||\mathsf{y} - \mathsf{Y}|| = \begin{cases} \left(\sum_{i=1}^{N} h|\mathsf{y}(\mathrm{x}_i, T) - \mathsf{Y}(\mathrm{x}_i, T)|^2 \right)^{1/2}, & L_2 \ Norm \\ max_{0 \le i \le N} |\mathsf{y}(\mathrm{x}_i, T) - \mathsf{Y}(\mathrm{x}_i, T)|, & L_\infty \ Norm, \end{cases} \tag{2.72}$$

where $\mathsf{y}(\mathrm{x}, \mathtt{t})$ and $\mathsf{Y}(\mathrm{x}, \mathtt{t})$ are exact and numerical solutions of the FDEs.

2.4.1 Numerical examples for the VO-TFRSDE

Example 2.1. Consider the VO-TFRSDE with homogeneous boundary conditions,

$$_0\mathrm{D}_{\mathtt{t}}^{\alpha(\mathrm{x},\mathtt{t})}\mathsf{y}(\mathrm{x}, \mathtt{t}) = \kappa_1 \mathsf{y}_{\mathrm{xx}}(\mathrm{x}, \mathtt{t}) + \kappa_2 \mathsf{y}(\mathrm{x}, \mathtt{t}) + \mathsf{f}(\mathrm{x}, \mathtt{t}), \quad (\mathrm{x}, \mathtt{t}) \in [0, 1] \times [0, 1], \tag{2.73}$$

$$\mathsf{y}(\mathrm{x}, 0) = 10\mathrm{x}^2(1 - \mathrm{x}), \quad \mathrm{x} \in [0, 1], \tag{2.74}$$

$$\mathsf{y}(0, \mathtt{t}) = \mathsf{y}(1, \mathtt{t}) = 0, \quad \mathtt{t} \in [0, 1]. \tag{2.75}$$

here we have $\kappa_1 = 1, \kappa_2 = -1,$ where the source function

$$\mathsf{f}(\mathrm{x}, \mathtt{t}) = 20\mathrm{x}^2(1 - \mathrm{x}) \left(\frac{\mathtt{t}^{2-\alpha(\mathrm{x},\mathtt{t})}}{\Gamma(3 - \alpha(\mathrm{x}, \mathtt{t}))} + \frac{\mathtt{t}^{1-\alpha(\mathrm{x},\mathtt{t})}}{\Gamma(2 - \alpha(\mathrm{x}, \mathtt{t}))} \right) - 20(\mathtt{t} + 1)^2(1 - 3\mathrm{x}) + 10\mathrm{x}^2(1 - \mathrm{x})(\mathtt{t} + 1)^2. \tag{2.76}$$

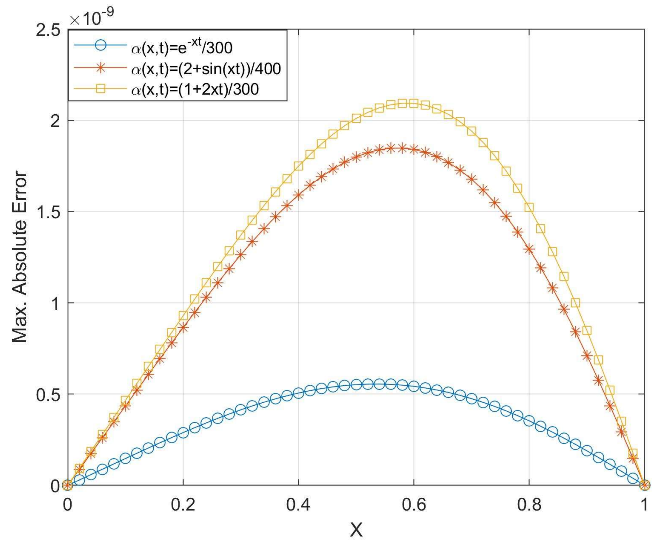

FIGURE 2.1: MAE of Ex. 2.1 for different $\alpha(\mathrm{x},\mathtt{t})$, $\tau = 1/40$, $h_{\mathrm{x}} = 1/50$.

The exact solution of VO-TFRSDE in Ex. 2.1 is,

$$\mathsf{y}(\mathrm{x},\mathtt{t}) = 10\mathrm{x}^2(1-\mathrm{x})(\mathtt{t}+1)^2. \tag{2.77}$$

The outcomes of Ex. 2.1 are as follows:

- Figure 2.1 represents the maximum absolute error (MAE) at time $\mathtt{t} = 1$ for distinct $\alpha(\mathrm{x},\mathtt{t})$, when $(h_{\mathrm{x}},\tau) = (1/50, 1/40)$. It can be observe that the MAE reached accuracy upto 10^{-9} on a very small grid point in x and $\mathtt{t}$ directions. The temporal order of convergence is reaches upto 3rd order (see Table 2.1).
- Figure 2.2 represents the maximum absolute error (MAE) at time $\mathtt{t} = 1$ for distinct $\alpha(\mathrm{x},\mathtt{t})$, when $h_{\mathrm{x}}^2 = \tau$ i.e. $(h_{\mathrm{x}},\tau) = (1/16, 1/256)$. In this case the accuracy reaches upto 10^{-12}. The spatial order of convergence reaches upto 6th order in this case (see Table 2.2).

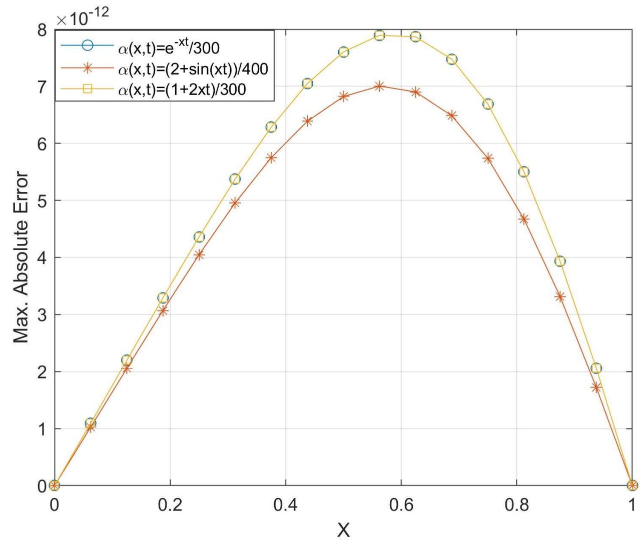

FIGURE 2.2: MAE of Ex. 2.1 for different $\alpha(\mathrm{x},\mathtt{t})$, ${h_{\mathrm{x}}}^2 = \tau$.

TABLE 2.1: L_2 error and L_∞ error of Ex. 2.1 at time $\mathtt{t} = 1$ when $h_{\mathrm{x}} = 1/50$.

$\alpha(\mathrm{x},\mathtt{t})$	τ	Scheme (2.51)		Scheme (2.51)	
		L_2 Error	Order	L_∞ Error	Order
$\frac{e^{-\mathrm{x}\mathtt{t}}}{300}$	1/4	4.413e-07	-	6.287e-07	-
	1/8	5.132e-08	3.104	7.311e-08	3.104
	1/16	6.205e-09	3.048	8.840e-09	3.048
	1/32	7.633e-10	3.023	1.087e-09	3.023
	1/64	9.472e-11	3.010	1.349e-10	3.010
	1/128	1.183e-11	3.001	1.685e-11	3.001
$\frac{2+\sin(\mathrm{x}\mathtt{t})}{400}$	1/4	1.464e-06	-	2.094e-06	-
	1/8	1.701e-07	3.104	2.434e-07	3.014
	1/16	2.057e-08	3.048	2.943e-08	3.048
	1/32	2.530e-09	3.023	3.620e-09	3.023
	1/64	3.139e-10	3.011	4.491e-10	3.011
	1/128	3.391e-11	3.004	5.593e-11	3.005

$\alpha(\mathrm{x}, \mathtt{t})$	τ	Scheme (2.51)		Scheme (2.51)	
		L_2 Error	Order	L_∞ Error	Order
$\dfrac{1+2\mathrm{x}\mathtt{t}}{300}$	1/4	1.652e-06	-	2.374e-06	-
	1/8	1.921e-07	3.104	2.759e-07	3.104
	1/16	2.321e-08	3.048	3.335e-08	3.048
	1/32	2.855e-09	3.023	4.102e-09	3.023
	1/64	3.541e-10	3.011	5.087e-10	3.011
$\dfrac{\mathrm{x}+\mathtt{t}}{2}$	1/4	3.495e-05	-	5.140e-05	-
	1/8	9.098e-06	1.941	1.304e-05	1.978
	1/16	1.145e-06	2.990	1.639e-06	2.991
	1/32	1.470e-07	2.960	2.105e-07	2.961
	1/64	1.951e-08	2.914	2.792e-08	2.914
$0.25+0.75\sin(\mathrm{x}\mathtt{t})$	1/4	7.834e-05	-	1.124e-04	-
	1/8	1.017e-05	2.944	1.458e-05	2.946
	1/16	1.235e-06	3.041	1.770e-06	3.042
	1/32	1.558e-07	2.987	2.231e-07	2.987
	1/64	2.011e-08	2.953	2.880e-08	2.953

TABLE 2.2: L_2 error and L_∞ error of Ex. 2.1 for different $\alpha(\mathrm{x},\mathtt{t})$ at time $\mathtt{t}=1$ when $h_{\mathrm{x}}^2=\tau$.

$\alpha(\mathrm{x},\mathtt{t})$	h	Scheme (2.51) L_2 Error	Order	Scheme (2.51) L_∞ Error	Order
$\frac{e^{-\mathrm{x}\mathtt{t}}}{300}$	1/4	6.489e-09	-	9.193e-09	-
	1/8	9.570e-11	6.083	1.356e-10	6.083
	1/16	1.471e-12	6.023	2.088e-12	6.020
$\frac{2+\sin(\mathrm{x}\mathtt{t})}{400}$	1/4	2.145e-08	-	2.976e-08	-
	1/8	3.173e-10	6.079	4.479e-10	6.053
	1/16	4.896e-12	6.018	7.004e-12	5.998
$\frac{1+2\mathrm{x}\mathtt{t}}{300}$	1/4	2.417e-08	-	3.310e-08	-
	1/8	3.581e-10	6.077	5.115e-10	6.015
	1/16	5.514e-12	6.020	7.894e-12	6.017
$\frac{\mathrm{x}+\mathtt{t}}{2}$	1/4	1.223e-06	-	1.754e-06	-
	1/8	1.984e-08	5.946	2.838e-08	5.949
	1/16	3.689e-10	5.748	5.271e-10	5.750
$0.25+0.75\sin(\mathrm{x}\mathtt{t})$	1/4	1.314e-06	-	1.874e-06	-
	1/8	2.043e-08	6.007	2.910e-08	6.008
	1/16	3.513e-10	5.861	5.004e-10	5.861

Example 2.2. Consider the VO-TFRSDE

$$ {}_0\mathrm{D}_{\mathtt{t}}^{\alpha(\mathrm{x},\mathtt{t})}\mathsf{y}(\mathrm{x},\mathtt{t}) = \kappa_1 \mathsf{y}_{\mathrm{xx}}(\mathrm{x},\mathtt{t}) + \kappa_2 \mathsf{y}(\mathrm{x},\mathtt{t}) + \mathsf{f}(\mathrm{x},\mathtt{t}), \quad (\mathrm{x},\mathtt{t}) \in [0,1]\times[0,1], \tag{2.78} $$

$$ \mathsf{y}(\mathrm{x},0) = \mathrm{x}(1-\mathrm{x}), \quad \mathrm{x}\in[0,1], \tag{2.79} $$

$$ \mathsf{y}(0,\mathtt{t}) = \mathsf{y}(1,\mathtt{t}) = 0, \quad \mathtt{t}\in[0,1]. \tag{2.80} $$

where the source function is

$$\mathtt{f}(\mathrm{x},\mathtt{t}) = \mathrm{x}(1-\mathrm{x})\frac{6t^{3-\alpha(\mathrm{x},\mathtt{t})}}{\Gamma(4-\alpha(\mathrm{x},\mathtt{t}))} + 2\mathtt{t}^3 + \mathrm{x}(1-\mathrm{x})\mathtt{t}^3 + \mathrm{x}(1-\mathrm{x})\mathtt{t}^3. \tag{2.81}$$

The exact solution of VO-TFRSDE in Ex. 2.2 is given by,

$$\mathtt{y}(\mathrm{x},\mathtt{t}) = \mathrm{x}(1-\mathrm{x})\mathtt{t}^3. \tag{2.82}$$

The outcomes of Ex. 2.2 are as follows:

- Figure 2.3 represents the maximum absolute error (MAE) at time $\mathtt{t} = 1$ for distinct $\alpha(\mathrm{x},\mathtt{t})$, when $(h_{\mathrm{x}}, \tau) = (1/50, 1/40)$. It can be observe that the MAE reached accuracy upto 10^{-11} on a very small grid point in x and $\mathtt{t}$ directions. The temporal order of convergence is reaches upto 4th order (see Table 2.3).
- Figure 2.4 represents the maximum absolute error (MAE) at time $\mathtt{t} = 1$ for distinct $\alpha(\mathrm{x},\mathtt{t})$, when $h_{\mathrm{x}}^2 = \tau$ i.e. $(h_{\mathrm{x}}, \tau) = (1/16, 1/256)$. In this case the accuracy reaches upto 10^{-13}. The spatial order of convergence reaches upto 8th order in this case (see Table 2.4).

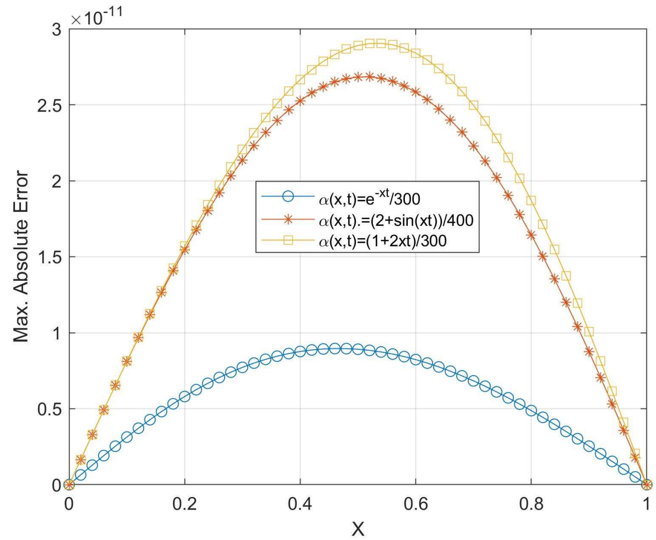

FIGURE 2.3: MAE of Ex. 2.2 for different $\alpha(\mathrm{x}, \mathtt{t})$, $\tau = 1/40$, $h_{\mathrm{x}} = 1/50$.

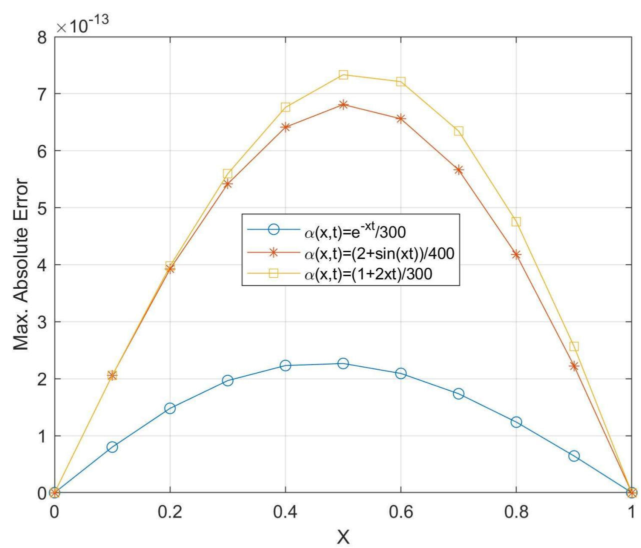

FIGURE 2.4: MAE of Ex. 2.2 for different $\alpha(\mathrm{x}, \mathtt{t})$, $h_{\mathrm{x}}^2 = \tau$.

TABLE 2.3: L_2 error and L_∞ error of Ex. 2.2 at time $\mathtt{t} = 1$ when $h_{\mathrm{x}} = 1/50$.

$\alpha(\mathrm{x},\mathtt{t})$	τ	Scheme (2.51)		Scheme (2.51)	
		L_2 Error	Order	L_∞ Error	Order
$\frac{e^{-\mathrm{x}\mathtt{t}}}{300}$	1/4	8.640e-08	-	1.121e-07	-
	1/8	4.469e-09	4.273	6.299e-09	4.273
	1/16	2.588e-10	4.110	3.648e-10	4.110
	1/32	1.562e-11	4.050	2.202e-11	4.050
	1/64	9.616e-13	4.021	1.355e-12	4.021
$\frac{2+\sin(\mathrm{x}\mathtt{t})}{400}$	1/4	2.592e-07	-	3.651e-07	-
	1/8	1.340e-08	4.273	1.888e-08	4.273
	1/16	7.758e-10	4.110	1.093e-09	4.110
	1/32	4.682e-11	4.050	6.596e-11	4.050
	1/64	2.878e-12	4.024	4.054e-12	4.024
$\frac{1+2\mathrm{x}\mathtt{t}}{300}$	1/4	2.799e-07	-	3.955e-07	-
	1/8	1.446e-08	4.274	2.044e-08	4.274
	1/16	8.372e-10	4.110	1.183e-09	4.110
	1/32	5.052e-11	4.050	7.138e-11	4.050
	1/64	3.105e-12	4.024	4.387e-12	4.024
$\frac{\mathrm{x}+\mathtt{t}}{2}$	1/4	4.364e-06	-	7.186e-06	-
	1/8	9.133e-07	2.256	1.300e-06	2.466
	1/16	4.942e-08	4.208	7.028e-08	4.209
	1/32	2.989e-09	4.047	7.028e-08	4.029
	1/64	1.913e-10	3.966	2.720e-10	3.966
$0.25+0.75\sin(\mathrm{x}\mathtt{t})$	1/4	9.866e-06	-	1.485e-05	-
	1/8	9.381e-07	3.394	1.334e-06	3.476
	1/16	5.230e-08	4.164	7.433e-08	4.165
	1/32	3.174e-09	4.042	4.511e-07	4.042
	1/64	2.015e-10	3.977	2.864e-10	3.977

TABLE 2.4: L_2 errors and L_∞ error of Ex. 2.2 at time $\mathtt{t}=1$ when $h_{\mathrm{x}}^2=\tau$.

$\alpha(\mathrm{x},\mathtt{t})$	h	Scheme (2.51)		Scheme (2.51)	
		L_2 Error	Order	L_∞ Error	Order
$\frac{e^{-\mathrm{x}\mathtt{t}}}{300}$	1/4	2.694e-10	-	3.769e-10	-
	1/8	9.702e-13	8.117	1.360e-12	8.114
	1/16	3.316e-15	8.192	4.902e-15	8.116
$\frac{2+\sin(\mathrm{x}\mathtt{t})}{400}$	1/4	8.084e-10	-	1.137e-09	-
	1/8	2.908e-12	8.118	4.094e-12	8.117
	1/16	1.136e-14	8.000	1.604e-14	7.995
$\frac{1+2\mathrm{x}\mathtt{t}}{300}$	1/4	8.725e-10	-	1.224e-09	-
	1/8	3.138e-12	8.119	4.409e-12	8.117
	1/16	1.195e-14	8.038	1.682e-14	8.034
$\frac{\mathrm{x}+\mathtt{t}}{2}$	1/4	5.276e-08	-	7.492e-08	-
	1/8	1.944e-10	8.084	2.760e-10	8.084
	1/16	8.614e-13	7.818	1.223e-12	7.817
$0.25+0.75\sin(\mathrm{x}\mathtt{t})$	1/4	5.564e-08	-	7.919e-08	-
	1/8	2.046e-10	8.086	2.910e-10	8.088
	1/16	8.719e-13	7.874	1.240e-12	7.874

2.5 Conclusion

In this chapter, we have proposed a higher order numerical scheme for VO-TFRSDE by using L-123 approximation in time. The proposed scheme is tested on two numerical examples for various $\alpha(\mathrm{x},\mathtt{t})$. The numerical results shown in Figures 2.1-2.4 and data presented in Tables 2.1-2.4 are observed to be highly accurate with higher order of convergence. The scheme reaches very high accuracy even on a very small number of grid points for some numerical examples. For the proposed numerical scheme, theoretical unconditional stability is shown. Several other variable-order time-fractional non-linear problems can be solved using the proposed numerical schemes, which is one of our objectives and a topic for further research.

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