Chapter 5

High-Order Approximation to Generalized Caputo Derivatives and Generalized Fractional Advection-Diffusion Equations

In this chapter, a high-order time-stepping scheme based on the cubic interpolation formula is considered to approximate the generalized Caputo fractional derivative of order $0 < \beta < 1$. Then, we adopt the developed scheme and central difference operator for approximation of spatial derivatives, to establish a difference scheme for the solution of generalized fractional advection-diffusion equation with Dirichlet boundary conditions. The stability and convergence of the difference scheme are discussed. Finally, some numerical examples are presented for the considered model.

5.1 Introduction

This chapter include, the numerical solution of following generalized fractional advection-diffusion equation:

$$\begin{cases} {}^{C}\mathcal{D}^{\beta}_{0,t;[\zeta(t),\omega(t)]}U(x,t) = D\frac{\partial^2 U(x,t)}{\partial x^2} - A\frac{\partial U(x,t)}{\partial x} + g(x,t), \quad x \in \Omega, \ t \in (0,T], \\ U(x,0) = U_0(x), \qquad x \in \overline{\Omega} = \Omega \cup \partial\Omega, \\ U(0,t) = \rho_1(t), \quad U(a,t) = \rho_2(t), \qquad t \in (0,T], \end{cases} \tag{5.1}$$

where $\Omega = (0,a)$ is a bounded domain with boundary $\partial\Omega$, and the notation ${}^{C}\mathcal{D}^{\beta}_{0,t,[\zeta(t),\omega(t)]}$ denotes the GCFD (defined in [11], and related references therein) with respect to t of order β is

$${}^{C}\mathcal{D}^{\beta}_{0,t;[\zeta(t),\omega(t)]}U(t) = \frac{[\omega(t)]^{-1}}{\Gamma(1-\beta)}\int_0^t \frac{[\omega(s)U(s)]'}{[\zeta(t)-\zeta(s)]^{\beta}}\,ds, \quad 0 < \beta < 1, \tag{5.2}$$

where parameters $D > 0$ is the diffusivity, $A > 0$ is the advection constant and U is the solute concentration, g, U_0, ρ_1 and ρ_2 are continuous functions on their respective domains with $U_0(0) = \rho_1(0)$ and $U_0(a) = \rho_2(0)$. Here scale, weight are sufficiently regular functions and our model (5.1) reduces to the diffusion problem when $A = 0$. Advection-diffusion equation is basically a transport problem that transport a passive scalar quantity in a fluid flow.

There are limited research papers available in the literature to approximate the GCFD. These works include the finite difference method, $L1$-method, $L1-2$-method and collocation method used to approximate GCFD. In this chapter, we presented the approximation of the GCFD based on cubic interpolation polynomials (say $L1-2-3$-method). Xu and Agrawal [104] considered the FDM for approximation of the GCFD for the generalized fractional Burgers equation. Kumar et al. [105], authors presented numerical scheme for the generalized fractional telegraph equation in time. Cao et al. [132] worked on the generalized time-fractional Kdv equation.They used the collocation method which is constructed using Jacobi–Gauss–Lobatto (JGL) nodes. Xu et al. [133] considered the FDM to approximate the GCFD of order $0 < \beta < 1$ and discussed the analytical and numerical solutions of the generalized fractional diffusion equation. Yadav et al. [106] discussed the Taylor expansion and finite difference approach to approximate the GCFD and then presented the numerical solution of the generalized fractional advection–diffusion equation. Motivated by all these works, the main focus of this chapter is to present a much higher-order numerical scheme to approximate the GCFD, and also establish the error analysis in both time and space discretization.

The main contributions of this work areas follows

- The approximation method discussed in [4], extended for the approximation of GCFD and obtain the convergence order $(4-\beta)$. Also, the obtained scheme reduces to the approximation scheme discussed by [4] for choice of the scale and the weight functions as $\zeta(t) = t$ and $\omega(t) = 1$.

- The error analysis of the presented higher-order numerical scheme for the generalized time fractional derivative by using Lagrange interpolation formula is established.

- The numerical results for different choices of scale and weight functions for the high-order time discretization scheme are provided with convergence order $\mathcal{O}(\tau^{4-\beta})$ for all $\beta \in (0,1)$ which achieves higher accuracy than the numerical methods developed in [3] for $\zeta(t) = t$ and $\omega(t) = 1$.

The remaining sections of the chapter is arranged as follows: In Section 5.2, we discuss the $(4-\beta)$-th order scheme to approximate the GCFD of order β of the function U. In Section 5.3, a higher order difference scheme to solve generalized fractional advection-diffusion equation is presented. Stability and convergence analysis are also discussed in this section. We present three numerical examples which illustrate the error and convergence order of our established numerical scheme in Section 5.4. Finally, Section 5.5, concludes some remarks.

5.2 Numerical Scheme for the Generalized Caputo Fractional Derivative

Motivated by the research carried out in [4, 105], this section is devoted to presenting a high-order approximation formula for the generalized Caputo-type fractional derivative using cubic interpolation polynomials.

Suppose that $U(t) \in C^4[0,T]$, and grid points $0 = t_0 < t_1 < ... < t_N = T$ with step length $\tau = t_n - t_{n-1}$ for $1 \leq n \leq N$. For simplicity, we use $g(s) = \omega(s)U(s)$, $U(t_l) = U_l$, $\omega(t_l) = \omega_l$, and $\zeta(t_l) = \zeta_l$. The generalized Caputo fractional derivative of order β of the function $U(t)$ at grid point t_n is given by,

$$ {}_0^C\mathcal{D}^{\beta}_{t;[\zeta(t),\omega(t)]}U_n = \frac{[\omega_n]^{-1}}{\Gamma(1-\beta)} \sum_{l=1}^{n} \int_{t_{l-1}}^{t_l} \frac{g'(s)}{[\zeta(t_n)-\zeta(s)]^{\beta}}\, ds. \tag{5.3} $$

On the first interval $[0, t_1]$ of domain, we use continuous linear polynomial $\Pi_1 g(t)$ to approximate the function $g(t)(= \omega(t)U(t))$. Let $g(t_l) = g_l$ and the difference operator $\nabla_\tau g_l = g_l - g_{l-1}$ for $l \geq 1$. Then, we have

$$ (\Pi_1 g(t))' = \frac{(g_1 - g_0)}{\tau} = \frac{(\nabla_\tau g_1)}{\tau}. $$

Thus, Eq. (5.3) yields,

$$\begin{aligned}&\frac{[\omega_n]^{-1}}{\Gamma(1-\beta)}\int_0^{t_1}[\zeta(t_n)-\zeta(s)]^{-\beta}\,g'(s)\,ds\\&\quad=\frac{[\omega_n]^{-1}}{\Gamma(1-\beta)}\,\frac{(\nabla_\tau g_1)}{\tau}\int_0^{t_1}[\zeta(t_n)-\zeta(s)]^{-\beta}ds+E_\tau^1=a_{n-1}(\nabla_\tau g_1)+E_\tau^1,\end{aligned}\tag{5.4}$$

where E_τ^1 is the truncation error on first interval and coefficients for this approximation are

$$a_{n-l}=\frac{[\omega_n]^{-1}}{\Gamma(2-\beta)}\left\{\frac{[\zeta(t_n)-\zeta(t_{l-1})]^{1-\beta}-[\zeta(t_n)-\zeta(t_l)]^{1-\beta}}{[\zeta(t_l)-\zeta(t_{l-1})]}\right\}.$$

Here, we denote notation $\beta_0^{(n)}=\frac{[\omega_n]^{-1}}{\Gamma(2-\beta)}$, $1\le n\le N$. Then

$$a_{n-l}=\beta_0^{(n)}\left\{\frac{[\zeta(t_n)-\zeta(t_{l-1})]^{1-\beta}-[\zeta(t_n)-\zeta(t_l)]^{1-\beta}}{[\zeta(t_l)-\zeta(t_{l-1})]}\right\},\quad 1\le l\le n.\tag{5.5}$$

Remark 5.2.1. If the scale function ζ is a positive strictly increasing functions on the domain $[0,T]$, then following inequality holds

$$0<[\zeta(t_n)-\zeta(t_l)]^{1-\beta}-[\zeta(t_n)-\zeta(t_{l-1})]^{1-\beta},\qquad 1\le l\le n.\tag{5.6}$$

Since, $t_l>t_{l-1}$, then it implies $\zeta(t_l)>\zeta(t_{l-1})$ for $1\le l\le n$, also $(1-\beta)>0$.

Remark 5.2.2. To estimate a_{n-1}, suppose that

$$\Theta=[\zeta(t_n)-\zeta(s)]\Rightarrow d\Theta=-\zeta'(s)\,ds=-\left(\frac{\zeta(t_l)-\zeta(t_{l-1})}{\tau}\right)ds,\quad s\in(t_{l-1},t_l),$$

therefore,

$$\int[\zeta(t_n)-\zeta(s)]^{-\beta}\,ds=\frac{-\tau}{\zeta(t_l)-\zeta(t_{l-1})}\frac{[\zeta(t_n)-\zeta(s)]^{1-\beta}}{(1-\beta)}.$$

On the second interval $[t_1, t_2]$, we use continuous quadratic polynomial $\Pi_2 g(t)$ to approximate the function $g(t)$, then we get

$$(\Pi_2 g(t))' = \frac{(2t - t_0 - t_1)}{2\tau^2} g_2 - \frac{(2t - t_0 - t_2)}{\tau^2} g_1 + \frac{(2t - t_1 - t_2)}{2\tau^2} g_0.$$

Thus from Eq. (5.3), we get,

$$\begin{aligned}
&\frac{[\omega_n]^{-1}}{\Gamma(1-\beta)} \int_{t_1}^{t_2} [\zeta(t_n) - \zeta(s)]^{-\beta}\, g'(s)\, ds \\
&\quad = \frac{[\omega_n]^{-1}}{\Gamma(1-\beta)} \int_{t_1}^{t_2} [\zeta(t_n) - \zeta(s)]^{-\beta}\, (\Pi_2 g(s))'\, ds + E_\tau^2 \\
&\quad = a_{n-2}(g_2 - g_1) + b_{n-2}(g_2 - 2g_1 + g_0) + E_\tau^2 \\
&\quad = a_{n-2}\big(\nabla_\tau g_2\big) + b_{n-2}\big(\nabla_\tau^2 g_2\big) + E_\tau^2. \qquad (5.7)
\end{aligned}$$

Here, truncation error on second interval is E_τ^2, and

$$\begin{aligned}
b_{n-l} = \beta_0^{(n)} \Bigg\{ &\frac{1}{(2-\beta)} \left[\frac{[\zeta(t_n) - \zeta(t_{l-1})]^{2-\beta} - [\zeta(t_n) - \zeta(t_l)]^{2-\beta}}{[\zeta(t_l) - \zeta(t_{l-1})]^2} \right] \\
&- \frac{1}{2} \left[\frac{[\zeta(t_n) - \zeta(t_{l-1})]^{1-\beta} + [\zeta(t_n) - \zeta(t_l)]^{1-\beta}}{[\zeta(t_l) - \zeta(t_{l-1})]} \right] \Bigg\}, \quad 2 \le l \le n. \qquad (5.8)
\end{aligned}$$

On the other sub domains $(l \ge 3)$, we use cubic interpolation polynomial $\Pi_l g(t)$ to approximate the function $g(t)$ using four points $(t_{l-3}, g_{l-3}), (t_{l-2}, g_{l-2}), (t_{l-1}, g_{l-1}), (t_l, g_l)$. As we know cubic interpolation polynomial is defined as,

$$\Pi_l g(t) = \sum_{r=0}^{3} g_{l-r} \prod_{s=0, s\neq r}^{3} \left(\frac{t - t_{l-s}}{t_{l-r} - t_{l-s}} \right),$$

then we get,

$$(\Pi_l g(t))' = g_{l-3} \frac{(t - t_{l-2})(t_l + t_{l-1} - 2t) + (t - t_{l-1})(t_l - t)}{6\tau^3}$$

$$
\begin{aligned}
&+ g_{l-2}\frac{(t-t_{l-3})(2t-t_{l-1}-t_l)+(t-t_l)(t-t_{l-1})}{2\tau^3} \\
&+ g_{l-1}\frac{(t-t_{l-2})(t_{l-3}+t_l-2t)+(t-t_{l-3})(t_l-t)}{2\tau^3} \\
&+ g_l\frac{(t-t_{l-2})(2t-t_{l-3}-t_{l-1})+(t-t_{l-1})(t-t_{l-3})}{6\tau^3}.
\end{aligned}
$$

Therefore from Eq. (5.3), we have

$$
\begin{aligned}
&\frac{[\omega_n]^{-1}}{\Gamma(1-\beta)}\sum_{l=3}^{n}\int_{t_{l-1}}^{t_l}[\zeta(t_n)-\zeta(s)]^{-\beta}g'(s)\;ds \\
&= \frac{[\omega_n]^{-1}}{\Gamma(1-\beta)}\sum_{l=3}^{n}\int_{t_{l-1}}^{t_l}[\zeta(t_n)-\zeta(s)]^{-\beta}(\Pi_l g(s))'\;ds + E_\tau^n \\
&= \sum_{l=3}^{n}\Bigg[A_{1,n-l}\;g_l + A_{2,n-l}\;g_{l-1} + A_{3,n-l}\;g_{l-2} + A_{4,n-l}\;g_{l-3}\Bigg] + E_\tau^n, \qquad (5.9)
\end{aligned}
$$

where E_τ^n $(3 \le n \le N)$ is the truncation error, and

$$
\begin{aligned}
A_{1,n-l} \;=\; & \beta_0^{(n)}\Bigg\{\frac{1}{6}\Bigg[\frac{2[\zeta(t_n)-\zeta(t_{l-1})]^{1-\beta}-11[\zeta(t_n)-\zeta(t_l)]^{1-\beta}}{[\zeta(t_l)-\zeta(t_{l-1})]}\Bigg] \\
&+\frac{1}{(2-\beta)}\Bigg[\frac{[\zeta(t_n)-\zeta(t_{l-1})]^{2-\beta}-2[\zeta(t_n)-\zeta(t_l)]^{2-\beta}}{[\zeta(t_l)-\zeta(t_{l-1})]^2}\Bigg] \\
&+\frac{1}{(2-\beta)(3-\beta)}\Bigg[\frac{[\zeta(t_n)-\zeta(t_{l-1})]^{3-\beta}-[\zeta(t_n)-\zeta(t_l)]^{3-\beta}}{[\zeta(t_l)-\zeta(t_{l-1})]^3}\Bigg]\Bigg\}, \qquad (5.10)
\end{aligned}
$$

$$
\begin{aligned}
A_{2,n-l} \;=\; & \beta_0^{(n)}\Bigg\{\frac{1}{2}\Bigg[\frac{6[\zeta(t_n)-\zeta(t_l)]^{1-\beta}+[\zeta(t_n)-\zeta(t_{l-1})]^{1-\beta}}{[\zeta(t_l)-\zeta(t_{l-1})]}\Bigg] \\
&+\frac{1}{(2-\beta)}\Bigg[\frac{5[\zeta(t_n)-\zeta(t_l)]^{2-\beta}-2[\zeta(t_n)-\zeta(t_{l-1})]^{2-\beta}}{[\zeta(t_l)-\zeta(t_{l-1})]^2}\Bigg] \\
&+\frac{3}{(2-\beta)(3-\beta)}\Bigg[\frac{[\zeta(t_n)-\zeta(t_l)]^{3-\beta}-[\zeta(t_n)-\zeta(t_{l-1})]^{3-\beta}}{[\zeta(t_l)-\zeta(t_{l-1})]^3}\Bigg]\Bigg\}, \qquad (5.11)
\end{aligned}
$$

$$
A_{3,n-l} \;=\; \beta_0^{(n)}\Bigg\{-\frac{1}{2}\Bigg[\frac{2[\zeta(t_n)-\zeta(t_{l-1})]^{1-\beta}+3[\zeta(t_n)-\zeta(t_l)]^{1-\beta}}{[\zeta(t_l)-\zeta(t_{l-1})]}\Bigg]
$$

$$+\frac{1}{(2-\beta)}\left[\frac{[\zeta(t_n)-\zeta(t_{l-1})]^{2-\beta}-4[\zeta(t_n)-\zeta(t_l)]^{2-\beta}}{[\zeta(t_l)-\zeta(t_{l-1})]^2}\right]$$
$$+\frac{3}{(2-\beta)(3-\beta)}\left[\frac{[\zeta(t_n)-\zeta(t_{l-1})]^{3-\beta}-[\zeta(t_n)-\zeta(t_l)]^{3-\beta}}{[\zeta(t_l)-\zeta(t_{l-1})]^3}\right]\Bigg\}, \quad (5.12)$$

$$\begin{aligned}A_{4,n-l} &= \beta_0^{(n)}\Bigg\{\frac{1}{6}\left[\frac{[\zeta(t_n)-\zeta(t_{l-1})]^{1-\beta}+2[\zeta(t_n)-\zeta(t_l)]^{1-\beta}}{[\zeta(t_l)-\zeta(t_{l-1})]}\right]\\ &+\frac{1}{(2-\beta)}\left[\frac{[\zeta(t_n)-\zeta(t_l)]^{2-\beta}}{[\zeta(t_l)-\zeta(t_{l-1})]^2}\right]\\ &+\frac{1}{(2-\beta)(3-\beta)}\left[\frac{[\zeta(t_n)-\zeta(t_l)]^{3-\beta}-[\zeta(t_n)-\zeta(t_{l-1})]^{3-\beta}}{[\zeta(t_l)-\zeta(t_{l-1})]^3}\right]\Bigg\}, \end{aligned}\quad (5.13)$$

where, $3 \leq l \leq n$. After simplifying the Eq. (5.9), we obtain the following form

$$\begin{aligned}&\frac{[\omega_n]^{-1}}{\Gamma(1-\beta)}\sum_{l=3}^{n}\int_{t_{l-1}}^{t_l}[\zeta(t_n)-\zeta(s)]^{-\beta}\ g'(s)\ ds\\ &=\sum_{l=3}^{n}\big[a_{n-l}(g_l-g_{l-1})+b_{n-l}(g_l-2g_{l-1}+g_{l-2})\\ &+c_{n-l}(g_l-3g_{l-1}+3g_{l-2}-g_{l-3})\big]+E_\tau^n\\ &=\sum_{l=3}^{n}\big[a_{n-l}(\nabla_\tau g_l)+b_{n-l}(\nabla_\tau^2 g_l)+c_{n-l}(\nabla_\tau^3 g_l)\big]+E_\tau^n.\end{aligned}\quad (5.14)$$

Here, we introduce another coefficient c_{n-l}, which is defined as

$$\begin{aligned}c_{n-l}=\beta_0^{(n)}&\Bigg\{\frac{1}{(2-\beta)(3-\beta)}\left[\frac{[\zeta(t_n)-\zeta(t_{l-1})]^{3-\beta}-[\zeta(t_n)-\zeta(t_l)]^{3-\beta}}{[\zeta(t_l)-\zeta(t_{l-1})]^3}\right]\\ &-\frac{1}{(2-\beta)}\left[\frac{[\zeta(t_n)-\zeta(t_l)]^{2-\beta}}{[\zeta(t_l)-\zeta(t_{l-1})]^2}\right]\\ &-\frac{1}{6}\left[\frac{[\zeta(t_n)-\zeta(t_{l-1})]^{1-\beta}+2[\zeta(t_n)-\zeta(t_l)]^{1-\beta}}{[\zeta(t_l)-\zeta(t_{l-1})]}\right]\Bigg\}, \quad 3\leq l\leq n,\end{aligned}\quad (5.15)$$

and coefficients a_{n-l}, b_{n-l} are defined in Eqs. (5.5), (5.8), respectively and Eq. (5.14) gives more compact form of Eq. (5.9). Such forms of coefficients were missing in [4].

From this we can easily discuss properties of coefficients.

Motivated by [4] (developed for Caputo derivative), a new numerical scheme for the generalized Caputo-type fractional derivative of order β of the function $U(t)$ at grid point t_n with the help of the equations (5.4), (5.7), (5.9), is defined by

$$\begin{aligned}
&{}^{\mathcal{H}}\mathcal{D}^{\beta}_{t;[\zeta(t),\omega(t)]}U(t)\mid_{t=t_n}\\
&\quad=\frac{[\omega_n]^{-1}}{\Gamma(1-\beta)}\int_0^{t_n}\frac{g'(s)}{[\zeta(t_n)-\zeta(s)]^{\beta}}\,ds\\
&\quad=\frac{[\omega_n]^{-1}}{\Gamma(1-\beta)}\left[\int_0^{t_1}\frac{g'(s)}{[\zeta(t_n)-\zeta(s)]^{\beta}}\,ds+\int_{t_1}^{t_2}\frac{g'(s)}{[\zeta(t_n)-\zeta(s)]^{\beta}}\,ds\right.\\
&\quad\left.+\sum_{l=3}^{n}\int_{l-1}^{t_l}\frac{g'(s)}{[\zeta(t_n)-\zeta(s)]^{\beta}}\,ds\right]\\
&\quad=\sum_{l=0}^{n}\lambda_l\;\omega_{n-l}\;U_{n-l}, \qquad (5.16)
\end{aligned}$$

with $g(s)=\omega(s)U(s)$.

Lemma 5.2.1. For any $\beta\in(0,1)$ and $U(t)\in\mathcal{C}^4[0,T]$, then

$${}^{C}_{0}\mathcal{D}^{\beta}_{t;[\zeta(t),\omega(t)]}U_n={}^{\mathcal{H}}\,\mathcal{D}^{\beta}_{t;[\zeta(t),\omega(t)]}U_n+E^n_{\tau},\;\; n=1,2,...,N,$$

where, ${}^{\mathcal{H}}\mathcal{D}^{\beta}_{t;[\zeta(t),\omega(t)]}$ is the approximation of GCFD and $\mid E^n_{\tau}\mid=\mathcal{O}(\tau^{4-\beta})$.

For distinct value of n, the coefficients in (5.16) can be expressed as below,

For $n=1$,

$$\begin{cases}\lambda_0=a_0,\\[2ex] \lambda_1=-a_0.\end{cases}$$

For $n=2$,

$$\begin{cases} \lambda_0 = a_0 + b_0, \\ \lambda_1 = a_1 - a_0 - 2b_0, \\ \lambda_2 = -a_1 + b_0. \end{cases}$$

For $n = 3$,

$$\begin{cases} \lambda_0 = A_{1,0}, \\ \lambda_1 = A_{2,0} + a_1 + b_1, \\ \lambda_2 = A_{3,0} + a_2 - a_1 - 2b_1, \\ \lambda_3 = A_{4,0} - a_2 + b_1. \end{cases}$$

For $n = 4$,

$$\begin{cases} \lambda_0 = A_{1,0}, \\ \lambda_1 = A_{1,1} + A_{2,0}, \\ \lambda_2 = A_{2,1} + A_{3,0} + a_2 + b_2, \\ \lambda_3 = A_{3,1} + A_{4,0} + a_3 - a_2 - 2b_2, \\ \lambda_4 = A_{4,1} - a_3 + b_2. \end{cases}$$

For $n = 5$,

$$\begin{cases} \lambda_0 = A_{1,0}, \\ \lambda_1 = A_{1,1} + A_{2,0}, \\ \lambda_2 = A_{1,2} + A_{2,1} + A_{3,0}, \\ \lambda_3 = A_{2,2} + A_{3,1} + A_{4,0} + a_3 + b_3, \\ \lambda_4 = A_{3,2} + A_{4,1} + a_4 - a_3 - 2b_3, \\ \lambda_5 = A_{4,2} - a_4 + b_3. \end{cases}$$

For $n \geq 6$,

$$\begin{cases} \lambda_0 = A_{1,0}, \\ \lambda_1 = A_{1,1} + A_{2,0}, \\ \lambda_2 = A_{1,2} + A_{2,1} + A_{3,0}, \\ \lambda_l = A_{1,l} + A_{2,l-1} + A_{3,l-2} + A_{4,l-3} \quad (3 \leq l \leq n-3), \\ \lambda_{n-2} = a_{n-2} + b_{n-2} + A_{2,n-3} + A_{3,n-4} + A_{4,n-5}, \\ \lambda_{n-1} = a_{n-1} - a_{n-2} - 2b_{n-2} + A_{3,n-3} + A_{4,n-4}, \\ \lambda_n = -a_{n-1} + b_{n-2} + A_{4,n-3}. \end{cases} \tag{5.17}$$

Lemma 5.2.2. If the scale function ζ fulfills (5.6), and the weight function ω is non-negative and non-decreasing on uniform time grids, then

(i)[132] Linear approximation coefficient satisfies, $1 \leq l \leq n$,

$$0 < a_{n-1} < ... < a_{n-l} < a_{n-l-1} < ... < a_1 < a_0.$$

(ii) Quadratic approximation coefficient satisfies, $2 \leq l \leq n$,

$$0 < b_{n-2} < ... < b_{n-l} < b_{n-l-1} < ... < b_1 < b_0.$$

Proof. (i) If scale function ζ is continuous on respective domain then by using mean-value theorem, there exist $\hat{x}_l \in (t_{l-1}, t_l)$ such that

$$\frac{1}{\tau} \int_{t_{l-1}}^{t_l} [\zeta(t_n) - \zeta(s)]^{-\beta} ds = [\zeta(t_n) - \zeta(\hat{x}_l)]^{-\beta}, \quad 1 \leq l \leq n. \tag{5.18}$$

Since, $[\zeta(t_n) - \zeta(s)]^{-\beta}$ is a monotone increasing function, we easily get our required result.

(ii) Let $\eta(s) = [\zeta(t_n) - \zeta(s)]^{1-\beta}$, suppose scale function ζ is sufficiently smooth on domain $[0, T]$ then mean-value theorem yields

$$\begin{aligned} 2\int_{t_{l-1}}^{t_l} \eta(s)ds - \big(\eta(t_{l-1}) + \eta(t_l)\big) \\ &= 2\eta(\tilde{x}_l) - \big(\eta(t_{l-1}) + \eta(t_l)\big), \quad \tilde{x}_l \in (t_{l-1}, t_l), \\ &= -\theta\big(\eta'(t_l) - \eta'(t_{l-1})\big), \qquad 0 < \theta < 1 \\ &= -\theta\tau\eta''(\nu_l), \qquad \nu_l \in (t_{l-1}, t_l), \\ &= \frac{\theta\beta(1-\beta)}{\tau}[\zeta(t_l) - \zeta(t_{l-1})]^2[\zeta(t_n) - \zeta(\nu_l)]^{-\beta-1} > 0. \end{aligned} \tag{5.19}$$

Using (5.19), we can easily get that $b_{n-l} > 0$ for positive strictly increasing weight function $\omega(t)$. Since $[\zeta(t_n) - \zeta(s)]^{-\beta-1}$ is a monotone increasing function on temporal domain $[0, T]$, so we get the desired result. ☐

Lemma 5.2.3. Suppose that the scale function ζ is positive and strictly increasing, and the weight function ω is non-negative and non-decreasing, then for each $\beta \in (0, 1)$, the following conditions hold for coefficients in (5.17)

(1) $\lambda_0 > 0, \quad \forall\ n \geq 1,$

(2) $\sum_{l=0}^{n} \lambda_l = 0.$

Proof. (1) If $n = 1$, then

$$\lambda_0 = a_0 = \beta_0^{(n)}[\zeta_1 - \zeta_0]^{-\beta} \quad ,$$

as scale function is strictly increasing, therefore $\zeta_{n-1} < \zeta_n$, for $n \geq 1$. Which implies that $\lambda_0 = a_0 > 0$.

If $n = 2$, then $\quad \lambda_0 = a_0 + b_0,$

since,

$$b_0 = \beta_0^{(n)} \left\{ \left(\frac{1}{2-\beta} - \frac{1}{2} \right) [\zeta_2 - \zeta_1]^{-\beta} \right\} > 0,$$

and we have already shown that $a_0 > 0$, therefore $\lambda_0 = a_0 + b_0 > 0$.

If $n \geq 3$, then for $0 < \beta < 1$,

$$\lambda_0 = A_{1,0} = \beta_0^{(n)} \left\{ \left(\frac{1}{3} + \frac{1}{(2-\beta)} + \frac{1}{(2-\beta)(3-\beta)} \right) [\zeta_n - \zeta_{n-1}]^{-\beta} \right\} > 0.$$

(2) If $n = 1$, then $\lambda_0 = -\lambda_1 = a_0 > 0$, for $0 < \beta < 1$. It implies that $\lambda_0 + \lambda_1 = 0$.

If $n = 2$, then there exist a $\beta \in (0, 1)$, by numerical analysis

$\lambda_0 + \lambda_1 + \lambda_2 = (a_0 + b_0) + (a_1 - a_0 + 2b_0) + (-a_1 + b_0) = 0.$

If $n \geq 3$, then

$$\begin{aligned}
\sum_{l=0}^{n} \lambda_l &= A_{1,0} + A_{1,1} + A_{2,0} + A_{1,2} + A_{2,1} + A_{3,0} \\
&+ \sum_{l=3}^{n-3} (A_{1,l} + A_{2,l-1} + A_{3,l-2} + A_{4,l-3}) + a_{n-2} + b_{n-2} \\
&+ A_{2,n-3} + A_{3,n-4} + A_{4,n-5} + a_{n-1} - a_{n-2} - 2b_{n-2} + A_{3,n-3} \\
&+ A_{4,n-4} - a_{n-1} + b_{n-2} + A_{4,n-3} \\
&= \sum_{j=0}^{n-3} (A_{1,j} + A_{2,j} + A_{3,j} + A_{4,j}) = 0.
\end{aligned}$$

□

Lemma 5.2.4. If scale function ζ is a Lipschitz function on interval $[t_{l-1}, t_l]$ with Lipschitz constant $\mathcal{L}$, then

$$\mid \zeta_l - \zeta_{l-1} \mid \leq \mathcal{L}\ \tau, \quad 1 \leq l \leq n.$$

5.2.1 Truncation Error for Generalized Caputo Derivative Term

For truncation error of approximation of the generalized Caputo derivative defined in (5.16), for simplicity, suppose that function $g(t) = \omega(t)U(t)$ such that $g(t) \in C^4((0,T])$, and $\zeta(t) = v$ this implies $t = \zeta^{-1}(v)$. Therefore, $g(v) = \omega(\zeta^{-1}(v))U(\zeta^{-1}(v))$.

Theorem 5.2.1. A triangle inequality gives the bound

$$|^C\, \mathcal{D}^{\beta}_{0,t;[\zeta(t),\omega(t)]} g_n - ^{\mathcal{H}} \mathcal{D}^{\beta}_{t;[\zeta(t),\omega(t)]} g_n | \leq \sum_{n=1}^{N} | E^n_\tau |,$$

with

$$E^n_\tau = \frac{[\omega_n]^{-1}}{\Gamma(1-\beta)} \int_{\zeta_{l-1}}^{\zeta_l} [\zeta_n - v]^{-\beta} [g(v) - \Pi_l g(v)]'\, dv, \quad 1 \leq l \leq n.$$

$$(1) \quad | E^1_\tau | \leq \frac{\beta[\omega_n]^{-1}}{\Gamma(1-\beta)} \left[\frac{1}{8\beta} + \frac{1}{2(1-\beta)(2-\beta)} \right] \max_{t_0 \leq v \leq t_1} | g^{(2)}(v) | L^{2-\beta} (\tau)^{2-\beta}, \quad n = 1,$$

$$(2) \quad | E^2_\tau | \leq \frac{\beta[\omega_n]^{-1}}{\Gamma(1-\beta)} \Bigg\{ \frac{1}{12} \max_{t_0 \leq v \leq t_1} | g^{(2)}(v) | (t_2 - t_1)^{-\beta-1} L^{2-\beta} (\tau)^3 + \Bigg[\frac{1}{12} + \frac{1}{3(1-\beta)(2-\beta)} \left(\frac{1}{2} + \frac{1}{(3-\beta)} \right) \Bigg] \max_{t_0 \leq v \leq t_2} | g^{(3)}(v) | L^{3-\beta} (\tau)^{3-\beta} \Bigg\}, \quad n = 2,$$

$$(3) \quad | E^n_\tau | \leq \frac{\beta[\omega_n]^{-1}}{\Gamma(1-\beta)} \Bigg\{ \frac{1}{72} \max_{t_0 \leq x \leq t_2} | g^{(3)}(x) | (t_n - t_2)^{-\beta-1} L^{3-\beta} (\tau)^4 + \Bigg[\frac{3}{128\beta} + \frac{1}{12(1-\beta)(2-\beta)} \left(1 + \frac{3}{(3-\beta)} + \frac{3}{(3-\beta)(4-\beta)} \right) \Bigg] \max_{t_0 \leq x_1 \leq t_n} | g^{(4)}(x_1) | L^{4-\beta} (\tau)^{4-\beta} \Bigg\}, \quad n \geq 3.$$

Proof. (1)[105] For $n = 1$, the scheme (5.16) is linear approximation of GCFD, and convergence order for this approximation formula is $\mathcal{O}(\tau^{2-\beta})$.

(2)[105] For $n = 2$, the scheme (5.16) is quadratic approximation of GCFD, and here convergence order of approximation formula is $\mathcal{O}(\tau^{3-\beta})$.

(3) For $n \geq 3$, By Lagrange interpolation remainder theorem, we use quadratic interpolation function $\Pi_2 g(v)$ to interpolate $g(v)$ using node points (ζ_0, g_0), (ζ_1, g_1), (ζ_2, g_2) on interval $[\zeta_0, \zeta_2]$ and cubic interpolation function $\Pi_l g(v)$ depends on (ζ_{l-3}, g_{l-3}), (ζ_{l-2}, g_{l-2}), (ζ_{l-1}, g_{l-1}), (ζ_l, g_l) to interpolate $g(v)$ on $[\zeta_{l-3}, \zeta_l]$ as follows

$$g(v) - \Pi_2 g(v) = \frac{g^{(3)}(\eta_1)}{3!}\,(v - \zeta_0)(v - \zeta_1)(v - \zeta_2),\ \ v \in [\zeta_0, \zeta_2],\ \ \eta_1 \in (\zeta_0, \zeta_2), \tag{5.20}$$

$$g(v) - \Pi_l g(v) = \frac{g^{(4)}(\eta_l)}{4!}\,(v - \zeta_{l-3})(v - \zeta_{l-2})(v - \zeta_{l-1})(v - \zeta_l),\ \ v \in [\zeta_{l-3}, \zeta_l], \tag{5.21}$$

where $3 \leq l \leq n$.

Now,

$$\begin{aligned}
E_\tau^n &= \frac{[\omega_n]^{-1}}{\Gamma(1-\beta)}\Bigg[\int_{\zeta_0}^{\zeta_2} [g(v) - \Pi_2 g(v)]'\ [\zeta_n - v]^{-\beta}\ dv \\
&\qquad + \sum_{l=3}^{n} \int_{\zeta_{l-1}}^{\zeta_l} [g(v) - \Pi_l g(v)]'\ [\zeta_n - v]^{-\beta}\ dv\Bigg] \\
&= \frac{[\omega_n]^{-1}}{\Gamma(1-\beta)}\Bigg\{[g(v) - \Pi_2 g(v)]\ [\zeta_n - v]^{-\beta}\ \Big|_{\zeta_0}^{\zeta_2} \\
&\qquad - \beta \int_{\zeta_0}^{\zeta_2} [g(v) - \Pi_2 g(v)]\ [\zeta_n - v]^{-\beta-1}\ dv \\
&\qquad + \sum_{l=3}^{n} \Bigg[g(v) - \Pi_l g(v)]\ [\zeta_n - v]^{-\beta}\ \Big|_{\zeta_{l-1}}^{\zeta_l} \\
&\qquad - \beta \int_{\zeta_{l-1}}^{\zeta_l} [g(v) - \Pi_l g(v)]\ [\zeta_n - v]^{-\beta-1}\ dv\Bigg]\Bigg\} \\
&= -\frac{\beta\ [\omega_n]^{-1}}{\Gamma(1-\beta)}\Bigg\{\int_{\zeta_0}^{\zeta_2} [g(v) - \Pi_2 g(v)]\ [\zeta_n - v]^{-\beta-1}\ dv \\
&\qquad + \sum_{l=3}^{n} \int_{\zeta_{l-1}}^{\zeta_l} [g(v) - \Pi_l g(v)]\ [\zeta_n - v]^{-\beta-1}\ dv\Bigg\}.
\end{aligned} \tag{5.22}$$

Since, from (2.18) and (2.19)

$$[g(v)-\Pi_2 g(v)]\ [\zeta_n-v]^{-\beta}\ |_{\zeta_0}^{\zeta_2} \ = \ \frac{g^{(3)}(\eta_1)}{3!}\ (v-\zeta_0)(v-\zeta_1)(v-\zeta_2)[\zeta_n-v]^{-\beta}\ |_{\zeta_0}^{\zeta_2}=0,$$
$$[g(v)-\Pi_l g(v)]\ [\zeta_n-v]^{-\beta}\ |_{\zeta_{l-1}}^{\zeta_l}=\frac{g^{(4)}(\eta_l)}{4!}\ (v-\zeta_{l-3})(v-\zeta_{l-2})(v-\zeta_{l-1})(v-\zeta_l)\ |_{\zeta_{l-1}}^{\zeta_l}=0.$$

Consider the first integration of Eq. (5.22)

$$\begin{aligned}
&\left|\int_{\zeta_0}^{\zeta_2}[g(v)-\Pi_2(v)]\ [\zeta_n-v]^{-\beta-1}\ dv\right|\\
&=\left|\int_{\zeta_0}^{\zeta_2}\frac{g^{(3)}(\eta_1)}{3!}(v-\zeta_0)(v-\zeta_1)(v-\zeta_2)[\zeta_n-v]^{-\beta-1}\ dv\right|\\
&\le\frac{1}{6}\max_{\zeta_0\le\eta_1\le\zeta_2}\ |\ g^{(3)}(\eta_1)\ |\ (\zeta_n-\zeta_2)^{-\beta-1}\frac{(\zeta_0-\zeta_2)^3(\zeta_0-2\zeta_1+\zeta_2)}{12}\\
&\le\frac{1}{6}\max_{\zeta_0\le\eta_1\le\zeta_2}\ |\ g^{(3)}(\eta_1)\ |\ (\zeta_n-\zeta_2)^{-\beta-1}\frac{(\zeta_0-\zeta_2)^3(\zeta_0-\zeta_1)}{12}.
\end{aligned}$$

Using Lamma (5.2.4), we have following inequality

$$\left|\int_{\zeta_0}^{\zeta_2}[g(v)-\Pi_2 g(v)]\ [\zeta_n-v]^{-\beta-1}\ dv\ \right|\le\frac{1}{72}\max_{t_0\le x\le t_2}\ |\ g^{(3)}(x)\big|(t_n-t_2)^{-\beta-1}L^{3-\beta}(\tau)^4. \tag{5.23}$$

Consider the second integration of Eq. (5.22)

$$\begin{aligned}
&\left|\sum_{l=3}^{n}\int_{\zeta_{l-1}}^{\zeta_l}[g(v)-\Pi_l g(v)]\ [\zeta_n-v]^{-\beta-1}\ dv\right|\\
&\le\left|\sum_{l=3}^{n-1}\int_{\zeta_{l-1}}^{\zeta_l}\frac{g^{(4)}(\eta_l)}{4!}\ (v-\zeta_{l-3})(v-\zeta_{l-2})(v-\zeta_{l-1})(v-\zeta_l)[\zeta_n-v]^{-\beta-1}\ dv\right|\\
&+\left|\int_{\zeta_{n-1}}^{\zeta_n}\frac{g^{(4)}(\eta_n)}{4!}\ (v-\zeta_{n-3})(v-\zeta_{n-2})(v-\zeta_{n-1})(v-\zeta_n)[\zeta_n-v]^{-\beta-1}\ dv\right|.
\end{aligned}\tag{5.24}$$

Consider first part of RHS of Eq. (5.24)

$$\begin{aligned}
&\left|\sum_{l=3}^{n-1}\int_{\zeta_{l-1}}^{\zeta_l}\frac{g^{(4)}(\eta_l)}{4!}\,(v-\zeta_{l-3})(v-\zeta_{l-2})(v-\zeta_{l-1})(v-\zeta_l)[\zeta_n-v]^{-\beta-1}\,dv\right|\\
&\le\frac{1}{24}\max_{\zeta_2\le\eta_2\le\zeta_{n-1}}\mid g^{(4)}(\eta_2)\mid\ \tilde{f}(\zeta_{l-3},\zeta_{l-2},\zeta_{l-1},\zeta_l)\int_{\zeta_2}^{\zeta_{n-1}}[\zeta_n-v]^{-\beta-1}\,dv\\
&\le\frac{1}{24\beta}\max_{\zeta_2\le\eta_2\le\zeta_{n-1}}\mid g^{(4)}(\eta_2)\mid\ \tilde{f}(\zeta_{l-3},\zeta_{l-2},\zeta_{l-1},\zeta_l)(\zeta_n-\zeta_{n-1})^{-\beta}.
\end{aligned}$$

Here, $\tilde{f}(\zeta_{l-3},\zeta_{l-2},\zeta_{l-1},\zeta_l)$ is the $\max\limits_{\zeta_2\le v\le\zeta_{n-1}}[(v-\zeta_{l-3})(v-\zeta_{l-2})(v-\zeta_{l-1})(v-\zeta_l)]$. Using Lemma (5.2.4), we get the following inequality

$$\begin{aligned}
&\left|\sum_{l=3}^{n-1}\int_{\zeta_{l-1}}^{\zeta_l}\frac{g^{(4)}(\eta_l)}{4!}\,(v-\zeta_{l-3})(v-\zeta_{l-2})(v-\zeta_{l-1})(v-\zeta_l)[\zeta_n-v]^{-\beta-1}\,dv\right|\\
&\le\frac{3}{128\beta}\max_{t_2\le x_1\le t_{n-1}}\mid g^{(4)}(x_1)\mid L^{4-\beta}(\tau)^{4-\beta}.
\end{aligned}\tag{5.25}$$

Remark 5.2.3. If scale function ζ fulfills the condition of Lipschitz function such that $\mid\zeta_l-\zeta_{l-1}\mid\le L\tau$ then, $\max_{\zeta_2\le v\le\zeta_{n-1}}\mid(v-\zeta_{l-3})(v-\zeta_{l-2})(v-\zeta_{l-1})(v-\zeta_l)\mid$ is obtained at $v=\zeta_{l-2}+\frac{L\tau}{2}$, therefore

$$\tilde{f}(\zeta_{l-3},\zeta_{l-2},\zeta_{l-1},\zeta_l)=\max_{\zeta_2\le v\le\zeta_{n-1}}[(v-\zeta_{l-3})(v-\zeta_{l-2})(v-\zeta_{l-1})(v-\zeta_l)]\le\frac{9}{16}L^4(\tau)^4.$$

Consider the second part of RHS of Eq. (5.24)

$$\begin{aligned}
&\left|\int_{\zeta_{n-1}}^{\zeta_n}\frac{g^{(4)}(\eta_n)}{4!}\,(v-\zeta_{n-3})(v-\zeta_{n-2})(v-\zeta_{n-1})(v-\zeta_n)[\zeta_n-v]^{-\beta-1}\,dv\right|\\
&\le\frac{1}{24}\max_{\zeta_{n-1}\le\eta_3\le\zeta_n}|g^{(4)}(\eta_3)\mid\ \left|\int_{\zeta_{n-1}}^{\zeta_n}(v-\zeta_{n-3})(v-\zeta_{n-2})(v-\zeta_{n-1})[\zeta_n-v]^{-\beta}\,dv\right|\\
&\le\frac{1}{12(1-\beta)(2-\beta)}\left(1+\frac{3}{(3-\beta)}\right.\\
&\left.+\frac{3}{(3-\beta)(4-\beta)}\right)\max_{\zeta_{n-1}\le\eta_3\le\zeta_n}\mid g^{(4)}(\eta_3)\mid(\zeta_n-\zeta_{n-1})^{4-\beta}
\end{aligned}$$

$$\leq \frac{1}{12(1-\beta)(2-\beta)}\Bigg(1+\frac{3}{(3-\beta)} + \frac{3}{(3-\beta)(4-\beta)}\Bigg)\max_{t_{n-1}\leq x_2\leq t_n} \mid g^{(4)}(x_2)\mid L^{4-\beta}(\tau)^{4-\beta}. \tag{5.26}$$

Now combining (5.23), (5.25) and (5.26), then error bound is

$$\begin{aligned} \mid E_{\tau}^{n}\mid \quad \leq \frac{\beta\ [\omega_n]^{-1}}{\Gamma(1-\beta)}\Bigg\{&\frac{1}{72}\max_{t_0\leq x\leq t_2}\mid g^{(3)}(x)\mid (t_n-t_2)^{-\beta-1}L^{3-\beta}(\tau)^4 \\ &+\Bigg[\frac{3}{128\beta}+\frac{1}{12(1-\beta)(2-\beta)}\Bigg(1+\frac{3}{(3-\beta)} \\ &+\frac{3}{(3-\beta)(4-\beta)}\Bigg)\Bigg]\max_{t_0\leq x_1\leq t_n}\mid g^{(4)}(x_1)\mid L^{4-\beta}(\tau)^{4-\beta}\Bigg\}. \end{aligned} \tag{5.27}$$

□

5.3 Numerical Scheme for the Generalized Fractional Advection-Diffusion Equation

In this section, we study the numerical scheme for solving the generalized fractional advection-diffusion defined by (5.1).

Let $u(x,t)=\omega(t)U(x,t)-\omega(0)U_0(x)$. We rewrite Eq. (5.1) to a similar equation using unity weight function and same scale function $\zeta(t)$. Then, Eq. (5.1) converts into the following form:

$$\begin{cases} {}_{0}^{C}\mathcal{D}_{t;[\zeta(t),1]}^{\beta}u(x,t)=D\frac{\partial^2 u(x,t)}{\partial x^2}-A\frac{\partial u(x,t)}{\partial x}+f(x,t), \quad x\in\Omega, \quad t\in(0,T], \\ u(x,0)=0, \qquad x\in\overline{\Omega}=\Omega\cup\partial\Omega, \\ u(0,t)=\phi_1(t), \quad u(a,t)=\phi_2(t), \qquad t\in(0,T], \end{cases} \tag{5.28}$$

where $\phi_1(t) = \omega(t)\rho_1(t) - \omega(0)\rho_1(0)$, $\phi_2(t) = \omega(t)\rho_2(t) - \omega(0)\rho_2(0)$, and $f(x,t) = D\omega(0)U_0''(x) - A\omega(0)U_0'(x) + \omega(t)g(x,t)$.

For the uniform spatial mesh, let $0 = x_0 < x_1 < ... < x_M = a$ of the interval $[0, a]$ with step size $h = \frac{a}{M}$ where M denotes number of subintervals, the grid points $x_0 + ih$ $(0 \leq i \leq M)$, and $\tau = \frac{T}{N}$ be the step-size in temporal direction with grids $t_n = n\tau$ $(0 = t_0 < t_1 < ... < t_n = T)$.

Now, we discretize our problem (5.28) at (x_i, t_n), then we get

$$ {}^{C}\mathcal{D}^{\beta}_{0,t;[\zeta(t),1]} u(x_i, t_n) = Du_{xx}(x_i, t_n) - Au_x(x_i, t_n) + f(x_i, t_n). \tag{5.29}$$

In Eq. (5.29), for fixed t_n and $1 \leq i \leq M - 1$, the first and second order spatial derivatives discretized by using the following central difference approximations:

$$\frac{\partial u(x_i, t_n)}{\partial x} = \frac{u_n^{i+1} - u_n^{i-1}}{2h} + \mathcal{O}(h^2), \tag{5.30}$$

$$\frac{\partial^2 u(x_i, t_n)}{\partial x^2} = \frac{u_n^{i+1} - 2u_n^i + u_n^{i-1}}{h^2} + \mathcal{O}(h^2). \tag{5.31}$$

With the help of Equation (5.16), we get an approximation of the generalized Caputo-type fractional derivative term in (5.29) as follows:

$$\begin{aligned} {}^{\mathcal{H}}\mathcal{D}^{\beta}_{t;[\zeta(t),1]} u(x_i, t_n) =& \lambda_0 u(x_i, t_n) + \lambda_1 u(x_i, t_{n-1}) + \sum_{l=3}^{n-2} \lambda_{n-l}\ u(x_i, t_l) \\ &+ \lambda_{n-2} u(x_i, t_2) + \lambda_{n-1} u(x_i, t_1) + \lambda_n u(x_i, t_0) + \mathcal{O}(\tau^{4-\beta}). \end{aligned} \tag{5.32}$$

where λ_l defined in Eq. (5.17). Next, we use Eq. (5.30), Eq. (5.31), and Eq. (5.32) in discretized equation (5.29), yields

$$\sum_{l=0}^{n} \lambda_{n-l}\ u(x_i, t_l) = Du_{xx}(x_i, t_n) - Au_x(x_i, t_n) + f_n^i + e_n^i. \tag{5.33}$$

where, $| e_n^i | \leq \tilde{c}(\tau^{4-\beta} + h^2)$ for some constant $\tilde{c}$.

Now, we use numerical approximation u_n^i of $u(x_i, t_n)$ to neglect the truncation error term in Equation (5.33), then we determine the following finite difference scheme:

$$\begin{aligned}\lambda_0 u_n^i + \sum_{l=3}^{n-1} \lambda_{n-l} u_l^i + \lambda_{n-2} u_2^i + \lambda_{n-1} u_1^i + \lambda_n u_0^i = D\, \frac{u_n^{i+1} - 2u_n^i + u_n^{i-1}}{h^2} \\ - A\, \frac{u_n^{i+1} - u_n^{i-1}}{2h} + f_n^i\,, \qquad (5.34)\end{aligned}$$

where $1 \leq n \leq N\,,\ \ 1 \leq i \leq M-1$. That is,

$$\begin{cases} (\frac{D}{h^2} + \frac{A}{2h}) u_1^{i-1} - (\lambda_0 + \frac{2D}{h^2}) u_1^i + (\frac{D}{h^2} - \frac{A}{2h}) u_1^{i+1} = \lambda_1 u_0^i - f_1^i, \quad n = 1, \\ (\frac{D}{h^2} + \frac{A}{2h}) u_2^{i-1} - (\lambda_0 + \frac{2D}{h^2}) u_2^i + (\frac{D}{h^2} - \frac{A}{2h}) u_2^{i+1} = \lambda_1 u_1^i + \lambda_2 u_0^i - f_2^i, \quad n = 2, \\ (\frac{D}{h^2} + \frac{A}{2h}) u_n^{i-1} - (\lambda_0 + \frac{2D}{h^2}) u_n^i + (\frac{D}{h^2} - \frac{A}{2h}) u_n^{i+1} \\ \qquad = \lambda_1 u_{n-1}^i + \sum\limits_{l=0}^{n-2} \lambda_{n-l} u_l^i - f_n^i, \quad n \geq 3. \\ u_0^i = 0, \qquad 0 \leq i \leq M, \\ u_n^0 = \phi_1(t_n), \quad u_n^M = \phi_2(t_n), \qquad 0 \leq n \leq N. \end{cases} \tag{5.35}$$

We rewrite the matrix form of above equation as follows:

$$\begin{cases} KU_1 = \lambda_1 U_0 - F_1 + H_1, \quad n = 1, \\ KU_2 = \lambda_1 U_1 + \lambda_2 U_0 - F_2 + H_2, \quad n = 2, \\ KU_n = \lambda_1 U_{n-1} + \sum\limits_{l=0}^{n-2} \lambda_{n-l} U_l - F_n + H_n, \quad n \geq 3, \end{cases} \tag{5.36}$$

where the matrix as well as vectors of Equation (5.36) are defined as follows:

$$K = tri\left[\tfrac{D}{h^2}+\tfrac{A}{2h},\ \ -\lambda_0-\tfrac{2D}{h^2},\ \tfrac{D}{h^2}-\tfrac{A}{2h}\right]_{(M-1)\times(M-1)},$$

$$U_n = (u_n^1, u_n^2, ..., u_n^{M-1})^T,$$

$$F_n = (f_n^1, f_n^2, ..., f_n^{M-1})^T,$$

$$H_n = \left((-\tfrac{D}{h^2}-\tfrac{A}{2h})u_n^0, 0, ..., 0, (-\tfrac{D}{h^2}+\tfrac{A}{2h})u_n^M\right)^T, \quad 1 \le n \le N.$$

Remark 5.3.1. Since the coefficient matrix A is a tridiagonal and strictly diagonally dominant then $det(A) \neq 0$ (Levy-Desplanques theorem). Therefore at each time level t_n, the proposed scheme (5.35) has an unique solution for $1 \le n \le N$.

Theorem 5.3.1. The local truncation error of difference scheme (5.34) at (x_i, t_n), $1 \le i \le M-1,\ 1 \le n \le N$ is

$$\mid e_n^i \mid \le C(h^2 + \tau^{4-\beta}), \tag{5.37}$$

where C is the positive constant independent of the time and space step sizes.

Proof. From Eqs. (4.7) the LTE of difference scheme (5.34) is

$$\begin{aligned}
& e_n^i \\
&= \lambda_0 u(x_i,t_n) + \lambda_1 u(x_i,t_{n-1}) + \sum_{l=3}^{n-2} \lambda_{n-l} u(x_i,t_l) + \lambda_{n-2} u(x_i,t_2) \\
&+ \lambda_{n-1} u(x_i,t_1) + \lambda_n u(x_i,t_0) - D\,\frac{u_n^{i+1} - 2u_n^i + u_n^{i-1}}{h^2} + A\,\frac{u_n^{i+1} - u_n^{i-1}}{2h} - f_n^i \\
&= \Bigg[\lambda_0 u(x_i,t_n) + \lambda_1 u(x_i,t_{n-1}) + \lambda_2 u(x_i,t_{n-2}) + \sum_{l=2}^{n-3} \lambda_{n-l} u(x_i,t_l) + \lambda_{n-1} u(x_i,t_1) \\
&+ \lambda_n u(x_i,t_0) - {}^{\mathcal{H}}\mathcal{D}^{\beta}_{t;[\zeta(t),1]} u(x_i,t_n)\Bigg] - D\left[\frac{u_n^{i+1} - 2u_n^i + u_n^{i-1}}{h^2} - \frac{\partial^2 u(x_i,t_n)}{\partial x^2}\right] \\
&+ A\left[\frac{u_n^{i+1} - u_n^{i-1}}{2h} - \frac{\partial u(x_i,t_n)}{\partial x}\right] \\
&= \mathcal{O}(\tau^{4-\beta}) - D\ \mathcal{O}(h^2) + A\ \mathcal{O}(h^2) = \mathcal{O}(\tau^{4-\beta} + h^2).
\end{aligned}$$

□

In next theorem we use $L^2(\Omega)$-space with the norm $\|.\|_2$ and the inner product $\langle ., . \rangle$.

Theorem 5.3.2. (see [104]) If tridiagonal matrix elements satisfy the inequality

$$\lambda_1 \leq (\lambda_0)^{M-3}\left(\lambda_0 + \frac{D}{h^2} - \frac{A}{2h}\right)\left(\lambda_0 + \frac{D}{h^2} + \frac{A}{2h}\right), \tag{5.38}$$

then finite difference scheme is stable.

Proof. The Eq. (5.36) can be rewritten as following, for $1 \leq n \leq N$

$$KU_n = \lambda_1 U_{n-1} + \sum_{l=0}^{n-2} \lambda_{n-l} U_l - F_n + H_n,$$

let, $$V_n = \sum_{l=0}^{n-2} \lambda_{n-l} U_l - F_n + H_n.$$

Since matrix K is invertible then above equation can be rewritten as

$$U_n = \lambda_1 K^{-1} U_{n-1} + K^{-1} V_n.$$

Using recurrence relation, we get the following equation

$$\begin{aligned} U_n =& (\lambda_1 K^{-1})^2 U_{n-2} + (\lambda_1 K^{-1}) K^{-1} V_{n-1} + K^{-1} V_n \\ =& (\lambda_1 K^{-1})^n U_0 + (\lambda_1 K^{-1})^{n-1} K^{-1} V_1 + (\lambda_1 K^{-1})^{n-2} K^{-1} V_2 + ... + K^{-1} V_n. \end{aligned} \tag{5.39}$$

Let $\widetilde{U}_n$ is the approximate solution of Eq. (5.36), then we define error at grid point $t_n = n\tau$

$$e_n = U_n - \widetilde{U}_n, \qquad 0 \leq n \leq N.$$

Then, we obtain

$$e_n = (\lambda_1 K^{-1})^n e_0, \qquad 1 \leq n \leq N.$$

From definition of compatible matrix norm

$$\|e_n\| \ \leq \ \|(\lambda_1 K^{-1})^n\| \ |e_0\|.$$

Since,

$$\|(\lambda_1 K^{-1})^n\| \leq \|(\lambda_1 K^{-1})\| \ \|(\lambda_1 K^{-1})^{n-1}\| \leq ... \leq \|(\lambda_1 K^{-1})\|^n$$

According to Ostrowski therorem ([134], Theorem 3.1), let $K = [a_{i,j}]_{M-1 \times M-1}$, then determinant of K satisfies

$$\begin{aligned} \mid det(K) \mid \ &\geq \prod_{i=1}^{M-1} \left(\mid a_{i,i} \mid - \sum_{j=1, j\neq i}^{M-1} \mid a_{i,j} \mid \right) \\ &= (\lambda_0)^{M-3} \left(\lambda_0 + \frac{D}{h^2} - \frac{A}{2h} \right) \left(\lambda_0 + \frac{D}{h^2} + \frac{A}{2h} \right). \end{aligned} \tag{5.40}$$

Using assumed inequality (5.38) into (5.40), we get $\mid det(K^{-1}) \mid \leq \frac{1}{\lambda_1}$,

it implies that,

$$\mid \lambda_1 \mid \ \|K^{-1}\| \leq 1.$$

Therefore, $\mid e_n \mid \leq \ \mid e_0 \mid .$

Thus, the numerical scheme is stable. □

Theorem 5.3.3. The solution U_n^i of the difference scheme (5.34) satisfies

$$\max_{i,n} \mid U(x_i, t_n) - \widetilde{U}(x_i, t_n) \mid \leq C(h^2 + \tau^{4-\beta}) \tag{5.41}$$

for some constant C.

Proof. The truncation error for difference scheme at $(x_i, t_n) \in [0, a] \times [0, T]$ is

$$\mid e_n^i \mid \leq C(h^2 + \tau^{4-\beta})$$

by Theorem 5.3.1. Now using theorem of stability, it implies

$$\max_{i,n} \mid U(x_i, t_n) - \widetilde{U}(x_i, t_n) \mid \leq C \mid U(x_i, t_0) - \widetilde{U}(x_i, t_0) \mid \tag{5.42}$$

We obtain the desired result easily after using the truncation error as discussed in Theorem 5.3.1. □

5.4 Numerical Examples

In this section, we will check the numerical accuracy of the proposed schemes (5.16) and difference scheme (5.35), also verify the theoretical convergence order discussed in Theorem (5.3.3). Here, we provide three examples to numerically support our theory; in the first example we check convergence order and absolute error of approximation for the GCFD, while last two problems get the form (5.28) to discribe accuracy and maximum absolute error of the difference scheme. We provide two Tables (5.7) and (5.10) to compare our scheme (5.35) with [3], Example 4.1 and [4], Example 5.1, respectively for particular choice of scale $\zeta(t) = t$ and weight $\omega(t) = 1$ functions. All numerical results are implemented in MATLAB R2018b.

To calculate the maximum absolute error $\|u^\tau - U^\tau\|_\infty$ and error $\|u^\tau - U^\tau\|_2$ corresponding to the L_2-norm, we use following formulas, respectively.

$$E_\infty(M, N) = \max_{1 \leq i \leq M-1} \mid U_N^i - u_N^i \mid,$$

$$E_2(M,N) = \left(h \sum_{i=1}^{M-1} \mid U_N^i - u_N^i \mid^2 \right)^{1/2},$$

where $\{U_n^i\}$ is the exact solution of advection diffusion equation and $\{u_n^i\}$ is the approximate solution at the point (x_i, t_n).

Moreover, the OC in space and time direction for the described difference scheme corresponding to L_∞-norm can be evaluated using the following formulas. OC_x is the order of convergence in space side and OC_t for temporal side.

$$OC_x = \frac{log(E(2M,N)) - log(E(M,N))}{log(2)},$$

and

$$OC_t = \frac{log(E(M,2N)) - log(E(M,N))}{log(2)}.$$

Example 5.4.1. [3] Take function $u(t) = t^{4+\beta}$, $t \in [0,1]$, for $0 < \beta < 1$. Determine the β order GCFD for $u(t)$ at $T = 1$ numerically.

The maximum absolute error and rate of convergence for the scheme (5.16) to approximate the GCFD of function $u(t)$ for $\beta = 0.2, 0.5, 0.8$ with uniform time steps 1/10, 1/20, 1/40, 1/80, 1/160 are calculated and shown in following tables.

Table 5.1 shows that the errors with respect to L_∞-norm and convergence rates in time, and these data are found after calculating the classical Caputo derivative (i.e. taking $\zeta(t) = t$, $\omega(t) = 1$) of function $u(t)$ with the help of scheme (5.16). From this Table, we can see that the errors of our scheme (5.16) obtaining from GCFD approximation are lesser in compare to scheme developed in [3] for approximation of Caputo derivative and convergence rate of our scheme is $(4 - \beta)$, while accuracy for time derivative in [3] is $(3 - \beta)$. In Table 5.2, to compute the maximum errors E_∞ and order of convergence in time direction, we take $\omega(t) = e^t$ while scale function is fixed with t. From Table 5.3, we validate the the convergence for $\zeta(t) = t$,

$\omega(t) = t + 1$ and the CPU time in seconds for $\beta = 0.8$ are discussed. Table 5.4 shows that maximum errors and order of convergence for different choice of weight $\omega(t) = t^{0.5}, \ t, \ t^4, \ e^{2t}$, scale is $\zeta(t) = t$ and $\beta = 1/3$. In all cases, accuracy in time is obtained as $(4 - \beta)$ for the scheme (5.16), which is higher than [106, 134].

TABLE 5.1: The maximum absolute errors and order of convergence in time for Example 5.4.1, when $\zeta(t) = t$, $\omega(t) = 1$ and compare our results with [3] for $\beta = 0.5$.

	$\beta = 0.2$		$\beta = 0.5$		[3]	
N	$\|u^\tau - U^\tau\|_\infty$	OC	$\|u^\tau - U^\tau\|_\infty$	OC	$\|u^\tau - U^\tau\|_\infty$	OC
10	1.6978E-04		1.5401E-03		1.3507E-02	
20	1.3130E-05	3.6928	1.4383E-04	3.4206	2.6121E-03	2.3704
40	9.9792E-07	3.7178	1.3116E-05	3.4550	4.8618E-04	2.4256
80	7.4966E-08	3.7346	1.1811E-06	3.4731	8.8645E-05	2.4554
160	5.5944E-09	3.7442	1.0560E-07	3.4835	1.5975E-05	2.4722

TABLE 5.2: The maximum absolute errors and order of convergence in time for Example 5.4.1, when $\zeta(t) = t$, $\omega(t) = e^t$ and different β's.

	$\beta = 0.2$		$\beta = 0.5$		$\beta = 0.8$		$\beta = 0.8$
N	$\|u^\tau - U^\tau\|_\infty$	OC	$\|u^\tau - U^\tau\|_\infty$	OC	$\|u^\tau - U^\tau\|_\infty$	OC	CPU (s)
10	7.5552E-04		6.3075E-03		3.2184E-02		11.354861
20	7.1075E-05	3.4101	6.7873E-04	3.2162	4.2148E-03	2.9328	33.306962
40	5.9370E-06	3.5815	6.6677E-05	3.3476	5.0375E-04	3.0647	82.772072
80	4.6934E-07	3.6610	6.2491E-06	3.4155	5.7497E-05	3.1312	180.064565
160	3.5650E-08	3.7186	5.7027E-07	3.4539	6.4090E-06	3.1653	419.978865

Table 5.3: The maximum absolute errors and order of convergence in time for Example 5.4.1, when $\zeta(t) = t$, $\omega(t) = t + 1$ and different β's.

	$\beta = 0.2$		$\beta = 0.5$		$\beta = 0.8$		$\beta = 0.8$
N	$\lVert u^\tau - U^\tau \rVert_\infty$	OC	$\lVert u^\tau - U^\tau \rVert_\infty$	OC	$\lVert u^\tau - U^\tau \rVert_\infty$	OC	CPU (s)
10	3.5019E-04		3.1752E-03		1.7251E-02		10.194270
20	3.0621E-05	3.5155	3.1426E-04	3.3368	2.0904E-03	3.0448	25.931157
40	2.4466E-06	3.6457	2.9499E-05	3.4132	2.3982E-04	3.1238	65.689186
80	1.8844E-07	3.6986	2.6976E-06	3.4509	2.6799E-05	3.1617	155.623905
160	1.4248E-08	3.7252	2.4326E-07	3.4711	2.9560E-06	3.1805	384.953845

Table 5.4: The maximum absolute errors and order of convergence in time for Example 5.4.1, when $\zeta(t) = t$, $\beta = \frac{1}{3}$ and different weight functions.

	$\omega(t) = t^{0.5}$		$\omega(t) = t$		$\omega(t) = t^4$		$\omega(t) = e^{2t}$
N	$\lVert u^\tau - U^\tau \rVert_\infty$	OC	$\lVert u^\tau - U^\tau \rVert_\infty$	OC	$\lVert u^\tau - U^\tau \rVert_\infty$	OC	OC
10	9.5450E-04		1.7465E-03		3.2557E-02		
20	8.4374E-05	3.4999	1.5027E-04	3.5388	2.0348E-03	4.0000	3.2332
40	7.0877E-06	3.5734	1.2845E-05	3.5483	1.2749E-04	3.9964	3.4225
80	5.8166E-07	3.6071	1.0650E-06	3.5923	1.1169E-05	3.5129	3.5220
160	4.7123E-08	3.6257	8.6813E-08	3.6167	9.3941E-07	3.5716	3.5764

Example 5.4.2. [4] We take the following generalized fractional advection-diffusion equation:

$$\begin{cases} {}_0^C\mathcal{D}^{\beta}_{t;[\zeta(t),\omega(t)]} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial x} + f(x,t), \quad (x,t) \in (0,1) \times (0,1), \\ u(x,0) = 0, \qquad x \in (0,1), \\ u(0,t) = t^{6+\beta}, \; u(1,t) = e t^{6+\beta}, \qquad t \in (0,1], \end{cases} \tag{5.43}$$

where $f(x,t) = e^x t^6 \frac{\Gamma(7+\beta)}{720}$. When $\zeta(t) = t$ and $\omega(t) = 1$, then $u(x,t) = e^x t^{6+\beta}$ is the exact solution.

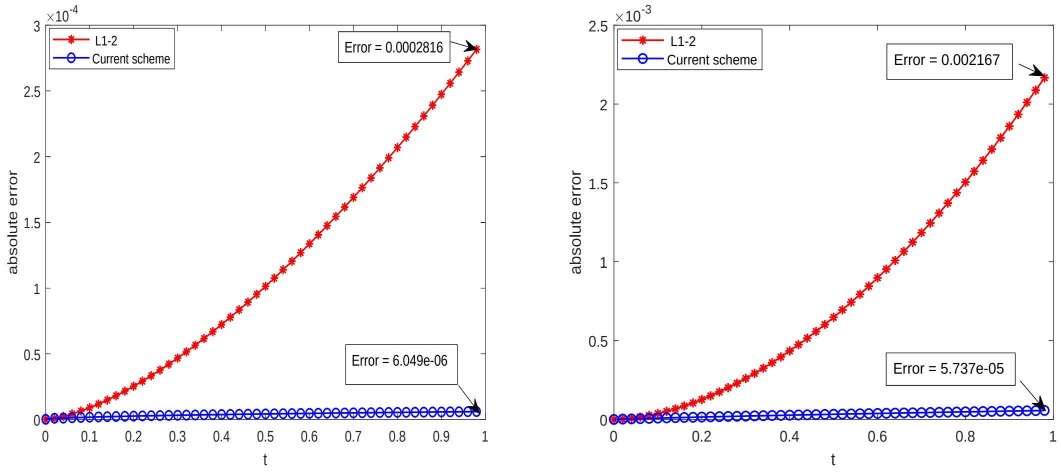
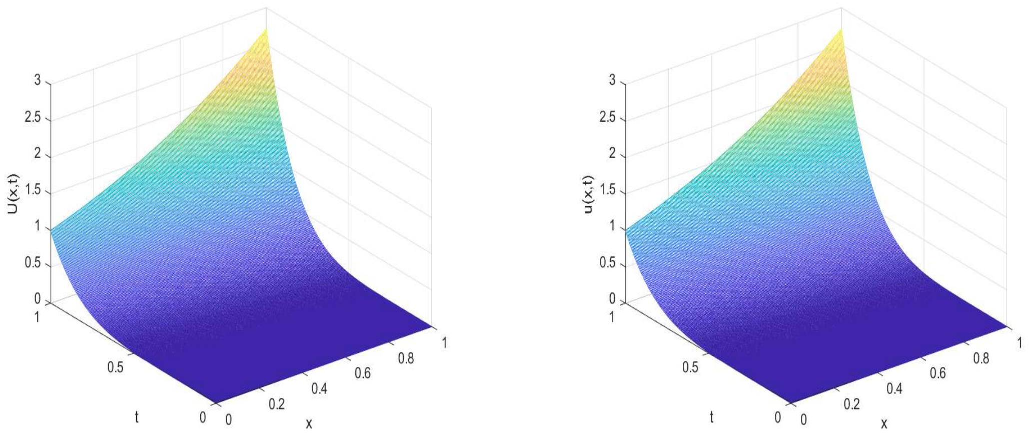
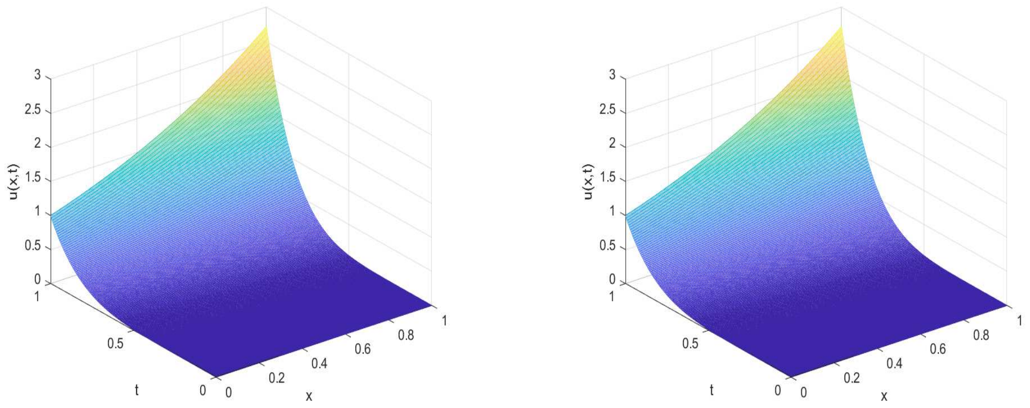


FIGURE 5.1: Errors plot of the numerical results for the Example (5.4.1) at final time $T = 1$ for different values of β (Left side for $\beta = 0.5$; Right side for $\beta = 0.8$).

To solve this Example 5.4.2, we use the numerical scheme defined in (5.35). The maximum errors at time $T = 1$ for different values of β with different step sizes, and rate of convergence in time direction and space direction are displayed in Tables 5.6 and 5.5. In Table 5.6, we set $h = \frac{1}{2000}$ and describe the numerical errors and OC in time for different values of N. In Table 5.5, we fix $\tau = \frac{1}{500}$ and present the numerical errors and spatial OC for different values of M. It is shown that our scheme (5.35) gives $(4-\beta)$-order convergence in temporal direction and second-order convergence in spatial direction.



(a) Exact solution for $\beta = 0.95$ **(b)** Numerical solution for $\beta = 0.95$



(c) Numerical solution for $\beta = 0.6$ **(d)** Numerical solution for $\beta = 0.45$

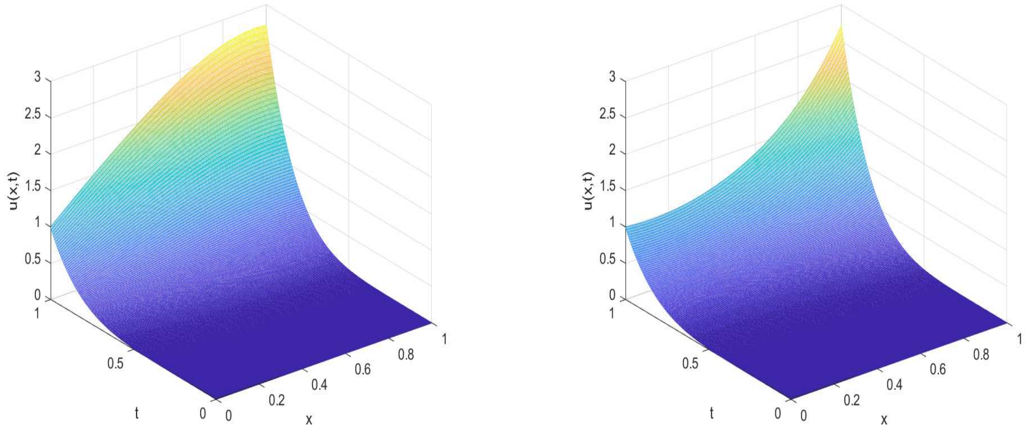
FIGURE 5.2: Exact and approximate solutions of Example 5.4.2 for different β's with $M = N = 200$ and $\omega(t) = 1$, $\zeta(t) = t$.

TABLE 5.5: The maximum absolute errors and order of convergence in space for Example 5.4.2, when $\tau = 1/500$ and different β's.

	$\beta = 0.2$		$\beta = 0.5$		$\beta = 0.8$	
M	$\|u^\tau - U^\tau\|_\infty$	OC	$\|u^\tau - U^\tau\|_\infty$	OC	$\|u^\tau - U^\tau\|_\infty$	OC
8	2.3437E-04		2.1275E-04		1.8078E-04	
16	5.8814E-05	1.9945	5.3334E-05	1.9960	4.5242E-05	1.9985
32	1.4733E-05	1.9971	1.3373E-05	1.9958	1.1319E-05	1.9990
64	3.6832E-06	2.0001	3.3408E-06	2.0010	2.7940E-06	2.0183
128	9.2088E-07	1.9999	8.3276E-07	2.0042	6.6258E-07	2.0762

TABLE 5.6: Errors with convergence rates in time for Example 5.4.2, when $h = 1/2000$ and different β's.

β	N	$\|u^\tau - U^\tau\|_\infty$	OC	$\|u^\tau - U^\tau\|_2$	OC
0.8	8	1.6385E-02		7.4195E-03	
	16	2.2920E-03	2.8377	1.3514E-03	2.4569
	32	2.8191E-04	3.0233	1.8556E-04	2.8645
	64	3.2609E-05	3.1119	2.2573E-05	3.0392
	128	3.6574E-06	3.1564	2.5937E-06	3.1215
0.5	8	3.7940E-03		1.8168E-03	
	16	4.2889E-04	3.1451	2.5900E-04	2.8104
	32	4.3064E-05	3.3160	2.8647E-05	3.1765
	64	4.0768E-06	3.4010	2.8348E-06	3.3371
	128	3.7173E-07	3.4551	2.6407E-07	3.4243
0.2	8	5.4498E-04		2.7293E-04	
	16	5.1093E-05	3.4150	3.1455E-05	3.1172
	32	4.2949E-06	3.5724	2.8822E-06	3.4480
	64	3.3828E-07	3.6663	2.3627E-07	3.6087
	128	2.2628E-08	3.9020	1.6173E-08	3.8687



(a) $\zeta(t) = e^t,\ \omega(t) = 1$ **(b)** $\zeta(t) = t^{1/2},\ \omega(t) = 1$

FIGURE 5.3: Numerical solutions of Example 5.4.2 with different choices of weight functions $\omega(t)$ and scale functions $\zeta(t)$, and $M = N = 200$ with $\beta = 0.9$.

Clearly visible from above figures that scale function $\zeta(t)$ can stretch or contract the domain.

TABLE 5.7: The maximum absolute errors and order of convergence in time of Example 4.1 in [3] (page no 43) for different values of β with $\tau = \frac{1}{2000}$.

		Current scheme		$L1-2$ [3]	
β	N	$\Vert u^\tau - U^\tau \Vert_\infty$	OC	$\Vert u^\tau - U^\tau \Vert_\infty$	OC
0.9	10	2.2536E-03		1.8600E-02	
	20	3.2727E-04	2.7837	4.7228E-03	1.9776
	40	4.1906E-05	2.9653	1.1488E-03	2.0395
	80	5.1072E-06	3.0366	2.7367E-04	2.0697
	160	6.1092E-07	3.0635	6.4520E-05	2.0846
0.5	10	2.5470E-04		2.4936E-03	
	20	2.6005E-05	3.2919	4.8366E-04	2.3662
	40	2.4615E-06	3.4012	9.0163E-05	2.4234
	80	2.2827E-07	3.4307	1.6457E-05	2.4539
	160	2.3662E-08	3.2701	2.9700E-06	2.4701

Example 5.4.3. Take the following fractional advection-diffusion equation:

$$\begin{cases} {}_0^C\mathcal{D}^{\beta}_{t,[\zeta(t),\omega(t)]}u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial x} + f(x,t), \quad (x,t) \in (0,1)\times(0,1), \\ u(x,0) = 0, \quad x \in (0,1), \\ u(0,t) = 0, \\ u(1,t) = t^7(sin\ 1), \quad t \in (0,1], \end{cases} \tag{5.44}$$

where $f(x,t) = \frac{\Gamma 8}{\Gamma(8-\beta)} t^{7-\beta} sin(x) + t^7(sin(x)+cos(x))$. When $\zeta(t) = t$ and $\omega(t) = 1$, the exact solution is $u(x,t) = t^7 sin(x)$.

To solve this PDE, we use our difference scheme (5.35). Here, two Tables 5.8, 5.9 are given in support of numerical results. In Table 5.8, we take fixed space step size $h = 1/2000$ and display maximum absolute errors and errors $\|u^\tau - U^\tau\|_2$ with respect to norm L_2, also rate of convergence for temporal direction with different time steps $N = 8, 16, 32, 64, 128$. In Table 5.9, we expressed maximum errors and convergence order in space direction to set time steps fix with $\tau = 1/500$ and taking different $M = 8, 16, 32, 64, 128$.

TABLE 5.8: Errors with convergence rates in time for Example 5.4.3, when $h = 1/2000$ and different β's.

β	N	$\|u^\tau - U^\tau\|_\infty$	OC	$\|u^\tau - U^\tau\|_2$	OC
0.8	8	5.2117E-03		2.2239E-03	
	16	7.4012E-04	2.8159	4.1857E-04	2.4095
	32	9.1743E-05	3.0121	5.8367E-05	2.8422
	64	1.0658E-05	3.1057	7.1562E-06	3.0279
	128	1.2018E-06	3.1486	8.2814E-07	3.1112
0.5	8	1.4554E-03		6.2805E-04	
	16	1.7116E-04	3.0880	9.7274E-05	2.6907
	32	1.7546E-05	3.2862	1.1191E-05	3.1198
	64	1.6840E-06	3.3812	1.1323E-06	3.3049
	128	1.5961E-07	3.3993	1.1013E-07	3.3620
0.2	8	2.5522E-04		1.1012E-04	
	16	2.5499E-05	3.3233	1.4486E-05	2.9264
	32	2.2221E-06	3.5204	1.4170E-06	3.3538
	64	1.8445E-07	3.5906	1.2401E-07	3.5144
	128	1.8801E-08	3.2944	1.3014E-08	3.2523

TABLE 5.9: The maximum absolute errors and order of convergence in space for Example 5.4.3, when $\tau = 1/500$ and different β's.

	$\beta = 0.2$		$\beta = 0.5$		$\beta = 0.8$	
M	$\|u^\tau - U^\tau\|_\infty$	OC	$\|u^\tau - U^\tau\|_\infty$	OC	$\|u^\tau - U^\tau\|_\infty$	OC
8	3.0308E-04		2.7311E-04		2.3102E-04	
16	7.6029E-05	1.9951	6.8533E-05	1.9946	5.8003E-05	1.9938
32	1.9050E-05	1.9968	1.7180E-05	1.9961	1.4560E-05	1.9941
64	4.7625E-06	2.0000	4.2962E-06	1.9996	3.6518E-06	1.9953
128	1.1909E-06	1.9997	1.0752E-06	1.9985	9.2444E-07	1.9820

TABLE 5.10: The maximum absolute errors and order of convergence in time for Example 5.1 in [4], when $h = 1/6000$, and maximum errors and order of convergence in space, when $\tau = 1/200$.

	$\beta = 0.368$		[4]		$\beta = 0.368$		[4]
N	$\|u^\tau - U^\tau\|_\infty$	OC	OC	M	$\|u^\tau - U^\tau\|_\infty$	OC	OC
8	8.1539E-04			4	8.8034E-04		
16	7.7720E-05	3.3911	3.4031	8	2.2559E-04	1.9643	1.9643
32	6.8845E-06	3.4969	3.4972	16	5.6583E-05	1.9953	1.9953
64	5.8739E-07	3.5510	3.5528	32	1.4173E-05	1.9972	1.9972
128	4.9904E-08	3.5571	3.5836	64	3.5359E-06	2.0030	2.0030

In Table 10, we validate the proposed scheme with [4] (Example 5.1). Firstly, we present the max error, the convergence order with fixed $h = \frac{1}{6000}$ and varying N with $8, 16, 32, 64, 128$ for $\beta = 0.368$. It is noted that the convergence order of our scheme (5.35) for temporal direction is almost the same as [4]. After that, we expressed maximum error and convergence rate for spatial dimension to set $\tau = 1/200$ and changing $M = 4, 8, 16, 32, 64$ for same value of β, we observe that the spatial convergence order of our numerical scheme is same as [4]. Thus, the scheme presented in [4] becomes a particular case of the proposed scheme (5.35) for $\zeta(t) = t$ and $\omega(t) = 1$. Also, we can compute numerical results to for different suitable choices of scale and weight functions.

5.5 Conclusion

A high-order numerical scheme based on cubic interpolation formula is discussed for approximation of GCFD of β-th order. Properties of discretized coefficients are analyzed and local truncation error in approximation of GCFD is also discussed. Further, we establish a difference scheme for generalized fractional advection-diffusion equation using the developed approximation formula for GCFD. The stability and convergence of this established scheme to approximate the time fractional generalized advection-diffusion equation are studied. Order of accuracy for the difference scheme is $\mathcal{O}(\tau^{4-\beta} + h^2)$, where step-sizes τ in time and h in space direction. The convergence order of the difference scheme is described by some numerical experiments.

Numerical calculations reveal that the proposed difference scheme has $(4 - \beta)-$th order convergence in time; and in space direction, it has second order convergence. The temporal rate of convergence of the scheme for generalized fractional advection-diffusion equation is highest accuracy to date. In addition, the developed scheme can be directly used to get the other Caputo-type of time fractional advection-diffusion equations by selecting the suitable scale $\zeta(t)$ and weight $\omega(t)$ functions in the generalized Caputo-type fractional derivative. The developed scheme is tested for the cases having smooth solutions of the considered fractional advection-diffusion equation. However, the case of the nonsmooth solutions will be presented in our future works.

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