Chapter 2

Numerical solution of two-dimensional nonlinear fractional order reaction-advection-diffusion equation by using collocation method

2.1 Introduction

Fractional diffusion equations are used in many fields. Researchers are paying attention on fractional calculus due to its increasing applications. it is not easy to find the analytical solution of fractional diffusion equation especially for nonlinear cases, so it is required to find the numerical solution of the problems. In the last few years, many numerical methods have been developed such as Agarwal and EI-Sayed [46] have used Non-standard finite difference collocation technique for solving diffusion equation of fractional order, Molla and Nova [47] have obtained the approximate solution of two-dimensional diffusion equation of fractional order, Zhang and Sun [48] have developed the ADI technique to solve two- dimensional time fractional sub-diffusion equation, Yang et al. [49] have attained numerical solution of two-dimensional diffusion equation of fractional order, Liu et al. [50] have discussed finite volume method for solving diffusion equation of fractional order, Meerschaert et al.[51] have solved two-dimensional dispersion equation of fractional order using finite difference scheme, Jaiswal et al. [52] have solved nonlinear PDE for porous media.

The main objective of this chapter is to find the approximate solutions of the following two-dimensional nonlinear time fractional order reaction-advection-diffusion equation (FRADE) given by

$$ {}_0^cD_t^{\alpha}u(x,y,t) = \vec{\nabla}\cdot\Big[u(x,y,t)\Big\{\hat{i}\frac{\partial u}{\partial x}+\hat{j}\frac{\partial u}{\partial y}\Big\}\Big] - v\Big(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\Big) + ku(x,y,t) + g(x,y,t), \tag{2.1} $$

where $0 < \alpha \leq 1$, and $\quad 0 < x, y, t \leq 1$

and the following multi-term time-fractional diffuson equation (MT-TFDE) as

$$ \sum_{i=1}^{q} d_i \ {}_0^cD_t^{\alpha_i}u(x,y,t) = \vec{\nabla}u(x,y,t) + g(x,y,t), 0 < \alpha_i \leq 1, \tag{2.2} $$

where $u(x, y, t)$ denotes the concentration of the solute, k is reaction coefficient, $g(x, y, t)$ is forcing term, v is the velocity representing the speed of movement, d_i's are some constant and i, j are the unit vectors in the positive x and y directions respectively. If $\alpha = 1$, the model is called the integer order two-dimensional nonlinear reaction-advection-diffusion equation (RADE). Here, we shall use the spectral collocation method to find the approximate solutions to the problems. The spectral method was investigated by Steven Orszag, which is similar to the finite element method. The difference is that in the spectral method, the basis function used is non-zero on the whole domain, while in the finite element, method function used, which is non-zero on the sub-domain. The spectral method has nice error properties as compared to the finite element method. The spectral collocation method is used to find the numerical solution of partial differential equations, ordinary differential equations, and eigenvalue problems related to differential equations. Here, the solution is written in combination with Lucas polynomial as $\sum c_{i,jk}\phi_i\phi_j\phi_k$, where ϕ's are the polynomial vector and $c_{i,jk}$ are coefficients which have to be determined. In the collocation method, the residual is collocated at certain collocation points together with prescribed conditions. The mathematical models (2.1) and model (2.2) are solved using the following initial and boundary conditions given by

$$\begin{aligned} u(x,y,0) &= \rho_1(x,y), \qquad 0 \leq x,y \leq 1, \\ u(0,y,t) &= \rho_2(y,t), \qquad 0 \leq y,t \leq 1, \\ u(1,y,t) &= \rho_3(y,t), \qquad 0 \leq y,t \leq 1, \\ u(x,0,t) &= \rho_4(x,t), \qquad 0 \leq x,t \leq 1, \\ u(x,1,t) &= \rho_5(x,t). \qquad 0 \leq x,t \leq 1. \end{aligned} \tag{2.3}$$

Here Lucas operational matrix (OM) is used to find the residual of the problem described in Eq.(2.1)- Eq.(2.2). OM method is better than the other methods because it has a sparse matrix, which reduces the computational time and increases the accuracy of the numerical solution. Also, OM method is simple to execute. Many OM methods have been developed in recent years to find approximate solutions to various problems. Bhrawy et al. [53] have developed Jacobi OM to obtain the numerical solution of the diffusion-wave equation of fractional order, Doha et al. [54] have found the approximate solution of fractional order IVP applying Chebyshev OM, Dehgan et al.[55] have solved telegraph equation of fractional order with the help of variational iteration method, Singh et al. [56] have obtained the approximate solution of two-dimensional fractional transport equation using Legendre OM, Saadatmandi and Dehghan [57] have discussed OM approach to deal with differential equation of fractional order, Elhameed and Youssri [58] construct Lucas OM for solving fractional differential equation, Tohidi et al. [59] have used OM of Bernoulli polynomial to obtain the approximate solution of generalized pantograph equation, Das [60] has solved diffusion equation of fractional order using variational iteration method, Das et al. [61] approximated the reaction-diffusion equation of fractional order, Vishal et al. [62] have used homotopy analytical method to solve time fractional nonlinear Swift Hohenberg equation, Kumar et al. [63] have used OM to find numerical solution of two-dimensional reaction-diffusion equation.

2.1.1 Approximation of function

Consider the function $g(x, y, t)$ defined in $L^2([0,1] \times [0,1] \times [0,1])$, then $g(x, y, t)$ can be expressed in combination of Lucas polynomial as

$$g(x, y, t) = \cong \sum_{i}^{n} \sum_{j}^{n} \sum_{k}^{n} u_{ijk} L_i(t) L_j(x) L_k(y) = \phi(t)^T U \psi(x, y),$$

where $\psi(x, y) = \phi(x) \otimes \phi(y)$ and $\phi(t) = [L_0(t), L_1(t), L_2(t), \cdots, L_n(t)]^T$, u_{ijk} are unknown coefficients and U is $(n+1) \times (n+1)^2$ dimensional unknown matrix defined as

$$U = \begin{pmatrix} u_{11} & u_{12} & \cdots & \cdots & u_{1(n+1)^2} \\ u_{21} & u_{22} & \cdots & \cdots & u_{2(n+1)^2} \\ \vdots & \vdots & & \ddots & \vdots \\ u_{(n+1)1} & u_{(n+2)2} & \cdots & \cdots & u_{(n+1)(n+1)^2} \end{pmatrix}_{(n+1)\times(n+1)^2} . \tag{2.4}$$

2.2 OM of derivative in terms of Lucas polynomials

According to [58], the derivative of the vector $\phi(t)$ is given by

$$\frac{d\phi(t)}{dt} = M^{(1)} \phi(t), \tag{2.5}$$

where $M^{(1)} = (m_{ij}^{(1)})$ is the Lucas OM of order $(n+1) \times (n+1)$ whose elements are given by

$$m_{ij}^{(1)} = \begin{cases} i(-1)^{\frac{i-j-1}{2}}, \text{if } \ i > j \ \text{ and } (i+j) \ \text{ is } \ \text{odd}, \\ 0, \qquad\qquad \text{elsewhere}, \end{cases},$$

and derivative of order k of $\phi(t)$ is given as

$$\frac{d^k \phi(t)}{dt^k} = M^k \phi(t) = (M^{(1)})^k \phi(t), \tag{2.6}$$

where k is a positive integer.

Now fractional derivative of order α of $\phi(t)$ is

$${}_0^c D_t^{\alpha} \phi(t) = t^{-\alpha} M^{(\alpha)} \phi(t), \ \ 0 < \alpha \leq 1, \tag{2.7}$$

where $M^{(\alpha)}$=$(m_{i,j}^{(\alpha)})$ is Lucas OM of order $(n+1) \times (n+1)$, which is lower triangular matrix and it is given as

$$M^{(\alpha)} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \eta_\alpha(\lceil \alpha \rceil, 0) & \eta_\alpha(\lceil \alpha \rceil, \lceil \alpha \rceil) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \eta_\alpha(i, 0) & \cdots & \eta_\alpha(i, i) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \eta_\alpha(n, 0) & \eta_\alpha(n, 1) & \eta_\alpha(n, 2) & \cdots & \eta_\alpha(n, n) \end{pmatrix}. \tag{2.8}$$

The elements $(m^{\alpha}_{i,j})$ are given explicitly by

$$m^{(\alpha)}_{ij} = \begin{cases} \eta_\alpha(i,j), & \text{if } i \geq \lceil \alpha \rceil \text{and } i \geq j, \\ 0, & \text{elsewhere}, \end{cases} ,$$

where

$$\eta_\alpha(i,j) = \sum_{\substack{k=\lceil \alpha \rceil \\ (i+k) even \\ (j+k) even}}^{i} \frac{(-1)^{\frac{k-j}{2}} \left(\frac{k+2+j}{2}\right)_{\frac{k-j}{2}} \delta_j \left(\frac{i-2+k}{2}\right)!}{\left(\frac{i-k}{2}\right)! \left(\frac{k-j}{2}\right)! \, \Gamma(1-\alpha+k)}. \tag{2.9}$$

2.3 Numerical method to solve two-dimensional fractional order diffusion model

Let us consider the following form of nonlinear Model 2.1 as

$${}^{c}_{0}D^{\alpha}_{t} u(x,y,t) = \Theta\Big(\frac{\partial^2 u(x,y,t)}{\partial x^2}, \frac{\partial^2 u(x,y,t)}{\partial y^2}, \frac{du(x,y,t)}{dx}, \frac{du(x,y,t)}{dy}, u(x,y,t), g(x,y,t)\Big), \tag{2.10}$$

and also the form of MT-TFDE Model 2.2 as

$$\sum_{i=1}^{q} d_i \ {}^{c}_{0}D^{\alpha_i}_{t} u(x,y,t) = \omega\Big(\frac{\partial^2 u(x,y,t)}{\partial x^2}, \frac{\partial^2 u(x,y,t)}{\partial y^2}, g(x,y,t)\Big), \tag{2.11}$$

where $0 < \alpha \leq 1$,

under the given initial and boundary conditions (2.3).

To approximate $u(x,y,t)$ with Lucas polynomial, let us consider

$$u(x,y,t) \cong \phi^T(t) U \psi(x,y).$$

The differentiation of order β(positive integer) of $\psi(x, y)$ w.r.t. x and y are given as

$$\begin{aligned}\frac{d^{\beta}\psi(x,y)}{dx^{\beta}} &= \frac{d^{\beta}(\phi(x)\otimes\phi(y))}{dx^{\beta}},\\ &= \frac{d^{\beta}\phi(x)}{dx^{\beta}}\otimes\phi(y),\\ &= (M^{(\beta)}\phi(x))\otimes(I\phi(y)),\\ &= (M^{\beta}\otimes I)(\phi(x)\otimes\phi(y),\\ &= M_x^{\beta}\psi(x,y),\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}\frac{d^{\beta}\psi(x,y)}{dy^{\beta}} &= \frac{d^{\beta}(\phi(x)\otimes\phi(y))}{dy^{\beta}},\\ &= \phi(x)\otimes\frac{d^{\beta}\phi(y)}{dy^{\beta}},\\ &= (I\phi(x))\otimes(M^{\beta}\phi(y)),\\ &= (I\otimes M^{\beta})(\phi(x)\otimes\phi(y)),\\ &= M_y^{\beta}\phi(x)\otimes\phi(y),\\ &= M_y^{\beta}\psi(x,y),\end{aligned} \tag{2.13}$$

where I is an identity matrix of order $(n+1)\times(n+1)$ and $M_x^{\beta} = (M^{\beta}\otimes I)$, $M_y^{\beta} = (I\otimes M^{\beta})$.

The derivative of $u(x, y, t)$ of order β (positive integer) w.r.t. x and y are given as

$$\begin{aligned}\frac{\partial^{\beta}u(x,y,t)}{\partial x^{\beta}} &\cong \frac{\partial^{\beta}}{\partial x^{\beta}}\phi^T(t)U\psi(x,y),\\ &= \phi^T(t)U\frac{\partial^{\beta}}{\partial x^{\beta}}\psi(x,y),\\ &= \phi^T(t)UM_x^{\beta}\psi(x,y),\end{aligned} \tag{2.14}$$

$$\begin{aligned}\frac{\partial^\beta u(x,y,t)}{\partial y^\beta} &\cong \frac{\partial^\beta}{\partial y^\beta}\phi^T(t)U\psi(x,y),\\ &= \phi^T(t)U\frac{\partial^\beta}{\partial y^\beta}\psi(x,y),\\ &= \phi^T(t)UM_y^\beta\psi(x,y),\end{aligned} \tag{2.15}$$

and derivative of $u(x,y,t)$ of order α, $0<\alpha<1$ w.r.t. t is given as

$$\begin{aligned}\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} &\cong \frac{\partial^\alpha}{\partial t^\alpha}\phi^T(t)U\psi(x,y),\\ &= \left(\frac{\partial^\alpha}{\partial t^\alpha}\phi^T(t)\right)U\psi(x,y),\\ &= \left(t^{-\alpha}M^{(\alpha)}\phi(t)\right)^T U\psi(x,y)\\ &= t^{-\alpha}\phi^T(t)(M^\alpha)^T U\psi(x,y).\end{aligned} \tag{2.16}$$

The Residual of the Model (2.1) is obtained by using Eqs.(2.12)-(2.16) as

$$\begin{aligned}&R(x,y,t) = t^{-\alpha}\phi^T(t)(M^\alpha)^T U\psi(x,y)\\ &-\Theta\Big(\phi^T(t)UM_x^2\psi(x,y), \phi^T(t)UM_y^2\psi(x,y), \phi^T(t)UM_x^{(1)}\psi(x,y), \phi^T(t)UM_y^{(1)}\psi(x,y), g(x,y,t)\Big),\end{aligned} \tag{2.17}$$

with the prescribed conditions

$$\begin{aligned}\phi^T(0)U\psi(x,y) &= \rho_1(x,y),\\ \phi^T(t)U\psi(0,y) &= \rho_2(y,t),\\ \phi^T(t)U\psi(1,y) &= \rho_3(y,t),\\ \phi^T(t)U\psi(x,0)) &= \rho_4(x,t),\\ \phi^T(t)U\psi(x,1) &= \rho_5(x,t).\end{aligned} \tag{2.18}$$

By choosing the collocation points as $x_i = y_i = t_i = \frac{2i-1}{2n+1}$, we may collocate the residual at $(n-1) \times (n-1) \times n$ points, so that the residual is zero.

$$R(x_i, y_j, t_k) = 0, \quad 1 \leq i \leq n-1, 1 \leq j \leq n-1, 1 \leq k \leq n. \tag{2.19}$$

Now collocating Eq.(2.19), we obtain $(n+1)^2 + 4n^2$ algebraic equations as

$$\begin{aligned}
\phi^T(0)U\psi(x_i, y_j) &= \rho_1(x_i, y_j), \quad 1 \leq i \leq n+1, 1 \leq j \leq n+1 \\
\phi^T(t_k)U\psi(0, y_j) &= \rho_2(y_j, t_k), \quad 1 \leq j \leq n, 1 \leq k \leq n \\
\phi^T(t_k)U\psi(1, y_j) &= \rho_3(y_j, t_k), \quad 1 \leq j \leq n, 1 \leq k \leq n \\
\phi^T(t_k)U\psi(x_i, 0)) &= \rho_4(x_i, t_k), \quad 1 \leq i \leq n, 1 \leq k \leq n \\
\phi^T(t_k)U\psi(x_i, 1) &= \rho_5(x_i, t_k), \quad 1 \leq i \leq n, 1 \leq k \leq n.
\end{aligned} \tag{2.20}$$

From Eqs.(2.19) and (2.20), total $(n+1)^3$ algebraic equations are obtained, which can be solved by well-known Newton's method. Similarly, the residual of Model (2.2) is obtained by collocating $(n-1) \times (n-1) \times n$ points and solved by Newton's method.

2.4 Numerical examples

This section discusses the accuracy, efficiency, and performance of the concerned numerical technique. For this purpose, the concerned method is applied to some problems that have already been solved analytically/ numerically by other researchers. A comparison has been made of the results obtained by the proposed scheme with the existing results through the absolute errors (AE) defined by

$$E_n(x_i, y_j, t_k) = |u(x_i, y_j, t_k) - \tilde{u}(x_i, y_j, t_k)|, \quad 0 \leq x_i, y_j, t_k \leq 1 \tag{2.21}$$

where $\tilde{u}(x_i, y_j, t_k)$ and $u(x_i, y_j, t_k)$ are approximate and exact solutions, respectively at (x_i, y_j, t_k) .

Example 2.1 Consider a two-dimensional FDE as

$$ {}_0^cD_t^\alpha u(x,y,t) = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} - \left(\frac{\partial u(x,y,t)}{\partial x} + \frac{\partial u(x,y,t)}{\partial y} \right) + g(x,y,t), \tag{2.22} $$

with

$$ g(x,y,t) = e^x \left(1 + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \right), $$

under the following conditions

$$ \begin{aligned} u(x,y,0) &= e^x y, \\ u(0,y,t) &= t + y, \\ u(1,y,t) &= e(t+y), \\ u(x,0,t) &= e^x t, \\ u(x,1,t) &= e^x (t+1), \end{aligned} \tag{2.23} $$

having the exact solution as $u(x, y, t) = (t + y)e^x$. The AEs of Example 2.1 for $\alpha = 0.4$ and $\alpha = 0.8$ are shown in tabular forms. From Table 2.1 and Table 2.2, it is clear that the numerical solution of the Example 2.1 by the proposed method is almost equal to exact solution for small values of n. The absolute error can be reduced with the increase in n, which shows higher convergence rate of the proposed scheme as compared to other existing methods.

TABLE 2.1: The absolute errors for $n = 8$ and $n = 10$ with $\alpha = 0.4$ at $t = 0.5$ for Example 2.1

(x, y)	$E_8(x, y, 0.5)$	$E_{10}(x, y, 0.5)$
$(0.1, 0.1)$	1.06509×10^{-10}	1.53988×10^{-13}
$(0.2, 0.2)$	4.11712×10^{-10}	5.96634×10^{-13}
$(0.3, 0.3)$	9.25109×10^{-10}	1.34692×10^{-12}
$(0.4, 0.4)$	1.64009×10^{-09}	2.39120×10^{-12}
$(0.5, 0.5)$	2.53065×10^{-09}	3.70126×10^{-12}
$(0.6, 0.6)$	3.56142×10^{-09}	5.21760×10^{-12}
$(0.7, 0.7)$	4.67059×10^{-09}	6.87717×10^{-12}
$(0.8, 0.8)$	5.87935×10^{-09}	8.64508×10^{-12}
$(0.9, 0.9)$	6.09956×10^{-09}	9.84990×10^{-12}

TABLE 2.2: The absolute errors for $n = 8$ and $n = 10$ with $\alpha = 0.8$ at $t = 0.5$ for Example 2.1

(x, y)	$E_8(x, y, 0.5)$	$E_{10}(x, y, 0.5)$
$(0.1, 0.1)$	5.41911×10^{-11}	7.74936×10^{-14}
$(0.2, 0.2)$	2.23101×10^{-10}	3.20965×10^{-13}
$(0.3, 0.3)$	5.52441×10^{-10}	8.02469×10^{-13}
$(0.4, 0.4)$	1.08100×10^{-09}	1.57407×10^{-12}
$(0.5, 0.5)$	1.83223×10^{-09}	2.67963×10^{-12}
$(0.6, 0.6)$	2.81421×10^{-09}	4.12514×10^{-12}
$(0.7, 0.7)$	3.98735×10^{-09}	5.88463×10^{-12}
$(0.8, 0.8)$	5.36519×10^{-09}	7.91944×10^{-12}
$(0.9, 0.9)$	5.84315×10^{-09}	9.55636×10^{-12}

Example 2.2 Consider the nonlinear diffusion equation given by

$$\frac{\partial u(x,y,t)}{\partial t} = \nabla^2 u(x,y,t) + u^2(1-u), \tag{2.24}$$

under initial and boundary conditions

$$\begin{aligned}
u(x,y,0) &= \left[1 + exp\left(\frac{x}{\sqrt{3}} + \frac{y}{\sqrt{6}}\right)\right]^{-1}, \\
u(0,y,t) &= \left[1 + exp\left(\frac{y}{\sqrt{6}} - \frac{t}{2}\right)\right]^{-1}, \\
u(1,y,t) &= \left[1 + exp\left(\frac{1}{\sqrt{3}} + \frac{y}{\sqrt{6}} - \frac{t}{2}\right)\right]^{-1}, \\
u(x,1,t) &= \left[1 + exp\left(\frac{x}{\sqrt{3}} + \frac{1}{\sqrt{6}} - \frac{t}{2}\right)\right]^{-1}, \\
u(x,0,t) &= \left[1 + exp\left(\frac{x}{\sqrt{3}} - \frac{t}{2}\right)\right]^{-1},
\end{aligned} \tag{2.25}$$

whose analytical solution is $u(x,y,t) = \left[1 + exp\left(\frac{x}{\sqrt{3}} + \frac{y}{\sqrt{6}} - \frac{t}{2}\right)\right]^{-1}$.

Table 2.3: The absolute errors for $n = 8$ and $n = 10$ at $t = 0.5$ for Example 2.2

(x,y)	$E_8(x,y,0.5)$	$E_{10}(x,y,0.5)$
$(0.1,0.1)$	3.15237×10^{-12}	1.26565×10^{-14}
$(0.2,0.2)$	1.11112×10^{-12}	4.62963×10^{-14}
$(0.3,0.3)$	2.45729×10^{-11}	1.04083×10^{-13}
$(0.4,0.4)$	4.23398×10^{-11}	1.80300×10^{-13}
$(0.5,0.5)$	6.16443×10^{-11}	2.62845×10^{-13}
$(0.6,0.6)$	7.84176×10^{-11}	3.31957×10^{-13}
$(0.7,0.7)$	8.75495×10^{-11}	3.66429×10^{-13}
$(0.8,0.8)$	8.59888×10^{-11}	3.59268×10^{-13}
$(0.9,0.9)$	7.12742×10^{-11}	3.41727×10^{-13}

The above Table 2.3 shows the AE between analytical results and numerical results. It is seen that for the nonlinear problem, the AE between exact solution and approximate solution is of order 10^{-12} and 10^{-14} for $n = 8$ and $n = 10$, respectively which can be decreased on increasing the value of n.

Example 2.3 Consider the time fractional two-dimensional nonlinear diffusion equation given as

$$ {}_0^cD_t^{0.6}u(x,y,t) = \nabla^2 u(x,y,t) + u^2(x,y,t) + g(x,y,t), \tag{2.26} $$

where

$$ g(x,y,t) = e^y\left(\frac{t^{0.4}}{\Gamma(1.4)} - e^y(t+x)^2 - (t+x)\right), $$

TABLE 2.4: The absolute errors for $n = 8$ and $n = 10$ at $t = 0.5$ for Example 2.3

(x, y)	$E_8(x, y, 0.5)$	$E_{10}(x, y, 0.5)$
$(0.1, 0.1)$	1.48230×10^{-10}	2.14273×10^{-13}
$(0.2, 0.2)$	5.71009×10^{-10}	8.33666×10^{-13}
$(0.3, 0.3)$	1.27240×10^{-09}	1.86429×10^{-12}
$(0.4, 0.4)$	2.22245×10^{-09}	3.26139×10^{-12}
$(0.5, 0.5)$	3.35403×10^{-09}	4.93916×10^{-12}
$(0.6, 0.6)$	4.56775×10^{-09}	6.75149×10^{-12}
$(0.7, 0.7)$	5.71915×10^{-09}	8.53184×10^{-12}
$(0.8, 0.8)$	6.74699×10^{-09}	1.01736×10^{-11}
$(0.9, 0.9)$	6.53286×10^{-09}	1.11613×10^{-11}

whose initial and boundary conditions can be obtained from the exact solution as in Example 2.1. The above equation has the analytical solution as $u(x, y, t) = e^y(t+x)$.

Table 2.4 shows the AE between analytical and numerical solutions.

Example 2.4 Consider the following MT-TFDE as

$$d_1 \, {}_0^c D_t^{\alpha_1} u(x,y,t) + d_2 \, {}_0^c D_t^{\alpha_2} u(x,y,t) = \nabla u(x,y,t) + g(x,y,t), \tag{2.27}$$

The approximate solution of the above mentioned two-dimensional MT-TFDE can be determined for $d_1 = 1$,$d_2 = 0.35$,$\alpha_1 = 0.60$ and $\alpha_2 = 0.40$ for the suitable choice of $g(x,y,t)$ such that its exact solution is $u(x,y,t) = t \ \sin x \ \sin y$ and the initial and boundary conditions can be obtained from the analytical solution.

The AEs for different choices of x and y are depicted in Table 2.5. From the above, one can analyze that the approximate results obtained by the presented approach is in nice agreement with the exact results.

Table 2.5: The absolute errors for $n = 8$ and $n = 10$ at $t = 0.5$ for Example 2.4

(x,y)	$E_8(x,y,0.5)$	$E_{10}(x,y,0.5)$
$(0.1, 0.1)$	2.93013×10^{-11}	3.30437×10^{-14}
$(0.2, 0.2)$	9.09550×10^{-11}	1.31839×10^{-13}
$(0.3, 0.3)$	2.07445×10^{-10}	3.01988×10^{-13}
$(0.4, 0.4)$	3.76958×10^{-10}	5.48950×10^{-13}
$(0.5, 0.5)$	6.04058×10^{-10}	8.81170×10^{-13}
$(0.6, 0.6)$	8.94985×10^{-10}	1.30523×10^{-12}
$(0.7, 0.7)$	1.24676×10^{-09}	1.82449×10^{-12}
$(0.8, 0.8)$	1.67331×10^{-09}	2.43866×10^{-12}
$(0.9, 0.9)$	1.77749×10^{-09}	2.93099×10^{-12}

Example 2.5 Considering $q = 2$, the Model (2.2) is reduced in the following MT-TFDE as

$$d_1\, {}_0^cD_t^{\alpha_1}u(x,y,t) + d_2\, {}_0^cD_t^{\alpha_2}u(x,y,t) = \nabla u(x,y,t) + g(x,y,t). \tag{2.28}$$

The approximate solution of the above MT-TFDE is obtained by using the proposed method for $d_1 = 1$, $d_2 = 1$, $\alpha_1 = 0.15$ and $\alpha_2 = 0.95$ for the proper choice of $g(x,y,t)$ so that its analytical solution is given as $u(x,y,t) = \sin x\ \sin y\ (t^3 + t^2 + t + 1)$ with the initial and boundary conditions obtained from the exact solution.
The errors between the existing analytical and the approximate solution by using the proposed approach are obtained for various values of x, y and $t = 0.5$, which are incorporated in Table 2.6. From the above results one can say that he approach introduced in this chapter produces excellent approximate results of the problems. Hence this approach is considered as the best tool to deal with such types of problems.

Table 2.6: The absolute errors for $n = 8$ and $n = 10$ at $t = 0.5$ for Example 2.5

(x,y)	$E_8(x,y,0.5)$	$E_{10}(x,y,0.5)$
$(0.1, 0.1)$	8.37133×10^{-11}	1.20706×10^{-13}
$(0.2, 0.2)$	3.33924×10^{-10}	4.81115×10^{-13}
$(0.3, 0.3)$	7.60437×10^{-10}	1.10353×10^{-12}
$(0.4, 0.4)$	1.38476×10^{-09}	2.00995×10^{-12}
$(0.5, 0.5)$	2.22477×10^{-09}	3.23391×10^{-12}
$(0.6, 0.6)$	3.30622×10^{-09}	4.80371×10^{-12}
$(0.7, 0.7)$	4.62109×10^{-09}	6.73306×10^{-12}
$(0.8, 0.8)$	6.22444×10^{-09}	9.01579×10^{-12}
$(0.9, 0.9)$	6.63281×10^{-09}	1.07920×10^{-11}

2.5 Solution of the problem

In this section, the solute concentration of the considered two-dimensional FRADE Model (2.1) is calculated by applying the proposed method for various values of α and k for $n = 2$ under the following initial and boundary conditions.

$$\phi^T(0)C\psi(x_i, y_i) = x + y, \quad 1 \leq i \leq n+1, \ 1 \leq j \leq n+1,$$
$$\phi^T(t_k)C\psi(0, y_j) = y, \quad 1 \leq j \leq n, \ 1 \leq k \leq n,$$
$$\phi^T(t_k)CM_x^1\psi(1, y_j) = e^t, \quad 1 \leq j \leq n, \ 1 \leq k \leq n,$$
$$\phi^T(t_k)CM_y^1\psi(x_i, 0) = 1, \quad 1 \leq i \leq n, \ 1 \leq k \leq n,$$
$$\phi^T(t_k)CM_y^1\psi(x_i, 1) = 1, \quad 1 \leq i \leq n, \ 1 \leq k \leq n.$$

The variations of the solute concentration versus x and y are determined for different fractional and integer values of α. The variations in the solute concentration are also exhibited graphically for conservative and non-conservative cases. The effects of the solute profile for two-dimensional FRADE are shown in Fig. 2.1 for different values of α. On increasing the values of α, the concentration decreases with the presence of sink term. The variations of solute concentration for the integer order ($\alpha = 1$) two-dimensional RADE are shown through Fig. 2.2 for both the conservative and non-conservative cases. It is observed that the concentration will be higher and lower than that of the conservative system ($k = 0$) due to the presence of source ($k = 1$) and sink ($k = -1$) terms, respectively, which are physically relevant.

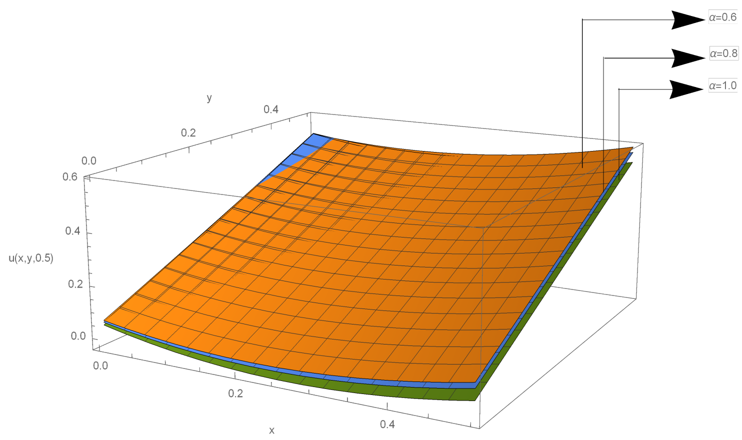

FIGURE 2.1: Plots of solute profile vs. x and y for non-conservative system with sink term ($k = -1$) for various values of α.

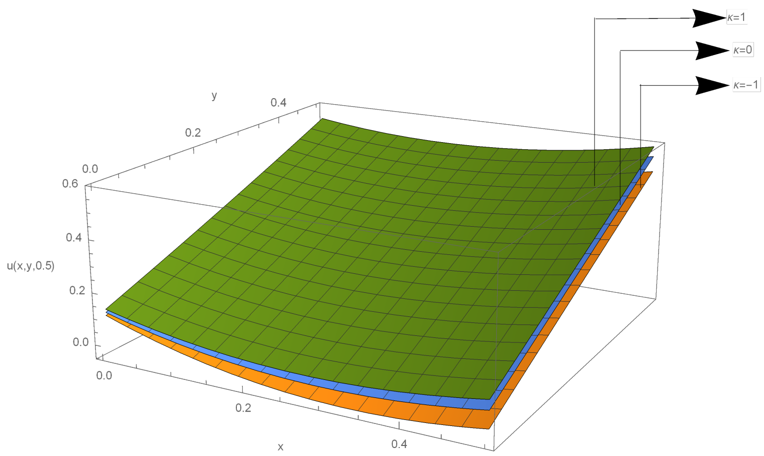

FIGURE 2.2: Plots of solute profile vs. x and y for conservative and non-conservative systems for α=1 at t=0.5.

2.6 Conclusion

The present chapter establishes a method to find the approximate solution of a two-dimensional nonlinear time-fractional diffusion equation. The Lucas polynomial and OM of the Lucas polynomial are used to find the solutions to the problems considered. Using the spectral collocation method, we used the Kronecker product property of Lucas polynomials to approximate the solution. In this approach, we apply the spectral collocation method to ensure that the residuals, as well as the initial and boundary conditions of the mathematical model, are collocated at specific collocation points. This process leads to a system of nonlinear algebraic equations, which we then solve using Newton's method. The efficiency of the proposed scheme is shown through absolute error when applied to a few existing problems. The accuracy can be improved by increasing the terms of polynomials. The author is confident that the method can be applied to find the approximate solutions of other types of two-dimensional multi-term fractional order PDEs.

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