Analysis of High-Order Methods for Time-Fractional Partial Differential Equations with Smooth and Non-smooth Solutions

Thesis submitted in partial fulfillment

for the award of degree of

Doctor of Philosophy

by

Anshima Singh

Department of Mathematical Sciences

Indian Institute of Technology

(Banaras Hindu University)

Varanasi – 221 005

Roll No.: 18121512

December 2023

Conclusions

When a differential equation becomes too complex - involving things like nonlocal effects, anomalous patterns, or time dependencies - mathematical analysis alone isn't enough. We have to turn to numerical simulations to find approximate solutions. This is especially true for FPDEs, where their nonlocal nature adds an extra layer of difficulty that makes analytical solutions very difficult to find. Thus, the primary emphasis of this thesis lies in developing numerical approximation of FPDEs arising from time-fractional derivatives, including the Caputo fractional derivative, the Caputo-Prabhakar fractional derivative, and generalized fractional derivatives. Further, the focus of this thesis extends beyond numerical methods formulation to include a comprehensive theoretical analysis. This analysis encompasses discussions of stability and convergence properties concerning the developed methods. Beyond this, one more key aspect of this thesis is the presentation of numerical outcomes that reinforce the theoretical findings.

This chapter provides a concise summary of the key finding, emphasizing the contributions of the thesis and the methodologies employed to achieve these results. The conclusion and notable observations are outlined as follows:

- Chapter 1 serves as the introduction to the thesis. Fundamental definitions, used consistently in the thesis, are brought together in this chapter. This chapter also provides an in-depth review of the historical progress made in the field of fractional problems and their numerical treatment. It also includes a concise literature survey related to the problems addressed in the thesis, along with a presentation of the thesis objectives.

- Chapter 2 focuses on developing and analyzing two high-order schemes, $\mathrm{CFD}_{g-\sigma}$ and $\mathrm{PQS}_{g-\sigma}$, for solving generalized variable coefficients fractional reaction-diffusion equations. These schemes use the $(3-\alpha)$th order Generalized Alikhanov's formula $(gL2-1_\sigma)$ to approximate the generalized time-fractional derivative of Caputo type, with α representing the order of the GFD. $\mathrm{CFD}_{g-\sigma}$ employs a fourth-order compact difference operator, while $\mathrm{PQS}_{g-\sigma}$ utilizes a parametric quintic spline scheme for spatial discretization. The stability and convergence analysis of both schemes are demonstrated thoroughly using the discrete energy method in the $\mathcal{L}_2$-norm. Numerical results align well with the proposed theory.

- Chapter 3 is devoted to constructing and analysing two new approximations (CPL2-1_σ and CPL-2 formulas) for the Caputo-Prabhakar fractional derivative. The newly developed approximations are then used in the numerical treatment of a reaction-diffusion problem with variable coefficients defined in the Caputo-Prabhakar sense. Moreover, the space variable in the developed numerical schemes, CFD_1 and CFD_2, is discretized using a fourth-order compact difference operator. Both schemes' stability and convergence analysis are demonstrated thoroughly using the discrete energy method. Numerical results are consistent with our theoretical findings.

- Chapter 4 aims to develop and analyze a robust fully discrete scheme for solving time-fractional reaction-diffusion equations with both smooth and non-smooth solutions. The method combines the L1 scheme for time discretization on uniform or graded meshes with a cubic spline difference scheme on a uniform mesh for spatial discretization. It incorporates a graded mesh in time to address the initial layer at $t = 0$. The chapter provides separate stability and convergence analyses for both smooth and non-smooth solutions. Two test problems are presented to validate the proposed method, one featuring a smooth solution and the other a non-smooth solution.

- Chapter 5 introduces a high-order non-polynomial spline method for non-linear time-fractional reaction-diffusion equations with non-smooth solutions. This method combines the L2-1_σ scheme on a graded mesh to approximate the Caputo fractional derivative and utilizes a parametric quintic spline for spatial discretization. The chapter demonstrates the solvability, stability, and convergence of the scheme and provides numerical results supporting the theory.

- Finally, Chapter 6 introduces an innovative high-order numerical method for solving the two-dimensional time-fractional convection-diffusion equation with non-smooth solutions. This approach combines Alikhanov's high-order scheme (L2-1_σ) for Caputo time-fractional derivative discretization on a non-uniform fitted mesh with a high-order two-dimensional compact difference operator on a uniform mesh for spatial discretization. The system is efficiently solved using a robust two-step Alternating Direction Implicit (ADI) approach. The chapter includes a comprehensive theoretical analysis, focusing on convergence and stability, employing Fourier analysis. It also presents numerical results, in the form of graphs and tables, validating the proposed theory.