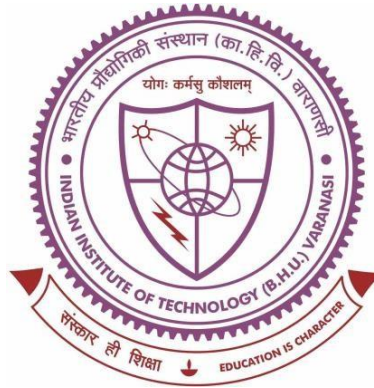


MATHEMATICAL MODELING ON VARIOUS  
STRUCTURES UNDER COUPLED  
DYNAMICAL THERMOELASTICITY



Thesis submitted in partial fulfillment for the  
award of degree

**Doctor of Philosophy**

by

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Dedicated  
to  
**My Family**

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*Bhagwan Singh*

**Bhagwan Singh**



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## LIST OF SYMBOLS

$\boldsymbol{\sigma} = (\sigma_{ij})$	Stress tensor
$\mathbf{u} = (u_i)$	Displacement vector
$\mathbf{e} = (e_{ij})$	Strain tensor
$\mathbf{q} = (q_i)$	Heat flux vector
$T_0$	Reference temperature
$\theta$	Temperature variation from the reference temperature
$T$	Temperature
$\beta$	Thermoelastic coupling parameter
$L$	Length of beam
$b$	Width of beam/plate
$h$	Thickness of beam/plate
$l$	Material length scale parameter
$\rho$	Mass density
$K$	Thermal conductivity
$K^*$	Thermal conductivity rate
$w$	Deflection
$\lambda, \mu$	Lame's constants of material

$\tau_q$	Phase-lag of heat-flux vector
$A$	Cross-section area
$\nabla$	Gradient operator
$\Delta/\nabla^2$	Laplacian operator
$\delta_{ij}$	Kronecker delta
$\hat{\Upsilon}_{K.E}$	Kinetic energy
$l_s$	Strain gradient parameter
$I$	Moment of inertia
$U_s$	Strain energy function
$\chi = \chi_{ij}$	Symmetric part of the rotation gradient tensor
$\omega$	Frequency of vibration
$Q^{-1}$	Inverse Quality factor
$R$	Heat source per unit mass
$\alpha_t$	Thermal expansion coefficient
$c_E$	Specific heat at constant volume
$E$	Young's modulus
$\Upsilon = (\Upsilon_{ij})$	Deviatoric part of the couple stress tensor
$\nu$	Poisson's ratio

**Note:** Throughout the thesis, the subscripted comma notations are used to denote the partial derivatives with respect to the space variables. The over-headed dots denote partial derivatives with respect to time variable,  $t$ . The bold notation is used for vector or tensor quantities. Subscripts  $i, j, k$  take the values  $x, y, z$  and summation is implied by index repetition.

# ABBREVIATIONS

<b>CTE</b>	Classical Coupled <b>T</b> hermoelasticity <b>T</b> heory
<b>CT</b>	Classical Elasticity <b>T</b> heory
<b>TED</b>	Thermoelastic <b>D</b> amping
<b>QF</b>	Quality <b>F</b> actor
<b>LS</b>	Lord- <b>S</b> hulman Thermoelasticity <b>T</b> heory
<b>DPL</b>	Dual- <b>P</b> hase- <b>L</b> ag Thermoelasticity <b>T</b> heory
<b>GN</b>	Green- <b>N</b> aghdi Thermoelasticity <b>T</b> heory
<b>TPL</b>	Three- <b>P</b> hase- <b>L</b> ag Thermoelasticity <b>T</b> heory
<b>MGT</b>	Moore- <b>G</b> ibson- <b>T</b> hompson Thermoelasticity <b>T</b> heory
<b>CCT</b>	Classical <b>C</b> ontinuum <b>T</b> heory
<b>MCST</b>	Modified <b>C</b> ouple <b>S</b> tress <b>T</b> heory
<b>NBT</b>	Nonlocal <b>B</b> eam <b>T</b> heory
<b>SGT</b>	Strain gradient <b>T</b> heory
<b>NSGT</b>	Nonlocal <b>S</b> train gradient <b>T</b> heory
<b>MEMS/NEMS</b>	Micro/ <b>N</b> ano- <b>E</b> lectromechanical <b>S</b> ystems



# PREFACE

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Micro-electromechanical systems (MEMS) and nano-electromechanical systems (NEMS), which operate at the micro- and nanoscale, combine mechanical components with sensors, actuators, and other electronic devices. The creation of nanoscale resonators has gained significant attention in recent years. Specially, beam and plate resonators are observed to have wide applications as wearable sensors, flexible electronic devices, semiconductor nanostructures etc. Many areas of engineering physics are benefited from high-quality resonating systems because they offer excellent frequency resolution and less energy dissipation rate during oscillation. However, there are many energy loss mechanisms (Thermoelastic damping, support loss, air damping, etc.) that contribute to reduce the performance of microstructure (resonator) which are based on extrinsic and intrinsic loss factors. Proper design and suitable operating conditions can minimize the rate of energy diminution. Thermoelastic damping (TED) is one of the dominant sources of intrinsic energy loss dissipation mechanism in thermoelastic structures due to irreversible heat flow. The performance assessment of TED in micro and nanobeam resonators is based on the quality factor (QF) that is determined by the stored energy in system and the energy dissipated by the system per vibration cycle. For introducing the concept of the theoretical estimation of TED in thin beams, the works reported by Zener (1937; 1938) and Lifshitz and Roukes (2000) are worth to be mentioned. Capturing the size effect for extremely thin beam and plate resonators is more challenging and conventional continuum theory is found to be incapable of predicting accurate es-

timation of TED. Therefore, the non-classical continuum theories have been developed to explain such micro and nanomechanical behavior in order to address this difficulty and describe the size effects in dynamic response for micro components.

An essential aspect of thermoelasticity is the interaction between the mechanical and thermal fields. It is an expansion of elasticity theory that considers thermal effects, including thermal stress, strain, and deformation. Essentially, it generalizes both elasticity theory and heat conduction theory. Biot (1956) gave the first comprehensive model of coupled thermoelasticity theory based on Fourier's law that has been widely applied to deal with the thermoelastic interactions due to applied thermal and mechanical disturbances in an elastic medium. However, this theory predicts an infinite speed behaviour of thermal disturbances, which is a physically unrealistic behavior in many situations. Therefore, the efforts to overcome this shortcoming of the conventional theory in simulating laser processing and the high-frequency response of materials gave rise to an effectively improved version of the classical theory. Two general concepts provide the foundation for the remodeling of the conventional thermoelasticity theory. First, by changing Fourier's law; second, by changing other constitutive laws while keeping Fourier's law unaltered. The generalized theory introduced by Lord and Shulman (1967), the dual phase-lag theory (1998), three phase-lag theory (2007), the modified thermoelasticity theory developed by Green and Naghdi (1991; 1992; 1993) are some of the well-studied generalized thermoelastic models that have drawn extensive research interest to understand the heat conduction in the material in a more realistic way. Recently, Quintanilla (2019) developed an innovative theory of thermoelasticity by employing the MGT heat conduction equation that gained the serious attention of researchers. The present thesis is aimed at the analysis of some aspects of the recent models of generalized thermoelasticity and their impacts on thermoelastic damping in micro and nano resonating systems.

First part of the thesis covers a theoretical analysis of the MGT thermoelastic model

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with Galerkin-type representations and fundamental solutions of field equations, discussed in **Chapters 2** and **3**. The second part of the thesis consists of three chapters, **Chapter 4**, **Chapter 5**, and **Chapter 6**, which address vibration analysis of microstructures (beam and plate resonators) based on nonclassical continuum theories. The purpose of the thesis is to theoretically analyze the MGT thermoelastic model and the resonant behavior of the micro/nano systems under nonclassical continuum theories in the frame of MGT thermoelasticity theory.

The work presented in the thesis is divided into seven chapters, out of which the **First Chapter** is the introduction. The design and development of MEMS/NEMS systems and their uses are briefly described in **this Chapter**. After that, the higher-order non-continuum theories that capture the size effects in the beam and plate resonators are elaborated comprehensively. This chapter also covers a brief description about the development of various generalized thermoelasticity theories. Additionally, this chapter summarizes the relevant literature review of the present study. Lastly, the chapter concludes with the objective of the thesis.

**Chapter 2** theoretically discusses the contemporary Moore-Gibson-Thompson thermoelastic model by deriving the Galerkin-type representation of the field equations of motion for an isotropic homogeneous elastic medium in the presence of body force and heat source. In order to achieve a fundamental solution to this theory, these Galerkin-type representations are the foundations. The Galerkin-type solution of basic governing equations is acquired in terms of the elementary functions. In the case of steady oscillations, an analogous approach is followed to establish the Galerkin-type representation of the solution in the context of elementary functions. Finally, the general solution of the system of equations in the case of steady oscillations is also derived in terms of metaharmonic functions.

**Chapter 3** of the thesis is devoted to obtain the fundamental solution of the MGT thermoelastic theory by exploring the Galerkin-type representation of the coupled field

equations as established in Chapter 2. Then fundamental solutions in the physical domain for the short-time approximated displacement components and temperature fields are derived analytically by applying the Laplace inversion technique. Lastly, the fundamental solutions in case of steady oscillations are obtained in the present context.

**Chapter 4** starts with the second part of the thesis. It has two subchapters which address the size-dependent vibration analysis for the two different categories of beam resonators: the Euler–Bernoulli beam model and Timoshenko beam resonator, respectively in the context of the MGT thermoelasticity theory. The **Subchapter 4.1** estimates the size-dependent expression of the thermoelastic damping in terms of quality factor for rectangular micro beam resonators using the frequency approach method. Effects of various parameters on quality factor of the beam are discussed in a detailed manner. **Subchapter 4.2** discusses the variational formulation of the dimensionless deflection and thermal moment for the Timoshenko beam resonator by considering Hamilton’s principle and modified couple stress theory which involves the material length scale parameter. The valid analytical formula for dimensionless deflection and the normalized thermal moment is derived by using Fourier series and Laplace transform technique with the appropriate boundary conditions. The findings are compared and analyzed with those obtained by the classical continuum theory (CCT) and the modified couple stress theory (MCST). MGT and LS models are shown to display a wider range of thermal moment, while the GN-III model displays a smaller range of thermal moment.

**Chapter 5** discusses about an analytical technique for measuring thermoelastic damping (TED) and dynamic behavior of micro/nano Kirchhoff plate resonators using Moore–Gibson–Thompson (MGT) generalized thermoelasticity theory. Modified couple stress theory (MCST) is employed here to take into account the size effect for micro/nano plate. This chapter comprehensively evaluates the frequency shift and normalized attenuation in the microplate resonator in the frame MCST. An explicit

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expression for determining TED in microplate is established by solving coupled partial differential equations, including small-scale effects and the effects of thermal relaxation parameter involved in MGT theory.

**Chapter 6** is devoted to predict the bending and temperature characteristics of the Euler–Bernoulli beam using Moore–Gibson–Thompson thermoelasticity theory in conjunction with nonlocal strain gradient theory (NSGT). Here the Legendre wavelet method is employed. Utilizing the Laplace transform technique and the wavelet approximation method, the coupled field equations for dimensionless deflection and temperature change are constructed and solved for the case of ramp-type temperature boundary conditions. By increasing the length scale parameter of nonlocal theory, a stiffness-softening effect is observed. On the contrary, an increase in the strain gradient parameter is noted to produce stiffness-hardening behavior.

**Chapter 7** highlights the summary of the current work as well as describes the possible directions of future scope of this work.

