# PSEUDO-DIFFERENTIAL OPERATORS INVOLVING THE WEINSTEIN TRANSFORM

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**by**

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From (1.4.1), the initial condition gives

$$\big(\mathcal{F}_\alpha u\big)(\xi,0)=\big(\mathcal{F}_\alpha f\big)(\xi). \tag{2.5.7}$$

Using (2.5.7) in (2.5.6), we find out $C(\xi)=\big(\mathcal{F}_\alpha f\big)(\xi)$.

Therefore, (2.5.6) becomes

$$(\mathcal{F}_\alpha u)(\xi,t)=\big(\mathcal{F}_\alpha f\big)(\xi)e^{-t\|\xi\|^2}. \tag{2.5.8}$$

Since $f\in S_*(\mathbb{R}^{n+1}_+)$, then from (1.4.6), we have $(\mathcal{F}_\alpha f)\in S_*(\mathbb{R}^{n+1}_+)$.

Taking inverse Weinstein transform in (2.5.8), we get

$$\begin{aligned}u(x,t)&=\int_{\mathbb{R}^{n+1}_+}e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})e^{-t\|\xi\|^2}(\mathcal{F}_\alpha f)(\xi)d\mu_\alpha(\xi)\\&=\int_{\mathbb{R}^{n+1}_+}e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma_t(\xi)(\mathcal{F}_\alpha f)(\xi)d\mu_\alpha(\xi).\end{aligned}$$

This shows that, $u(x,t)=(T_{\sigma_t}f)(x)$ is the solution of heat equation (6.4.1) for any initial function $f$.

**Conclusion.**

The Weinstein transform contains strong calculus and it is used in many problems of mathematics, particularly in wavelets, partial-differential equations and distributions. Taking the theory of aforesaid transform, many authors found useful observations in harmonic analysis and other areas. In their research works, Chettaoui-Trimèche [10], Hleili [24], Hleili-Hleili [25], Saudi [66] expounded different problems in their papers. In the present chapter, authors found the theory of pseudo-differential operators associated with the Weinstein transform and studied its properties. Utilizing the aforesaid results, the Sobolev type of spaces and associated results were

discussed in this chapter. The solution of the heat equation is obtained and it is expressed in the form of pseudo-differential operators by using the Weinstein transform technique.

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For $p > \frac{1}{2}(\alpha + \frac{n}{2} + r + 3)$, there is a constant $C_{\alpha,p,r,t}$ such that

$$\left\| \left[\mathcal{P}'(x,D), \mathcal{B}(\infty,D)\right] u \right\|_{\mathcal{H}_\alpha^{r,2}} \leq C_{\alpha,p,r,t} \|u\|_{\mathcal{H}_\alpha^{r+t,2}}.$$

□

**Conclusion**: On the basis of rich calculus of the Weinstein transform and results of Zaidman [82], Pathak and Upadhyay [56], Upadhyay [78], Upadhyay and Chauhan [79] and others pseudo-differential operators $\mathcal{P}(x,D)$ and $\mathcal{Q}(x,D)$ associated with the homogeneous class of symbol of order zero are introduced and discussed boundedness properties on $\mathcal{H}_\alpha^{r,2}$ - type Sobolev space by considering theory of the Weinstein transform. Product of symbols and commutators of two pseudo-differential operators associated with homogeneous class of symbol are defined and its properties on certain type of Sobolev space are discussed. The aforesaid results are useful to find the weak solution of a pseudo-differential equation of pseudo-differential operators $\mathcal{P}(x,D)$ and $\mathcal{Q}(x,D)$ associated with homogeneous symbol $\sigma(x,\xi)$.

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which are linear. Then, from Theorem 4.3.4, the operators $J_{-r,\alpha} : H^{r,p}_\alpha \to H^{0,p}_\alpha$ and $J_{r-m,\alpha} : H^{0,p}_\alpha \to H^{r-m,p}_\alpha$ are bounded. Also, by Theorem 4.2.5, $T_\sigma J_{m,\alpha} : L^p_\alpha(\mathbb{R}^{n+1}_+) \to L^p_\alpha(\mathbb{R}^{n+1}_+)$ is the bounded linear operator. Hence, $T_\sigma : H^{r,p}_\alpha \to H^{r-m,p}_\alpha$ is a bounded linear operator. □

**Conclusion**: Taking concepts of the papers of Fefferman [17], Illner [30], Cato [31], Nagase [43], Hwang-Lee [29], Wong [81] and Pathak and Upadhyay [57], the $L^p_\alpha(\mathbb{R}^{n+1}_+)$ - boundedness of pseudo-differential operators associated with the Weinstein transform on a certain class of symbol $S^0$ is discussed. Applying the theory of $L^p_\alpha(\mathbb{R}^{n+1}_+)$ - boundedness results, various properties of pseudo-differential operators associated with $L^p_\alpha(\mathbb{R}^{n+1}_+)$ - Sobolev space are investigated. The Weinstein transform has rich calculus and well established theory. The results of $L^p_\alpha(\mathbb{R}^{n+1}_+)$ - boundedness of pseudo-differential operators contains useful applications in Sobolev spaces which are given in the present chapter. The aforesaid theory acts as the bridge between pseudo-differential operators and maximal - minimal pseudo-differential operators. This theory will also be useful in functional analysis, partial-differential equations and other areas of mathematics.

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## 5.5 Conclusion

Employing the theory of the Fourier transform technique Kohn-Nirenberg [32], Hörmander [26], Grushin [22], Kumano-go [34], Taylor [70], Wong [81], others made a proper foundation for the theory of pseudo-differential operators and Coriasco-Rodino [11], Nicola [47], Nicola-Rodino [48], Cappiello-Rodino [5] and Dasgupta-Wong [14] gave adequate development regarding the theory of SG pseudo-differential operators. The importance of the aforesaid theory is that they construct the proper smell of pseudo-differential operators related to the non-compact manifolds. Motivated by the above works, the symmetrically global pseudo-differential operator is introduced and found many properties in Sobolev space $H_{\alpha}^{r,s,p}$ by exploiting theory of the Weinstein transform. Using the aforesaid theory, various properties of minimal-maximal operators for symmetrically global pseudo-differential operators are obtained. This theory will make the future development in the theory of pseudo-differential operators.

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**Conclusion**: Product of pseudo-differential operators played an important role to find the various properties by using asymptotic expansions. In this regards, Beals Richard [2], Egorov [15], Wong [81], Cappiello and Toft [6], made a proper contribution by using the Fourier transform technique. Utilizing aforesaid transform theory various estimations are possible for the composition of two pseudo-differential operators. Composition operators are heavily used by Wong [81], to discuss various properties of an elliptic pseudo-differential operators, boundedness and compactness properties of pseudo-differential operators in Sobolev-type spaces. Using the different integral transform techniques, estimations of the composition of operators were done by Pathak [55], Prasad and Mahto [58], Prasad and Manish [59] and others in their various research papers and found important observations. From the motivations of the results [55, 58, 59, 81], the composition of two pseudo-differential operators involving the Weinstein transform is provided by the authors and found various estimates. In the future by using the aforesaid theory, other properties of pseudo-differential operators involving the Weinstein transform can be obtained.

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