Chapter 5

Hough transform generalization for detecting fuzzy lines and fuzzy circles

5.1 Introduction

This chapter attempts to detect fuzzy lines and fuzzy circles by a voting procedure. An accumulator array of the α-cuts of fuzzy parameters is used in FHT. We can identify the α-cuts of the fuzzy lines and circles through the proposed accumulator array. The standard voting procedure for the detection of fuzzy lines and fuzzy circles is represented in Algorithms 5.2.2 and 5.2.4, respectively. The drawback may be seen in Remark 5.2.1, which asserts that FHT may not be able to map the sets of crisp lines within the fuzzy lines that are located in bounded regions into points within bounded regions in parameter space. To resolve this problem, we use the alternative Definition 5.2.1 of fuzzy line and corresponding voting procedure is given in Algorithm 5.2.3. Also, a generalized version of FHT to detect fuzzy curves is defined. In a sequel, successful implementation of the proposed FHT is provided. These are the novelty of paper.

The basic advantage of the FHT is that it can be used to identify the parameters of a fuzzy geometric shape that best fits a set of given fuzzy points. FHT gives the answer to the following questions:

(i) Given a fuzzy point that belongs to a fuzzy line, what is the corresponding fuzzy line?

(ii) How many such fuzzy lines are there?

(iii) Which fuzzy point belongs to which fuzzy line?

5.2 Fuzzy Hough transform

In the following section we describe a generalized version of the fuzzy Hough transform (FHT) to detect fuzzy shapes using fuzzy geometry. Notably, the FHT for the detection of line and circle, when they are given imprecisely, are elaborated here.

5.2.1 Generalized version of fuzzy Hough transform

Let us consider a fuzzy image space, say $\widetilde{I}$, is defined by

$$\mu\left((x,y)\middle|\widetilde{I}\right) = \{\text{intensity value of } (x,y) : (x,y) \in I\},$$

where I is the image space defined by the set

$$\{(x,y) : x_{\min} \leq x \leq x_{\max},\ y_{\min} \leq y \leq y_{\max}\}.$$

Let a fuzzy shape $\widetilde{F}$ in $\widetilde{I}$ with parameter $\widetilde{\theta}$ be prescribed by a set of points S^{α} such that

$$S^{\alpha} = \{(x,y) : f(x,y,\theta^{\alpha}) = 0 \text{ with } \mu\left(\theta^{\alpha}\middle|\widetilde{\theta}\right) = \alpha\},$$

where $f(\cdot)$ is the parametric form of a shape $F \in \widetilde{F}(0)$, $\alpha \in [0,1]$. Let the fuzzy object region ($\widetilde{R} \subset \widetilde{I}$) be defined as

$$\widetilde{R}(\alpha) = \{(x,y) : \mu\left((x,y)\middle|\widetilde{F}\right) \geq \alpha\}.$$

Then fuzzy Hough transform can be considered as a mapping $\widetilde{T} : \widetilde{R} \to \widetilde{A}$, where $\widetilde{A}$ is a fuzzy set specified as

$$\widetilde{A}(\alpha) = \{(\theta_k^\alpha, |S_k^\alpha|), k = 1, \ldots, n\}.$$

The vector θ_k^α is the kth quantization slot in accumulator space with membership value

$$\mu\left(\theta_k^\alpha\middle|\widetilde{A}\right) = \sup\left\{\mu\left((x,y)\middle|\widetilde{F}\right) : (x,y) \in \widetilde{R}(\alpha) \text{ and } f(x,y,\theta_k^\alpha) = 0\right\}$$

and n represents the maximum number of such slots. The notion $|S_k^\alpha|$ is the cardinality of the set S_k^α which is determined as

$$S_k^\alpha = \{(x,y) : (x,y) \in R \text{ and } f(x,y,\theta_k^\alpha) = 0\}.$$

We describe a step-by-step procedure for detecting any arbitrary fuzzy shape $\widetilde{F}$ using *FHT* as follows.

Algorithm 5.2.1: To detect fuzzy shape $\widetilde{F}$ using *FHT*

Step 1: For $\alpha \in (0,1]$, quantize the parameter space $(\widetilde{\theta}(\alpha))$ and form an array $\widetilde{A}(\alpha)$ with appropriate quantization levels. This quantized space is often referred to as accumulator cells.

Step 2: Initialize the array $\widetilde{A}(\alpha)$ to zero.

Step 3: Increment by 1 each element of $\widetilde{A}(\alpha)$ which corresponds to a point on $F \in \widetilde{F}(\alpha)$.

Step 4: Count the number of times a shape $F \in \widetilde{F}(\alpha)$ intersects a cell $\theta_k^\alpha \in \widetilde{A}(\alpha)$.

Step 5: A histogram or a voting matrix shows the frequency of points corresponding to $\theta_k^\alpha \in \widetilde{A}(\alpha)$ values, for $\alpha \in (0,1]$.

Step 6: $\widetilde{A}(\alpha)$ is thresholded such as only the large valued elements are taken. These elements correspond to the α-cuts of the fuzzy shape $\widetilde{F}$ in the image.

5.2.2 Fuzzy line detection using FHT

In fuzzy plane geometry, the α-cuts of the slope-intercept form of the fuzzy line is

$$\begin{aligned}\widetilde{L}_{SI}(\alpha) =& \{l^\alpha : l^\alpha \text{ is a line with slope } m^\alpha \in \widetilde{m}(\alpha) \text{ and } y\text{-intercept } c^\alpha \in \widetilde{c}(\alpha) \\ & \text{with } \mu(m^\alpha|\widetilde{m}) = \mu(c^\alpha|\widetilde{c}) = \alpha\}.\end{aligned}$$

We know that the α-cuts of the $\widetilde{L}_{SI}$ is a collection of crisp lines $l^\alpha : y = m^\alpha x + c^\alpha$ with $\mu\left(m^\alpha|\widetilde{m}\right) = \mu\left(c^\alpha|\widetilde{c}\right) = \alpha$, where $\widetilde{m}$ and $\widetilde{c}$ are fuzzy numbers. Our attempt is to detect the α-cuts of the $\widetilde{L}_{SI}$ by using the fuzzy Hough transform technique.

Let a set of given points $\widetilde{X}(\alpha) = \left\{(x_i^\alpha, y_i^\alpha) : \mu\left((x_i^\alpha, y_i^\alpha)\middle|\widetilde{L}_{SI}\right) \geq \alpha\right\}$ describes the α-cuts of the fuzzy line in the fuzzy image, $i \in I$, where I is an index set. The fuzzy Hough transform maps $\widetilde{X}(\alpha)$ of the fuzzy image to the points $(m_k^\alpha, c_k^\alpha) \in \widetilde{A}(\alpha)$ in the $2D$ parameter space, such that $y_i^\alpha = m_k^\alpha x_i^\alpha + c_k^\alpha$. Each point (x_i^α, y_i^α) determines a line in the $2D$ parameter space so that the intersections of these lines determine a point that is the parameter of the line that best fit the points (x_i^α, y_i^α) of the image. So the search for local maximums in the parameter space provides the parameter lists of the crisp lines $l^\alpha \in \widetilde{L}_{SI}(\alpha)$ present in the fuzzy image.

To find the α-cuts of the fuzzy line using the fuzzy Hough transform, we need a two-dimensional accumulator array. Form an array

$$\widetilde{A}(\alpha) = \{(m_k^\alpha, c_k^\alpha), k = 1, \ldots, n\}.$$

The vector (m_k^α, c_k^α) is the kth quantization slot in accumulator space with membership value

$$\mu\left((m_k^\alpha, c_k^\alpha)\middle|\widetilde{A}\right) = \sup\left\{\mu\left((x, y)\middle|\widetilde{L}_{SI}\right) : (x, y) \in \widetilde{R}(\alpha) \text{ and } f(x, y, m_k^\alpha, c_k^\alpha) = 0\right\}$$

and n represents the maximum number of such slots.

The set $\widetilde{X}(\alpha) = \left\{(x_i^\alpha, y_i^\alpha) : \mu\left((x_i^\alpha, y_i^\alpha)\middle|\widetilde{L}_{SI}\right) \geq \alpha, i = 1, 2, \ldots, k\right\}$ with k edge points extracted from fuzzy image will be the entry for the usual implementation of the voting process.

The following algorithm explains the standard voting procedure.

Algorithm 5.2.2: $\widetilde{A}(\alpha) = Hough(\widetilde{X}(\alpha))$

Step 1: For each $\alpha \in (0, 1]$, quantize the parameter space $((\widetilde{m}(\alpha), \widetilde{c}(\alpha))$-space) with appropriate quantization levels.

Step 2: Form an array $\widetilde{A}(\alpha)$, for $\alpha \in (0, 1]$. This array is often referred to as accumulator cells.

Step 3: Initialize the array $\widetilde{A}(\alpha)$ to zero.

Step 4: For each $(x_i^\alpha, y_i^\alpha) \in \widetilde{X}(\alpha)$, increment those elements $(m_k^\alpha, c_k^\alpha) \in \widetilde{A}(\alpha)$ by 1 which satisfy the equation $c_k^\alpha = y_i^\alpha - m_k^\alpha x_i^\alpha$.

Step 5: $\widetilde{A}(\alpha)$ is thresholded such as only the large valued elements are taken. These elements correspond to the α-cuts of the fuzzy line in the fuzzy image.

For the explicit view, consider the points (x_i, y_i) which lie on the same line $l^\alpha \in \widetilde{L}_{SI}(\alpha)$ with the membership value α, $i \in I$. Note that for every point (x_i, y_i), all the straight lines passing through that point satisfy equation $y_i = m_k^\alpha x_i + c_k^\alpha$ for varying values of line slope m_k^α and intercept c_k^α, for each $\alpha \in [0, 1]$, $k = 1, \ldots, n$.

Now if we reverse the variables then, the equation $y_i = m_k^\alpha x_i + c_k^\alpha$ becomes equation $c_k^\alpha = y_i - m_k^\alpha x_i$ which describes a straight line in $(\widetilde{m}(\alpha), \widetilde{c}(\alpha))$-space. For each points (x_i, y_i), we can represent all the possible lines through it by a single line in the parameter space.

Thus a line in (x, y)-space that passes through the points (x_i, y_i), must lie on the intersection of the lines in $(\widetilde{m}(\alpha), \widetilde{c}(\alpha))$-space representing these points (x_i, y_i). This means that all points (x_i, y_i) which lie on the same line in the (x, y)-space are represented by lines that all pass through a single point in $(\widetilde{m}(\alpha), \widetilde{c}(\alpha))$-space. For varying $\alpha \in [0, 1]$, we can detect each line on the α-cuts of the $\widetilde{L}_{SI}$.

Apparently, let $l_1^\alpha : y = m_1^\alpha x + c_1^\alpha$ and $l_2^\alpha : y = m_2^\alpha x + c_2^\alpha$ be the boundaries of the α-cuts of the $\widetilde{L}_{SI}$, where m_1^α, m_2^α, c_1^α and c_2^α are slope and intercept of l_1^α and l_2^α, respectively (see Figure 5.1). Let (x_i, y_i) and (x_j, y_j) be the points lying on l_1^α and l_2^α, respectively, $i, j = 1, 2, \ldots, n$. The accumulator cells (m_1^α, c_1^α) and (m_2^α, c_2^α) in $\widetilde{A}(\alpha)$ must have maximum number of elements (see Figure 5.2).

Note that the FHT can be perceived as a mapping $\widetilde{T}$ from $\widetilde{L}_{SI}(\alpha) \to \widetilde{A}(\alpha)$, for $\alpha \in (0, 1]$. Such that for each line $l^\alpha \in \widetilde{L}_{SI}(\alpha)$, there is an accumulator cell $(m_k^\alpha, c_k^\alpha) \in \widetilde{A}(\alpha)$ which has maximum number of elements, for some $k = 1, 2, \ldots, n$.

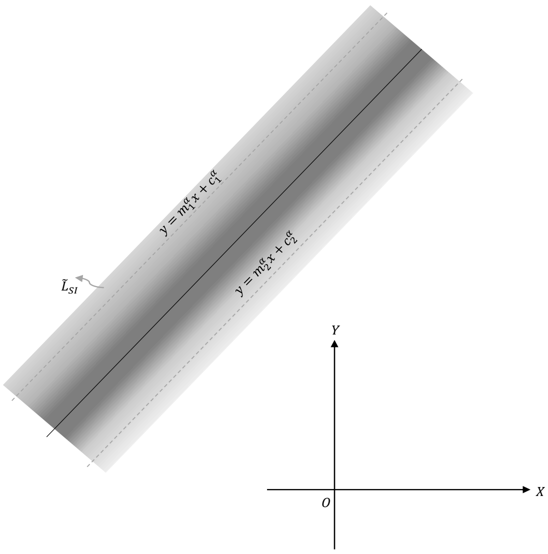


FIGURE 5.1: Fuzzy line $\widetilde{L}_{SI}$

$A^\alpha(m_k^\alpha, c_k^\alpha)$

					1				1
						1		1	
							n		
						1		1	
1				1	1				1
	1		1						
		n							
	1		1						
1				1					

FIGURE 5.2: Accumulator cell $\widetilde{A}(\alpha)$

Theorem 5.2.1. The fuzzy Hough transform maps the α-cuts of $\widetilde{L}_{SI}$ in (x, y)-space to a fuzzy number in $\widetilde{A}(\alpha)$. Also, there is a one-to-one correspondence between the α-cuts of $\widetilde{L}_{SI}$ to the α-cuts of a fuzzy number, for $\alpha \in [0, 1]$.

Proof. Lets begin by proving that the α-cuts of $\widetilde{L}_{SI}$ in (x, y)-space is the α-cuts of a fuzzy number in $\widetilde{A}(\alpha)$, for $\alpha \in [0, 1]$.

We know that the fuzzy Hough transform is a map $\widetilde{T}$ from $\widetilde{L}_{SI}(\alpha) \to \widetilde{A}(\alpha)$. There always exists a unique accumulator cell $(m_k^1, c_k^1) \in \widetilde{A}(\alpha)$ with maximum number of elements for the core line $\widetilde{L}_{SI}(1)$, for some $k = 1, 2, \ldots, n$. Let l_1^α and l_2^α be the boundaries of the α-cuts of the $\widetilde{L}_{SI}$ with slope m_1^α, m_2^α, and intercept c_1^α, c_2^α, respectively. Let $N_1^\alpha = (m_1^\alpha, c_1^\alpha) \in \widetilde{A}(\alpha)$ and $N_2^\alpha = (m_2^\alpha, c_2^\alpha) \in \widetilde{A}(\alpha)$ are the corresponding cells that have maximum number of elements. We see that $\widetilde{N}(\alpha) = [N_1^\alpha, N_2^\alpha]$ is the α-cuts of a fuzzy number, say $\widetilde{N}$, whose core is $(m_k^1, c_k^1) \in \widetilde{A}(\alpha)$. For any $\beta > \alpha$, $\widetilde{L}_{SI}(\beta) \subseteq \widetilde{L}_{SI}(\alpha)$ (see Definition 3.3.1 in [2]). We note that $[N_1^\beta, N_2^\beta] \subseteq [N_1^\alpha, N_2^\alpha]$, for $0 < \alpha < \beta \leq 1$. Thus, $\widetilde{N}$ is a fuzzy number in $(\widetilde{m}(\alpha), \widetilde{c}(\alpha))$-space.

Next, we show that $\widetilde{T}$ maps $\widetilde{L}_{SI}(\alpha)$ to unique α-cut of a fuzzy number $\widetilde{N}$ in the parameter space, for each $\alpha \in [0, 1]$. Also, for each α-cut $\widetilde{N}(\alpha) = [N_1^\alpha, N_2^\alpha]$ there is unique $\widetilde{L}_{SI}(\alpha)$ such that $\widetilde{T}$ maps $\widetilde{L}_{SI}(\alpha)$ to $\widetilde{N}(\alpha)$, where $N_1^\alpha = (m_1^\alpha, c_1^\alpha) \in \widetilde{A}(\alpha)$ and $N_2^\alpha = (m_2^\alpha, c_2^\alpha) \in \widetilde{A}(\alpha)$ are corresponding cells that have maximum number of elements. Let $\widetilde{X}(\alpha) = \left\{(x_i^\alpha, y_i^\alpha) : \mu\left((x_i^\alpha, y_i^\alpha)\middle|\widetilde{L}_{SI}\right) \geq \alpha, i = 1, 2, \ldots, k\right\}$ be a set of points of α-cuts of $\widetilde{L}_{SI}$ in the fuzzy image space. Consider (x_i^α, y_i^α) are points lying on $l^\alpha \in \widetilde{L}_{SI}(\alpha)$, for some $\alpha \in [0, 1]$. Let $(m_k^\alpha, c_k^\alpha) \in \widetilde{A}(\alpha)$ be a accumulator cell having maximum number of elements, say p, voted by the points (x_i^α, y_i^α), for some $k = 1, \ldots, n$. Suppose there exist another accumulator cell $({m'}_k^\alpha, {c'}_k^\alpha) \in \widetilde{A}(\alpha)$ having same number of elements p, voted by the points (x_i^α, y_i^α). This implies that there is two crisp lines passing through the co-linear points (x_i^α, y_i^α), which is not possible, for some $\alpha \in [0,1]$, $i = 1, 2, \ldots, k$. Thus, for each $l^\alpha \in \widetilde{L}_{SI}(\alpha)$, there is only one possible accumulator cell $(m_k^\alpha, c_k^\alpha) \in \widetilde{A}(\alpha)$ which has maximum number of elements, for some $k = 0, \ldots, n$. Similarly, we can see that for each accumulator cell $(m_k^\alpha, c_k^\alpha) \in \widetilde{A}(\alpha)$ having maximum number of elements, there is unique crisp line $l^\alpha \in \widetilde{L}_{SI}(\alpha)$ in fuzzy image space. This proves that there is a one-to-one correspondence between the α-cuts of $\widetilde{L}_{SI}$ in (x, y)-space to the α-cuts of a fuzzy number $\widetilde{N}$ in $(\widetilde{m}(\alpha), \widetilde{c}(\alpha))$-space, for $\alpha \in [0, 1]$. □

Proposition 5.2.1. The fuzzy Hough transform cannot map a set of fuzzy numbers lying in a fuzzy line in (x, y)-space to the fuzzy lines in $(\widetilde{m}(\alpha), \widetilde{c}(\alpha))$-space, for $\alpha \in [0, 1]$.

Proof. Consider the fuzzy numbers $\widetilde{a}_i$ along the lines l_i perpendicular to the core line of the fuzzy line $\widetilde{L}_{SI}$, respectively, for $i = 1, 2, \ldots, n$. If we take a point $(x_{ji}^\alpha, y_{ji}^\alpha) \in \widetilde{a}_i(\alpha)$ in the image, all lines which pass through that point have the form $y_{ji}^\alpha = m_k^\alpha x_{ji}^\alpha + c_k^\alpha$ for varying values of m_k^α and c_k^α, for some $j = 1, 2, \ldots, p$, $i = 1, 2, \ldots, q$ and $k = 1, \ldots, n$. However, the equation $y_{ji}^\alpha = m_k^\alpha x_{ji}^\alpha + c_k^\alpha$ can also be written as $c_k^\alpha = -m_k^\alpha x_{ji}^\alpha + y_{ji}^\alpha$, where we now consider x_{ji}^α and y_{ji}^α to be constants, and m_k^α and c_k^α as varying. This is a straight line in $(\widetilde{m}(\alpha), \widetilde{c}(\alpha))$-space. Also, each point $(x_{1i}^\alpha, y_{1i}^\alpha)$, $(x_{2i}^\alpha, y_{2i}^\alpha)$, …, $(x_{ji}^\alpha, y_{ji}^\alpha)$ of $\widetilde{a}_i(\alpha)$ corresponds to a straight line in $(\widetilde{m}(\alpha), \widetilde{c}(\alpha))$-space (by the fuzzy Hough transform), for $j = 1, 2, \ldots, k$. Now, we proceed to show that the lines corresponding to the points $(x_{ji}^\alpha, y_{ji}^\alpha)$ are concurrent, which implies that the collections of these lines cannot be the support of a fuzzy line (see Definition 1.3.7) in $(\widetilde{m}(\alpha), \widetilde{c}(\alpha))$-space. It is easily noted that the lines

$$\begin{cases} y_{1i}^{\alpha} = m_k^{\alpha} x_{1i}^{\alpha} + c_k^{\alpha} \\ y_{2i}^{\alpha} = m_k^{\alpha} x_{2i}^{\alpha} + c_k^{\alpha} \\ \quad \vdots \\ y_{ni}^{\alpha} = m_k^{\alpha} x_{ni}^{\alpha} + c_k^{\alpha} \end{cases}$$

are concurrent in $(\widetilde{m}(\alpha), \widetilde{c}(\alpha))$-space, since the points $(x_{ji}^{\alpha}, y_{ji}^{\alpha}) \in \widetilde{a}_i(\alpha)$ are collinear points, for each $\alpha \in [0, 1]$, $k = 1, \ldots, n$. Therefore, the fuzzy Hough transform of the fuzzy numbers $\widetilde{a}_i$ does not correspond to the fuzzy lines in $(\widetilde{m}(\alpha), \widetilde{c}(\alpha))$-space. This completes the proof. □

Remark 5.2.1. The fuzzy Hough transform may not map the collection of crisp lines $l^{\alpha} \in \widetilde{L}_{SI}(\alpha)$ that lie in a bounded region into the set of accumulator cells $(m_k^{\alpha}, c_k^{\alpha}) \in \widetilde{A}(\alpha)$ that lie in a bounded region in $(\widetilde{m}(\alpha), \widetilde{c}(\alpha))$-space, for $\alpha \in [0, 1]$, $k = 1, 2, \ldots, n$.

To observe this, let $\widetilde{X}(\alpha) = \left\{(x_i^{\alpha}, y_i^{\alpha}) : \mu\left((x_i^{\alpha}, y_i^{\alpha}) \middle| \widetilde{L}_{SI}\right) \geq \alpha, i = 1, 2, \ldots, k\right\}$ be a set of points of the $\widetilde{L}_{SI}(\alpha)$ in (x, y)-space. Let us consider $(x_i^{\alpha}, y_i^{\alpha})$ are points lying on $l^{\alpha} \in \widetilde{L}_{SI}(\alpha)$, for some $\alpha \in (0, 1]$. Assume that $(x_i^{\alpha}, y_i^{\alpha}) \in l^{\alpha}$ is a point on the image; therefore, the image is intersected in intervals by lines l_i^{α} through $(x_i^{\alpha}, y_i^{\alpha})$. So the image contains many collinear points that lie on l_i^{α}. All these collinear points create concurrent lines through the point, $(m_k^{\alpha}, c_k^{\alpha}) \in \widetilde{A}(\alpha)$ say. Since the l_i^{α}'s are concurrent lines, and by the Theorem 5.2.1, the points $(m_k^{\alpha}, c_k^{\alpha}) \in \widetilde{A}(\alpha)$ must be collinear, say lying on the line F^{α}. Moreover, by the Theorem 5.2.1, every point of F^{α} arises in this way from some line through $(x_i^{\alpha}, y_i^{\alpha})$. As a result, any point $(m_k^{\alpha}, c_k^{\alpha}) \in \widetilde{A}(\alpha)$ of F^{α} is a possible peak location. Therefore, the peaks that arise from all the possible sets of collinear points in the image could lie anywhere on the line F^α, so they could not be confined to any bounded region.

The Remark 5.2.1 says that the fuzzy Hough transform is practically used for the fuzzy line $\widetilde{L}_{SI}$ in which the slopes of crisp lines $l^\alpha \in \widetilde{L}_{SI}(\alpha)$ are sufficiently away from vertical (e.g., at most $\pm 45°$). Note that the Hough transform maps points (x_i^α, y_i^α) that lie on a vertical line $l^\alpha \in \widetilde{L}_{SI}(\alpha)$ into parallel lines (intersect at infinity) in the parameter space. So this technique cannot be practically used to detect sets of lines $l^\alpha \in \widetilde{L}_{SI}(\alpha)$ that is close to vertical.

Note 16. There are some disadvantages of the classical Hough transformation: it does not give information about the position and length of a line segment. Also, co-linear line segments cannot be separated. Due to this disadvantage, it may not be possible to detect non-symmetric fuzzy lines (see Definition 1.3.8). Therefore, we only focus on the detection of symmetric fuzzy lines.

Analogous to the classical Hough transform, detection of the crisp lines $y_i^\alpha = m_k^\alpha x_i^\alpha + c_k^\alpha$ break down for lines, when m_k^α becomes a large value, for $k = 1, \ldots, n$. To overcome this problem, we use the following alternative definition of the fuzzy line.

Definition 5.2.1. Let $x\cos\theta + y\sin\theta = \rho$ be a line, where (ρ, θ) represent the parameters describing a straight line. The parameter ρ describes the distance perpendicular to the origin, and θ represents the angle subtended by the normal relative to the axis. Let $\tilde{\rho}$ be a fuzzy number. A symmetric fuzzy line, denoted as $\widetilde{L}_S$, can be defined with the membership function by

$$\mu\left((x,y)\middle|\widetilde{L}_S\right) = \mu(\rho^\alpha|\tilde{\rho}),$$

where $\rho^\alpha = x\cos\theta + y\sin\theta$. For any $\alpha \in [0,1]$, the α-cuts of the $\widetilde{L}_S$ is given by

$$\widetilde{L}_S(\alpha) = \bigvee\{(x, y) : \rho^\alpha = x\cos\theta + y\sin\theta,\ \text{where } \rho^\alpha \in \widetilde{\rho}(\alpha)\}.$$

Let $(x_i^\alpha, y_i^\alpha) \in \widetilde{L}_S(\alpha)$ be the points of a crisp line $x\cos\theta + y\sin\theta = \rho^\alpha$, where ρ^α describes the perpendicular distance from the origin, and θ describes the angle subtended by the normal relative to the x-axis, for some $\alpha \in (0, 1]$, $i = 1, 2, \ldots, n$. The set of all crisp lines in the $\widetilde{L}_S(\alpha)$ constitutes a two-parameter family. If we fix a parametrization for the family, then a crisp line can be represented by a single point in $(\theta, \widetilde{\rho}(\alpha))$-space. This parametrization specifies a crisp line $x\cos\theta+y\sin\theta = \rho^\alpha$ by the angle θ of its normal and its algebraic distance ρ^α from the origin. If we restrict θ to the interval $[0, \pi]$, every crisp line in (x, y)-space corresponds to a unique point in $(\theta, \widetilde{\rho}(\alpha))$-space.

Suppose that we have a set of n points $\{(x_1^\alpha, y_1^\alpha), (x_2^\alpha, y_2^\alpha), \ldots, (x_n^\alpha, y_n^\alpha)\}$ in $\widetilde{I}$ and we want to fit them by a crisp line in $\widetilde{L}_S(\alpha)$. We transform the points (x_i^α, y_i^α) into the curves in $(\theta, \widetilde{\rho}(\alpha))$-space defined by

$$x_i^\alpha \cos\theta + y_i^\alpha \sin\theta = \rho^\alpha.$$

It is noted that a point (x_i^α, y_i^α) in (x, y)-space is represented by a curve in $(\theta, \widetilde{\rho}(\alpha))$-space rather than a straight line. It is easy to see that the curves corresponding to co-linear points have a common point of intersection. This point of intersection, say $(\theta_k, \rho_k^\alpha) \in \widetilde{A}(\alpha)$, defines a crisp line $x\cos\theta_k + y\sin\theta_k = \rho_k^\alpha$ passing through the co-linear points $(x_i^\alpha, y_i^\alpha) \in \widetilde{L}_S(\alpha)$, for $i = 1, 2, \ldots, n$.

The properties of the fuzzy Hough transform are as follows.

(i) A point $(x_i^\alpha, y_i^\alpha) \in \widetilde{L}_S(\alpha)$ in (x, y)-space represents a curve in $(\theta, \widetilde{\rho}(\alpha))$-space.

(ii) A point $(\theta_k, \rho_k^\alpha)$ in $\widetilde{A}(\alpha)$ represents a crisp line in $\widetilde{L}_S(\alpha)$ in (x, y)-space.

(iii) The points lying on a crisp line in $\widetilde{L}_S(\alpha)$ correspond to curves through a common point in $(\theta, \widetilde{\rho}(\alpha))$-space.

(iv) The points lying on the same curve in $(\theta, \widetilde{\rho}(\alpha))$-space correspond to crisp lines through the same point in (x, y)-space.

To find the α-cuts of the fuzzy line using the fuzzy Hough transform, we need a two-dimensional accumulator array. Form an array

$$\widetilde{A}(\alpha) = \{(\theta_k, \rho_k^\alpha), k = 1, \ldots, n\}.$$

The vector $(\theta_k, \rho_k^\alpha)$ is the kth quantization slot in accumulator space with membership value

$$\mu\left((\theta_k, \rho_k^\alpha)\middle|\widetilde{A}\right) = \sup\left\{\mu\left((x, y)\middle|\widetilde{L}_S\right) : (x, y) \in \widetilde{R}(\alpha) \text{ and } f(x, y, \theta_k, \rho_k^\alpha) = 0\right\}$$

and n represents the maximum number of such slots.

The set $\widetilde{X}(\alpha) = \left\{(x_i^\alpha, y_i^\alpha) : \mu\left((x_i^\alpha, y_i^\alpha)\middle|\widetilde{L}_S\right) \geq \alpha, i = 1, 2, \ldots, k\right\}$ with k edge points extracted from image will be the entry for the usual implementation of the voting process.

The standard voting procedure is detailed in the following Algorithm 5.2.3.

Algorithm 5.2.3: $\widetilde{A}(\alpha) = Hough(\widetilde{X}(\alpha))$

Step 1: For each $\alpha \in (0, 1]$, quantize the parameter space $((\theta, \widetilde{\rho}(\alpha))$-space) with appropriate quantization levels.

Step 2: Form an array $\widetilde{A}(\alpha)$, for $\alpha \in (0, 1]$. This array is often referred to as accumulator cells.

Step 3: Initialize the array $\widetilde{A}(\alpha)$ to zero.

Step 4: For each $(x_i^\alpha, y_i^\alpha) \in \widetilde{X}(\alpha)$, increment those elements $(\theta_k, \rho_k^\alpha) \in \widetilde{A}(\alpha)$ by 1 which satisfy the equation $\rho_k^\alpha = x_i^\alpha \cos\theta_k + y_i^\alpha \sin\theta_k$.

Step 5: In thresholding $\widetilde{A}(\alpha)$, only the elements of $\widetilde{A}(\alpha)$ with maximum number of votes are considered. These elements correspond to the α-cuts of the fuzzy lines in the fuzzy image.

In the following subsection, we examine the detection of fuzzy circles using the fuzzy Hough transform.

5.2.3 Fuzzy circle detection using FHT

In fuzzy plane geometry, the α-cuts of the fuzzy circle is

$$\widetilde{C}(\alpha) = \{(x, y) : (x, y) \text{ lies on the circle that passes through the same three points on } \widetilde{P}_1(\alpha),\ \widetilde{P}_2(\alpha) \text{ and } \widetilde{P}_3(\alpha)\}.$$

The fuzzy circle is the union of all the crisp circles that pass through the same three points on the supports of $\widetilde{P}_1(\alpha)$, $\widetilde{P}_2(\alpha)$ and $\widetilde{P}_3(\alpha)$ (see Definition 3.2 in [3]). We want to detect the α-cuts of $\widetilde{C}$ by using the fuzzy Hough transform.

Let a set of given points $\widetilde{X}(\alpha) = \left\{(x_i^\alpha, y_i^\alpha) : \mu\left((x_i^\alpha, y_i^\alpha)\middle|\widetilde{C}\right) \geq \alpha\right\}$ describes the α-cuts of the $\widetilde{C}$ in the fuzzy image. The fuzzy Hough transform maps $\widetilde{X}(\alpha)$ of the fuzzy image to the points $(u_k^\alpha, v_k^\alpha, r_k^\alpha) \in \widetilde{A}(\alpha)$ in the $3D$ parameter space, such that

$(u_k^\alpha - x_i^\alpha)^2 + (v_k^\alpha - y_i^\alpha)^2 = (r_k^\alpha)^2$, for $\alpha \in [0,1]$. Apparently, each point (x_i^α, y_i^α) determines a surface in the 3D parameter space so that the intersections of these surfaces determine points that are the parameters of the circles that best fit to the points of the image.

To find the α-cuts of the fuzzy circle using the fuzzy Hough transform, we need a three-dimensional accumulator array. Form an array

$$\widetilde{A}(\alpha) = \{(u_k^\alpha, v_k^\alpha, r_k^\alpha), k = 1, \ldots, n\}.$$

The vector $(u_k^\alpha, v_k^\alpha, r_k^\alpha)$ is the kth quantization slot in accumulator space with membership value

$$\mu\left((u_k^\alpha, v_k^\alpha, r_k^\alpha)\middle|\widetilde{A}\right) = \sup\left\{\mu\left((x,y)\middle|\widetilde{C}\right) : (x,y) \in \widetilde{R}(\alpha) \text{ and } f(x,y,u_k^\alpha, v_k^\alpha, r_k^\alpha) = 0\right\}$$

and n represents the maximum number of such slots.

The set $\widetilde{X}(\alpha) = \left\{(x_i^\alpha, y_i^\alpha) : \mu\left((x_i^\alpha, y_i^\alpha)\middle|\widetilde{C}\right) \geq \alpha, i = 1, 2, \ldots, k\right\}$ with k edge points extracted from image will be the entry for the usual implementation of the voting process.

The standard voting procedure is detailed in the following Algorithm 5.2.4.

Algorithm 5.2.4: $\widetilde{A}(\alpha) = Hough(\widetilde{X}(\alpha))$

Step 1: For each $\alpha \in (0, 1]$, quantize the parameter space with appropriate quantization levels.

Step 2: Form an array $\widetilde{A}(\alpha)$, for $\alpha \in (0, 1]$. This array is often referred to as accumulator cells.

Step 3: Initialize the array $\widetilde{A}(\alpha)$ to zero.

Step 4: For each $(x_i^\alpha, y_i^\alpha) \in \widetilde{X}(\alpha)$, increment those elements $(u_k^\alpha, v_k^\alpha, r_k^\alpha) \in \widetilde{A}(\alpha)$ by 1 which satisfy the equation $(r_k^\alpha)^2 = (u_k^\alpha - x_i^\alpha)^2 + (v_k^\alpha - y_i^\alpha)^2$.

Step 5: In thresholding $\widetilde{A}(\alpha)$, only the elements of $\widetilde{A}(\alpha)$ with maximum number of votes are considered. These elements correspond to the α-cuts of the fuzzy circle in the fuzzy image.

Note 17. For each circle $c^\alpha \in \widetilde{C}(\alpha)$, the 3D array is searched for the peak that give circle positions and radii. If the radius of the fuzzy circle is known in advance, then the 3D accumulator array reduces to the 2D accumulator array. In practice, the fuzzy Hough transform technique takes a rapidly increasing amount of time as the number of parameters increases. Hence, the fuzzy Hough transform is computationally complex for the objects with many parameters. So, for the efficiency in the detection of the fuzzy circle, we consider a fuzzy circle whose radius is known. This will reduce the 3D parameter space $(u_k^\alpha, v_k^\alpha, r_k^\alpha)$ to the 2D parameter space (u_k^α, v_k^α).

A geometrical view for the detection of a crisp circle $c^\alpha \in \widetilde{C}(\alpha)$ (see Figure 5.3) is described below.

Let the circle c^α of known radius, say 2, is to be detected in an image (see Figure 5.3). Let the equation of the circle c^α be $(x - u^\alpha)^2 + (y - v^\alpha)^2 = 4$ in the (x, y)-space, where (u^α, v^α) is unknown (see Figure 5.4). In Figure 5.4, $A = (0, 3)$, $B =$

$(3.99, 3.21)$, $C = (1.99, 5)$ and $D = (2.06, 1)$ represents the points on the circle c^α. The Hough transform maps each point A, B, C and D to the circles of radius 4 in (u^α, v^α)-space since x and y are fixed, and the parameters u^α and v^α are variable in $(u^\alpha - x)^2 + (u^\alpha - y)^2 = 4$ (see Figure 5.5). In Figure 5.5, A, B, C and D are circles in (u^α, v^α)-space corresponding to each point $A = (0, 3)$, $B = (3.99, 3.21)$, $C = (1.99, 5)$ and $D = (2.06, 1)$ in (x, y)-space. The loci of the center for each point A, B, C and D must intersect at a single point $(2, 3)$ in the (u^α, v^α)-space (see Figure 5.5). The point $(2, 3)$ represents the coordinates of the center of the circle c^α in the image.

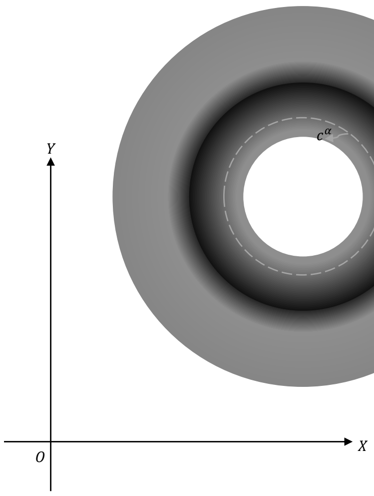


FIGURE 5.3: Fuzzy circle $\widetilde{C}(\alpha)$

The following Theorem demonstrates that the fuzzy circles corresponding to each fuzzy point (situated on the fuzzy circle in (x, y)-space) intersect in a fuzzy number in the parameter space.

Theorem 5.2.2. All the fuzzy points in a fuzzy circle (with known radius $\widetilde{r}$) form a collection of fuzzy circles in $\widetilde{A}(\alpha)$ which will intersect in a fuzzy number, for $\alpha \in [0, 1]$.

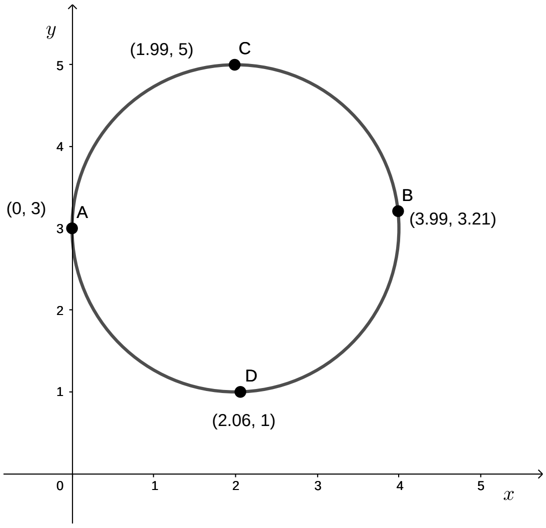


FIGURE 5.4: Crisp circle c^α

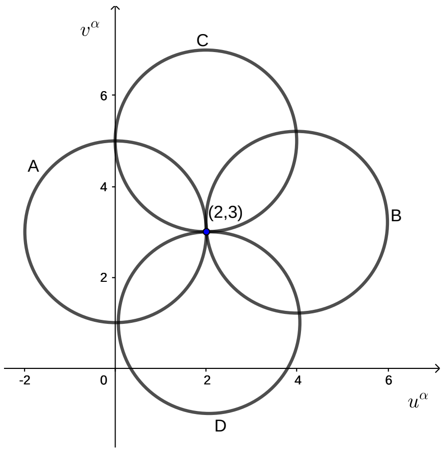


FIGURE 5.5: The Hough transform of $c^\alpha \in \widetilde{C}(\alpha)$

Proof. Consider the fuzzy points $\widetilde{P}_i$ lying in the fuzzy circle $\widetilde{C}$, $i = 1, 2, \ldots, n$. Let $(x_i^\alpha, y_i^\alpha) \in \widetilde{P}_i(\alpha)$ be the points of $c_i^\alpha : (x - u_k^\alpha)^2 + (y - v_k^\alpha)^2 = (r_i^\alpha)^2$ in the $\widetilde{C}(\alpha)$, where $r_i^\alpha \in \widetilde{r}(\alpha)$, for some $\alpha \in [0, 1]$. The fuzzy Hough transform maps $\widetilde{P}_i(\alpha)$ to the $(u_k^\alpha, v_k^\alpha) \in \widetilde{A}(\alpha)$ such that $(u_k^\alpha - x_i^\alpha)^2 + (v_k^\alpha - y_i^\alpha)^2 = (r_i^\alpha)^2$ for varying values of u_k^α and v_k^α, for $k = 1, \ldots, n$. Thus each point $(x_i^\alpha, y_i^\alpha) \in c_i^\alpha$ forms a circle in $\widetilde{A}(\alpha)$. Let $\widetilde{a}_i$ be the fuzzy numbers lying on the support of $\widetilde{P}_i$, for $i = 1, 2, \ldots, n$. Now since $\widetilde{P}_i$ lie on the fuzzy circle $\widetilde{C}$ and by the Theorem 2.1 (in [3]), there exist fuzzy numbers

$\widetilde{b}$'s in $\widetilde{A}(\alpha)$ such that $\widetilde{D}(\widetilde{a}_i, \widetilde{b}) = \widetilde{D}(\widetilde{b}, \widetilde{a}_i) = \widetilde{r}$, where the fuzzy numbers $\widetilde{a}_i$ and $\widetilde{r}$ are known, for a fix $i = 1, 2, \ldots, n$. This implies that the fuzzy Hough transform of the fuzzy point $\widetilde{P}_i$ is a fuzzy circle in $\widetilde{A}(\alpha)$. Now since $(x_i^\alpha, y_i^\alpha) \in \widetilde{P}_i(\alpha)$ are the points of a circle in the image, the circles

$$\begin{cases} (r_i^\alpha)^2 = (u_k^\alpha - x_1^\alpha)^2 + (v_k^\alpha - y_1^\alpha)^2 \\ (r_i^\alpha)^2 = (u_k^\alpha - x_2^\alpha)^2 + (v_k^\alpha - y_2^\alpha)^2 \\ \quad \vdots \\ (r_i^\alpha)^2 = (u_k^\alpha - x_n^\alpha)^2 + (v_k^\alpha - y_n^\alpha)^2 \end{cases}$$

are concurrent in $\widetilde{A}(\alpha)$, for each $\alpha \in [0, 1]$. Thus, the corresponding fuzzy circles in $\widetilde{A}(\alpha)$ for each fuzzy point $\widetilde{P}_i$ are concurrent, for $\alpha \in [0, 1]$. Therefore, the fuzzy Hough transform of the fuzzy points $\widetilde{P}_i$ in which $(x_i^\alpha, y_i^\alpha) \in \widetilde{P}_i(\alpha)$ lie in a circle c_i^α, are fuzzy circles, and pass through a fuzzy number in $\widetilde{A}(\alpha)$. This completes the proof. □

The following section demonstrates the idea of similarity measures between two fuzzy shapes.

5.3 Similarity measure between two fuzzy shapes

Before studying the concept of similarity measure between two fuzzy shapes, we give the idea of fuzzy shape descriptor in the fuzzy image. This is a method for describing the texture of fuzzy shapes. It will be used for extracting the feature points or interest points of fuzzy shapes from the fuzzy image. Once the feature points have been extracted from two or more images, the next step is establishing the similarity measure between two fuzzy shapes. The approach of similarity measure consists of detecting a set of feature points associated with fuzzy shape descriptors of fuzzy image.

5.3.1 Fuzzy shape descriptor

Fuzzy shapes (fuzzy lines and circles) descriptor is a step-wise procedure that takes a fuzzy image and outputs feature points with membership value $\alpha \in [0, 1]$.

(i) A step-wise procedure of fuzzy line descriptor is given below.

Step: 1 A fuzzy line $\widetilde{L}$ in the fuzzy image $\widetilde{I}$ is given.

Step: 2 Assign the membership value of pixel $a_{ij} \in \widetilde{L}$ by

$$\mu\left(a_{ij}\middle|\widetilde{L}\right) = \begin{cases} 0 & \text{if } a_{ij} \leq a \\ \dfrac{a_{ij} - a}{b - a} & \text{if } a \leq a_{ij} \leq b \\ \dfrac{c - a_{ij}}{c - b} & \text{if } b \leq a_{ij} \leq c \\ 0 & \text{if } c \leq a_{ij} \end{cases} \tag{5.1}$$

with crossover point $b = \frac{a+c}{2}$ lying on the $\widetilde{L}(1)$ and the bandwidth $b - a = c - b$.

Step: 3 By varying b along $\widetilde{L}(1)$, assign membership value $\alpha \in [0, 1]$ of each pixel in $\widetilde{L}$ by equation (5.1).

Step: 4 For $\alpha \in [0, 1]$, extract and store all the coordinates of the pixels $(x, y) \in \widetilde{L}(\alpha)$.

(ii) Next, we provide a step-wise procedure of fuzzy circle descriptor.

Step: 1 A fuzzy circle $\widetilde{C}$ in the fuzzy image $\widetilde{I}$ is given.

Step: 2 Assign the membership value of pixel $a_{ij} \in \widetilde{C}$ by

$$\mu\left(a_{ij}\middle|\widetilde{C}\right) = \begin{cases} 0 & \text{if } a_{ij} \leq a \\ \dfrac{a_{ij}-a}{b-a} & \text{if } a \leq a_{ij} \leq b \\ \dfrac{c-a_{ij}}{c-b} & \text{if } b \leq a_{ij} \leq c \\ 0 & \text{if } c \leq a_{ij} \end{cases} \tag{5.2}$$

with crossover point $b = \frac{a+c}{2}$ lying on the $\widetilde{C}(1)$ and the bandwidth $b-a=c-b$.

Step: 3 By varying b along $\widetilde{C}(1)$, assign membership value $\alpha \in [0,1]$ of each pixel in $\widetilde{C}$ by equation (5.2).

Step: 4 For $\alpha \in [0,1]$, extract and store all the coordinates of the pixels $(x,y) \in \widetilde{C}(\alpha)$.

Now, we propose a distance measure between two fuzzy shapes. Then, we define the similarity measure between two fuzzy shapes based on the proposed fuzzy distance. After that, the proposed similarity measure of fuzzy shapes has been used to deal with the fuzzy image such that a fuzzy shape (fuzzy line or fuzzy circle) is present therein. We show that the similarity and distance measures between two fuzzy shapes are closely related concepts. The similarity measure is inversely related to the distance measure.

5.3.2 Distance measure between two fuzzy shapes in the fuzzy image

We define distance measure between two fuzzy shapes $\widetilde{F}_1$ and $\widetilde{F}_2$ based on average distance. It takes all elements into account.

Definition 5.3.1. Let $\widetilde{F}_1$ and $\widetilde{F}_2$ be two fuzzy shapes in the fuzzy image. The membership function of distance measure between $\widetilde{F}_1$ and $\widetilde{F}_2$, say $\widetilde{D}(\widetilde{F}_1, \widetilde{F}_2)$, can be defined by

$$\begin{aligned} &\mu\left(d' \middle| \widetilde{D}(\widetilde{F}_1, \widetilde{F}_2)\right) \\ = \ &\sup\{\alpha : d' = \frac{\sum_{a_1^\alpha \in l_1^\alpha} \sum_{a_2^\alpha \in l_2^\alpha} d(a_1^\alpha, a_2^\alpha)}{|l_1^\alpha||l_2^\alpha|}, \text{ where } l_1^\alpha \in \widetilde{F}_1(\alpha),\ l_2^\alpha \in \widetilde{F}_2(\alpha) \\ &\qquad \text{with membership value } \alpha\}. \end{aligned}$$

Here, 'd' is the Euclidean distance and $|l_i^\alpha|$ denotes the cardinality of the set $\{a_i^\alpha \in l_i^\alpha\}$, for $i = 1, 2$.

Now, the similarity measure between two fuzzy shapes $\widetilde{F}_1$ and $\widetilde{F}_2$, say $\widetilde{S}(\widetilde{F}_1, \widetilde{F}_2)$, can be defined by

$$\widetilde{S}(\widetilde{F}_1, \widetilde{F}_2) = 1 - \widetilde{D}(\widetilde{F}_1, \widetilde{F}_2). \tag{5.3}$$

Note 18. The similarity measure $\widetilde{S}(\widetilde{F}_1, \widetilde{F}_2)$ is a fuzzy number by (5.3). Also, the above distance based similarity measure $\widetilde{S}(\widetilde{F}_1, \widetilde{F}_2)$ satisfies the following properties.

1. $\widetilde{S}(\widetilde{F}_1, \widetilde{F}_2)$=$\widetilde{S}(\widetilde{F}_2, \widetilde{F}_1)$.
2. $\widetilde{S}(\widetilde{F}_1, \widetilde{F}_2)$=1 if $\widetilde{F}_1 = \widetilde{F}_2$.
3. If $\widetilde{F}_1 \leq \widetilde{F}_2 \leq \widetilde{F}_3$, then $\widetilde{S}(\widetilde{F}_1, \widetilde{F}_3) \leq \min\{\widetilde{S}(\widetilde{F}_1, \widetilde{F}_2), \widetilde{S}(\widetilde{F}_2, \widetilde{F}_3)\}$.

Next, we develop an Algorithm 5.3.1 to find the similarity measure between two fuzzy shapes $\widetilde{F}_1$ and $\widetilde{F}_2$ in the fuzzy image.

Algorithm 5.3.1: To get the similarity measure $\widetilde{S}(\widetilde{F}_1, \widetilde{F}_2)$

Step 1: Input the query image and scene image.

Step 2: Apply fuzzy shape descriptor on both images for extracting feature points.

Step 3: Extract feature points $a_1^\alpha \in \widetilde{F}_1(\alpha)$, $a_2^\alpha \in \widetilde{F}_2(\alpha)$ with membership value α from both images, respectively, for $\alpha \in (0, 1]$.

Step 4: Apply similarity measure formula by equation (5.3) to compare query image feature points and scene image feature points.

Step 5: For $\alpha \in (0, 1]$, return the α-cut of $\widetilde{S}(\widetilde{F}_1, \widetilde{F}_2)$.

5.3.3 Similarity measure between a crisp line and a symmetric fuzzy line

We begin with the definition of the distance between a crisp line and a symmetric fuzzy line.

Definition 5.3.2. Let l and $\widetilde{L}_S$ be a crisp line and a symmetric fuzzy line, respectively. The membership function of fuzzy distance between l and $\widetilde{L}_S$, say $\widetilde{D}(l, \widetilde{L}_S)$, can be defined by

$$\mu\left(d' \middle| \widetilde{D}(l, \widetilde{L}_S)\right) = \sup\{\alpha : d' = \sqrt{\frac{\sum_{i=1}^{n} {d_i}^2}{|l_i^\alpha|}}, \text{ where } d_i = d(a_i^\alpha, l),\ a_i^\alpha \in l^\alpha$$
$$\text{and } l^\alpha \in \widetilde{L}_S(\alpha) \text{ with membership value } \alpha\}.$$

The similarity measure between l and $\widetilde{L}_S$, say $\widetilde{S}(l, \widetilde{L}_S)$, can be defined by

$$\widetilde{S}(l, \widetilde{L}_S) = 1 - \widetilde{D}(l, \widetilde{L}_S), \text{ for } 0 \leq \widetilde{D}(l, \widetilde{L}_S) \leq 1. \tag{5.4}$$

Note 19. Note that a fuzzy number $(a_1/a_2/a_3)$ can be reduced between 0 and 1 by applying the transformation $T(x) = \frac{x-a_1}{a_3-a_1}$.

The following Algorithm 5.3.2 demonstrates how to find the similarity measure between l and $\widetilde{L}_S$ in the fuzzy image.

Algorithm 5.3.2: To get the similarity measure $\widetilde{S}(l, \widetilde{L}_S)$

Step 1: Input the query image and scene image.

Step 2: Apply fuzzy shape descriptor on scene image for extracting feature points.

Step 3: Extract feature points $a_i^\alpha \in \widetilde{L}_S(\alpha)$ with membership value α from scene image, for $\alpha \in (0, 1]$.

Step 4: Apply similarity measure formula by equation (5.4).

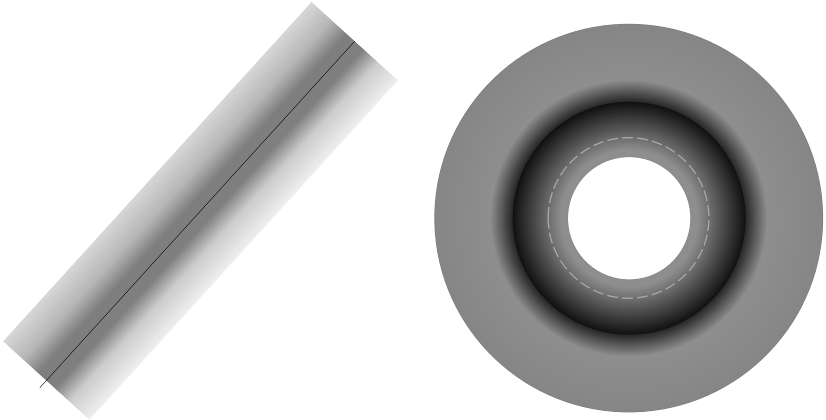
Step 5: For $\alpha \in (0, 1]$, return the α-cut of $\widetilde{S}(l, \widetilde{L}_S)$.

In the following section, we present experimental results for detecting fuzzy lines and circles and evaluating similarity measures between a crisp line and a symmetric fuzzy line.

5.4 Experimental results

As shown in Figure 5.6, different synthetically generated images are used to illustrate the effectiveness of the proposed fuzzy Hough transform Algorithms 5.2.3 and 5.2.4. Also, the proposed Algorithm 5.3.2 is applied on two sample images (see Figure 5.9) to evaluate similarity measures between a crisp line and a symmetric fuzzy line.

Figure 5.8 (a) demonstrates the performance of the proposed FHT Algorithm 5.2.3 in identifying crisp lines $l^\alpha \in \widetilde{L}_S(\alpha)$ on a gray image as shown in Figure 5.6 (a), for $\alpha = 0.6$ and 1. The parameter values (ρ^α, θ) are quantized in steps of unity, where $\rho^\alpha = [-4297, 4297]$ and $\theta \in [-90^\circ, 90^\circ]$, for $\alpha = 0.6$ and 1. Accumulated votes in parameter space are shown in Figure 5.7 with arrow highlighting maximums. The



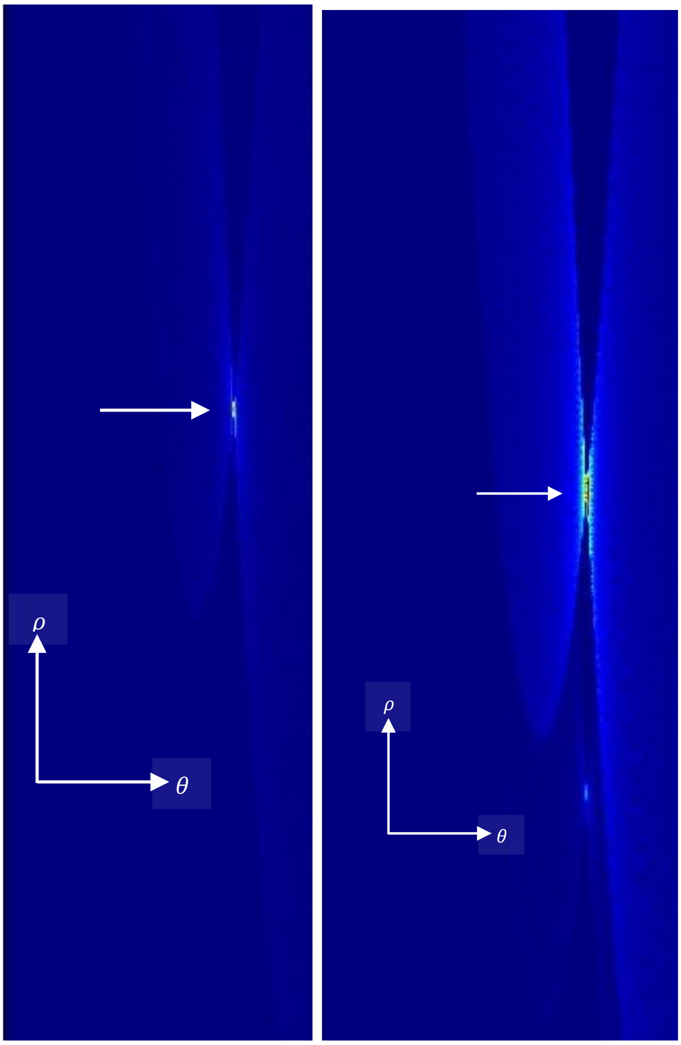
(a) Fuzzy line $\widetilde{L}_S$ (b) Fuzzy circle $\widetilde{C}$

FIGURE 5.6: Synthetically generated images of fuzzy line $\widetilde{L}_S$ and fuzzy circle $\widetilde{C}$

parameters indicated by the maximums are used to reconstruct the lines (detected in red color) in the image (see Figure 5.8 (a)).

In Algorithm 5.2.4, the parameter value $r^\alpha \in [3, 200]$ is quantized in steps of unity, for $\alpha = 0.8$ and 0.6. The parameters $(u^\alpha, v^\alpha, r^\alpha) = (265, 199, 114)$ and $(u^\alpha, v^\alpha, r^\alpha) = (302, 266, 3)$ has maximum number of voting, for $\alpha = 0.8$ and 0.6, respectively. The crisp circles (shown in blue color in Figure 5.8 (b)) is detected by the proposed FHT Algorithm 5.2.4, for $\alpha = 0.8$ and 0.6.

Next, we apply Algorithm 5.3.2 in Figure 5.9 to evaluate similarity measure between images (a) and (b). To extract the features points in the images (a) and (b), in the equation (5.1), we take the distance between the corner pixels of the images to find the bandwidth of the fuzzy line. By applying Algorithm 5.3.2 in Figure 5.9, we find the points $(0.9932), (0.9928), (0.4665) \in \widetilde{S}(l, \widetilde{L}_S)(\alpha)$ with $\mu\left((0.9932)\middle|\widetilde{S}(l, \widetilde{L}_S)\right) = 0.0850$, $\mu\left((0.9928)\middle|\widetilde{S}(l, \widetilde{L}_S)\right) = 0.1833$ and $\mu\left((0.4665)\middle|\widetilde{S}(l, \widetilde{L}_S)\right) = 0.7534$.

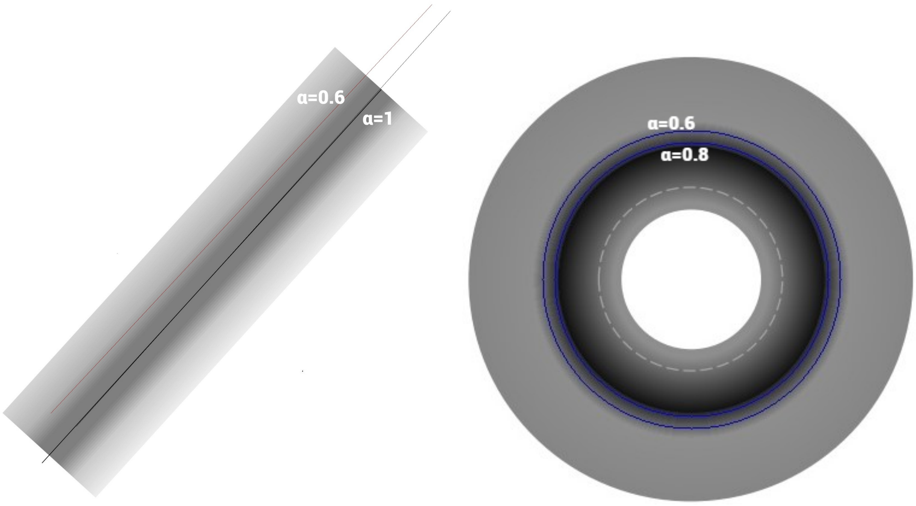


(a) Accumulated votes for $\alpha = 1$ (b) Accumulated votes for $\alpha = 0.6$

FIGURE 5.7: Accumulated votes in parameter space

5.5 Conclusion

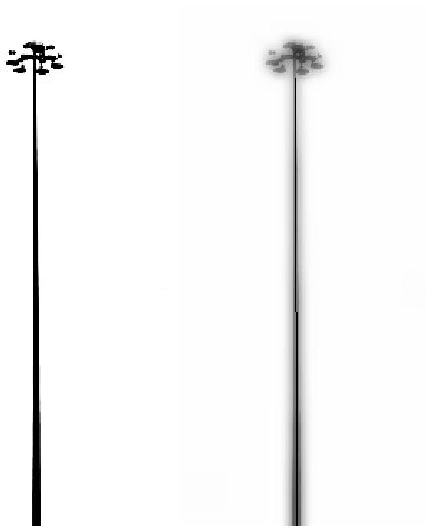
This paper includes the study of the application of fuzzy plane geometrical elements like fuzzy lines and fuzzy circles. We have introduced a technique, say fuzzy Hough transform (FHT), for detecting fuzzy lines and fuzzy circles in the fuzzy image space. This technique aims to find imprecise objects within a certain class of imprecise shapes by a voting procedure. Also, the fuzzy Hough transform can be applied to detect generalized fuzzy geometrical curves (fuzzy triangles and fuzzy conics) by



(a) Detected crisp lines in $\widetilde{L}_S(0)$, for $\alpha = 0.6$ and 1

(b) Detected crisp circles in $\widetilde{C}(0)$, for $\alpha = 0.8$ and 0.6

FIGURE 5.8: Detected fuzzy lines and circles using Algorithms 5.2.3 and 5.2.4



(a) Crisp line l

(b) Fuzzy line $\widetilde{L}_S$

FIGURE 5.9: Sample images of a crisp line l and a symmetric fuzzy line $\widetilde{L}_S$

extending the dimension of the accumulator array. A brief study on the generalized version of the fuzzy Hough transform is also described.

One of the basic cons of the FHT is that it is not suitable for the detection of non-symmetric fuzzy lines since a point in the accumulator cell with a maximum number of elements represents an infinite line in the (x, y)-space. Hence FHT is valid only for detecting symmetric fuzzy lines.

Future research will be based on the detection of generalized fuzzy curves (fuzzy triangles, fuzzy conics, etc.). An extension of the fuzzy Hough transform to extract the features of space fuzzy lines [117] and fuzzy spheres will also be included in future study.

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