Chapter 2

Introduction

In the previous Chapter 1, we studied some fundamental concepts related to semigroup theory, FC, required function spaces, and preliminary facts that will be used in subsequent chapters to investigate the considered problems. We discuss semigroup, fractional powers of operators, and resolvent operators. Additionally, we discuss fractional integrals and derivatives and analyze their fundamental features.

In this chapter, we present a brief introduction to the fundamental concepts and the context of the study, including a historical background. We discuss an overview of key topics such as fractional differential equations, delay differential equations, evolution equations, and nonlocal conditions. Additionally, this chapter presents a concise introduction to the problems addressed in the thesis and includes a detailed literature survey.

2.1 Fractional Differential Equations

The theory of FC is not a modern concept; its origins are almost as far as the birth of differential calculus. Interest in this field arose soon after the foundational ideas of classical calculus were established, reflecting the curiosity of early mathematicians about extending calculus beyond integer orders. FC originated alongside classical calculus in the 17th century. Its first mention is attributed to a 1695 correspondence between Leibniz and L'Hôpital, where the idea of derivatives of arbitrary order was posed as a theoretical question.

Euler initiated the study of fractional derivatives in 1738, observing the noninteger-order derivative of x^η. In 1812, Laplace extended this idea to integrals, and Lacroix discussed fractional derivatives in 1819, though his work remained theoretical. Fourier mentioned derivatives of arbitrary order in 1822, also without applications. The first practical application came in 1823 when Abel used FC to solve the Tautochrone problem. This marked the beginning of its use in addressing real-world problems alongside theoretical advancements. However, these ideas remained speculative until the 19th century, when Riemann and Liouville formally introduced fractional integrals and derivatives, laying the groundwork for the systematic study of FC. FC, now a crucial subject in mathematical analysis, evolved with contributions from many notable mathematicians, including Euler, Leibniz, Liouville, Abel, Fourier, Riemann, Caputo, Grünwald, Letnikov, Weyl, and Hadamard.

In the last few decades, FC have grown into a robust mathematical field with diverse applications across science and engineering. Modern approaches integrate FC with nonlinear dynamics, control systems, and anomalous diffusion models, providing tools to study complex phenomena like memory effects and hereditary systems. This evolution reflects its journey from theoretical exploration to a versatile framework for modeling real-world systems. FC extend classical calculus to fractional orders and provide a robust toolkit for dealing with numerous complex challenges involving fractional derivatives and integrals [9, 15]. Apart from this, the field has garnered increasing attention in the research community and has found significant applications in control theory [24, 25], biology [26, 27, 28], neural networks [29, 30, 31], population dynamics [32, 33], and the references cited there, and other disciplines of sciences and mathematics [11, 34, 35, 36, 37].

2.2 Delay Differential Equations

The concept of DDEs has existed for 200 years, but significant progress occurred in the late 20th and early 21st centuries. Volterra [38] advanced the theory with applications in predator-prey models and viscoelasticity, while Minorsky [39] emphasized delays in feedback mechanisms for ship stabilization and automatic steering. The rise of control theory during this era further accelerated interest in DDEs. Foundational works, including Myshkis' book [40] on linear systems with delays and contributions by Bellman, Danskin Jr [41], and Cook [42] on stability theory, shaped the field's development.

Despite being significantly efficient in solving real-life problems, in some cases where complex temporal behavior is involved, such as studying the spread of disease depending on the incubation period, fractional delay differential equations (FDDEs) have a considerable edge over FDEs. The delay differential equation theory enables us to deal with physical problems that rely on both the present state and the initial history function, for example, the modeling of a ferromagnetic material [43]. The magnetization of the material depends not only on the current magnetic field but also on the previous history of the magnetic field. Such behavior can be effectively captured using delay differential equations, where the rate of change of magnetization relies on both the current magnetic field and its past values. A variety of real-world issues can be solved using delay differential equations (DDEs), including those in biology, life science, population dynamics, classical electrodynamics, epidemiology, the dynamics of diabetes, etc. Interested readers can find a thorough discussion of DDEs and their applications in [44, 45, 46, 47]. By involving fractional derivatives with DDEs, we get a more accurate representation of complex dynamics, systems with memory effects, and phenomena where the system evolution depends on past states with fractional delay. FDDEs are more suitable for capturing intricate temporal patterns and behaviors that may be challenging for classical functional differential equations. Moreover, it can offer greater flexibility in adjusting time delays.

2.3 Evolution Equations

Evolution equations describe the time-dependent behavior of physical, biological, or other dynamic systems. These equations often take the form of PDEs or integro-differential equations, capturing how a system evolves over time. They are fundamental in modeling processes such as heat conduction, fluid flow, population dynamics, and quantum mechanics. Evolution equations can be studied to understand the existence, uniqueness, and stability of solutions and long-term behavior, providing critical insights into the dynamics of the systems they represent. For example, nonlinear evolution equations appear in various fields, such as nonlinear reaction-diffusion equations in heat transfer and biological sciences, nonlinear Klein-Gordon and nonlinear Schrödinger equations in quantum mechanics, Navier-Stokes and Euler equations in fluid mechanics, and Cahn-Hilliard equations in material science.

These are just a few instances where nonlinear evolution equations play a critical role.

Evolution equations are crucial in chemistry for modeling time-dependent processes like reaction-diffusion systems, chemical kinetics, and heat transfer. They help predict how chemical concentrations, temperatures, and other properties change over time. These equations are especially valuable for understanding complex behaviors like oscillations and reaction pattern formations. Additionally, they play a crucial role in quantum chemistry and material science, aiding in analyzing particle dynamics and phase transitions. Overall, evolution equations enhance the ability to model, predict, and control chemical processes effectively. In physics, evolution equations are used to describe how physical systems change over time, such as in quantum mechanics with the Schrödinger equation, which governs particle behavior. They model processes like wave propagation, heat conduction, and fluid dynamics, including equations like the Navier-Stokes and heat equations. Evolution equations are essential for predicting the behavior of systems in fields like electromagnetism, general relativity, and material science. They provide a framework for understanding the dynamics of physical systems, from subatomic particles to large-scale phenomena like fluid flow and gravitational waves. In biology, evolution equations model processes like population dynamics and the spread of diseases, using equations such as reaction-diffusion models for pattern formation in ecosystems. They describe how species populations change over time, incorporating factors like birth rates, death rates, and migration. These equations are also applied in modeling the growth and spread of infections or diseases. Additionally, they are used to understand neural activity and patterns in biological systems. Evolution equations are used in engineering to model and analyze dynamic systems such as heat transfer, fluid flow, and structural vibrations. They describe how temperature, pressure, and displacement change over time in response to various inputs. These equations are crucial for designing and optimizing systems like engines, pipelines, and building structures. They also play a role in control engineering, where they help design controllers for stable and efficient system performance. Overall, evolution equations are essential for understanding and improving engineering processes and systems.

2.4 Nonlocal Conditions

Extending the preceding framework, we now turn our attention to the nature of initial and boundary conditions. Classical models typically impose local conditions, where the state of the system is defined at a single point in time or space. However, numerous real-world phenomena require a more comprehensive perspective. In systems characterized by memory and hereditary effects, such as those modeled by fractional or delay differential equations, nonlocal conditions provide a more accurate and realistic framework. These conditions capture information over intervals rather than isolated points and often involve integral expressions or functionals that depend on the system's past states. This approach aligns well with the inherent memory properties of fractional derivatives and the delayed responses found in many dynamic systems.

Nonlocal conditions significantly enhance the modeling of processes wherein the current state is influenced by historical or spatially distributed data. In the context of evolution equations, they can effectively represent extended spatial and temporal dependencies. Consequently, nonlocal conditions offer greater flexibility and accuracy in describing complex phenomena across a wide range of disciplines, including heat conduction with memory, population dynamics with hereditary structures, viscoelasticity, and control systems with delayed feedback. For instance, a nonlocal condition might define the initial state as an integral over a previous time interval, thereby incorporating cumulative past effects instead of relying solely on instantaneous values. The increasing adoption of such models underscores the importance of investigating fractional differential equations under nonlocal conditions, both from theoretical and application oriented perspectives. We also discuss nonlocal conditions in Section 3.1.

2.5 Model Problems

In this section, we present a concise overview of the model problems investigated throughout this thesis. To provide a clear understanding, comprehensive explanations of these problems are included at the beginning of the respective chapters. Below, we offer a brief synopsis of the models explored in this research, serving as a foundational guide for the detailed discussions that follow.

A brief overview of the model problems investigated throughout this thesis has been provided in this section. Moreover, comprehensive explanations of these problems are provided in the subsequent chapters. This thesis is organized into seven chapters to present the current work effectively. An outline of each model problem is provided below.

2.5.1 Nonautonomous Fractional Integro-Differential System with Nonlocal Condition

In Chapter 3, we first establish the existence and uniqueness of mild solutions for the following fractional integro-differential equations:

$$\begin{cases} {}^C_0D^\eta_t \mathsf{x}(t) = \mathcal{A}(t)\mathsf{x}(t) \\ \qquad\qquad + f\big(t, x(\mu_1(t)), \mathsf{x}(\mu_2(t)), \dots, \mathsf{x}(\mu_n(t)), \mathcal{H}\mathsf{x}(\mu_{n+1}(t))\big), \ t \in J, \\ \mathsf{x}(0) + \mathsf{g}(\mathsf{x}) = \mathsf{x}_0, \quad \mathsf{x}_0 \in X, \end{cases} \tag{2.1}$$

where ${}^C_0D^\eta_t$ stands for the Caputo operator of order $\eta \in (0,1)$, $J = [0,T]$, and X is a Banach space. The terms $\mu_i : J \to J$ are continuous functions satisfying the conditions $0 \le \mu_i(t) \le t$, $(i = 1, 2, \dots, n, n+1)$, $n \in \mathbb{N}$. Moreover, $\mathsf{f} : J \times X^{n+1} \to X$ is a continuous function. Further, $\{\mathcal{A}(t)\}_{t \ge 0}$ is a family of closed linear operators defined on the domain $D(\mathcal{A})$ dense in X, and the integral operator $\mathcal{H}$ is defined by

$$\mathcal{H}\mathsf{x}(t) = \int_0^t \mathsf{h}\big(t, p, \mathsf{x}(\mu_{n+1}(p))\big)dp,$$

where $\mathsf{h} : \Delta \times X \to X$ is nonlinear and continuous such that $\Delta = \{(t, \mathsf{x}) : 0 \le \mathsf{x} \le t \le T\}$.

Further, we explore the controllability result of the following nonautonomous fractional integro-differential control system:

$$\begin{cases} {}^C_0D^\eta_t \mathsf{x}(t) = \mathcal{A}(t)\mathsf{x}(t) + \mathcal{G}\mathsf{u}(t) \\ \qquad\qquad + \mathsf{f}\big(t, x(\mu_1(t)), \mathsf{x}(\mu_2(t)), \dots, \mathsf{x}(\mu_n(t)), \mathcal{H}\mathsf{x}(\mu_{n+1}(t))\big), \ t \in J, \\ \mathsf{x}(0) + \mathsf{g}(\mathsf{x}) = \mathsf{x}_0, \quad \mathsf{x}_0 \in X, \end{cases} \tag{2.2}$$

where $\mathcal{G} \in BL(U, X)$. We note that U is a Banach space and $\mathsf{u}(t)$ is a control function in a Banach space of admissible control functions $L^2(J, U)$.

2.5.2 Nonautonomous Fractional Control System with Nonlocal Condition

In Chapter 4, we extend the results of Chapter 3 for the fractional order control system. We explore the existence of mild solutions and approximate controllability for the following fractional order control system:

$$\begin{cases} {}_0^C D_t^\eta \mathsf{x}(t) = \mathcal{A}(t)\mathsf{x}(t) + \mathcal{G}\mathsf{u}(t) \\ \qquad\qquad + \mathsf{f}\big(t, x(\mu_1(t)), \mathsf{x}(\mu_2(t)), \ldots, \mathsf{x}(\mu_n(t)), \mathcal{H}\mathsf{u}(t)\big), \ t \in J \\ \mathsf{x}(0) + \mathsf{g}(\mathsf{x}) = \mathsf{x}_0, \quad \mathsf{x}_0 \in X, \end{cases} \tag{2.3}$$

where ${}_0^C D_t^\eta$ denotes the Caputo fractional operator of order $\eta \in (0,1)$, $J = [0, T]$, and X, U are Banach spaces. The delay terms $\mu_i : J \to J$ are continuous satisfying the conditions $0 \leq \mu_i(t) \leq t$, $(i = 1, 2, \ldots, n, n+1)$, $n \in \mathbb{N}$. The function $\mathsf{f} : J \times X^n \times U \to X$ and the function $\mathsf{g} : C(J, X) \to C(J, X)$ are continuous. Further, $\{\mathcal{A}(t)\}_{t \geq 0}$ is a family of closed linear operators defined on the domain $D(\mathcal{A})$ dense in X. Additionally, the integral operator $\mathcal{H}$ is defined by

$$\mathcal{H}\mathsf{x}(t) = \int_0^t \mathsf{h}\big(t, p, \mathsf{u}(p))\big) dp,$$

where $\mathsf{h} : \Delta \times U \to U$ is continuous such that $\Delta = \{(t, \mathsf{x}) : 0 \leq \mathsf{x} \leq t \leq T\}$. Further, $\mathcal{G} \in BL(U, X)$ and $\mathsf{u}(t)$ is a control function in a Banach space of admissible control functions $L^2(J, U)$.

2.5.3 Fractional Delay Differential Equations with Riesz-Caputo Derivative

In Chapters 3 and 4, we explore the existence, uniqueness, and controllability results for the FDEs with Caputo fractional derivatives. However, the incorporation of the Riesz-Caputo derivative in FDDEs brings a significant advancement in modeling systems that exhibit nonlocal dependencies and complex temporal behaviors. It provides a more efficient and accurate framework for capturing the evolution of phenomena in various fields, including geophysics, stock price options, control theory, and mechanics. In scenarios where physical processes initiated in the past but are also influenced by their future evolution, the Riesz-Caputo derivative stands out as a more suitable choice compared to the traditional right or left derivatives. This preference arises due to its ability to effectively account for synchronous impacts from both the past and the future within the domain.

In Chapter 5, we investigate the existence results for the following fractional differential equations:

$$\begin{cases} {}^{RC}_{0}D^{\eta}_{t}\mathsf{x}(t) = \mathsf{f}\big(t, \mathsf{x}(\mu_1(t)), \mathsf{x}(\mu_2(t)), \ldots, \mathsf{x}(\mu_n(t))\big), \, t \in J, \\ \mathsf{x}(0) = \mathsf{x}_0, \quad \mathsf{x}(T) = \mathsf{x}_T, \end{cases} \tag{2.4}$$

where ${}^{RC}_{0}D^{\eta}_{t}$ is the Riesz-Caputo fractional derivative of order $\eta \in (0, 1)$, $J = [0, T]$ X is a Banach space. The delay terms $\mu_i : [0, T] \to [0, T]$ are continuous functions satisfying the conditions $0 \leq \mu_i(t) \leq t$ for $i = 1, 2, \ldots, n$, $n \in \mathbb{N}$. Further, the function $\mathsf{f} : [0, T] \times X^n \to X$ is continuous.

2.5.4 Fractional Delay Integro-Differential Equations with Riesz-Caputo Derivative

In Chapter 6, we generalize the concept of Chapter 5 for fractional order integro-differential equations. We investigate the existence and uniqueness results for the following fractional order integro-differential equations:

$$\begin{cases} {}_0^{RC}D_t^{\eta}\mathsf{x}(t) = \mathsf{f}\big(t, x(\mu_1(t)), \mathsf{x}(\mu_2(t)), \ldots, \mathsf{x}(\mu_n(t)), Q\mathsf{x}(\mu_{n+1}(t))\big), \ t \in J, \\ \mathsf{x}(0) = \mathsf{x}_0, \ \ \mathsf{x}(T) = \mathsf{x}_T, \end{cases} \tag{2.5}$$

where ${}_0^{RC}D_t^{\eta}$ is the Riesz-Caputo fractional derivative of order $\eta \in (0,1)$, $J = [0,T]$, and X is a Banach space. The delay terms $\mu_i : [0,T] \to [0,T]$ are continuous functions satisfying the conditions $0 \le \mu_i(t) \le t$ for $i = 1, 2, \ldots, n, n+1$, $n \in \mathbb{N}$. Moreover, $\mathsf{f} : [0,T] \times X^{n+1} \to X$ is a continuous function and $\mathcal{H}$ is an integral operator defined as:

$$Q\mathsf{x}(t) = \int_0^t q\big(t, p, \mathsf{x}(\mu_{n+1}(p))\big)dp,$$

where $q : \Delta \times X \to X$ is a continuous function such that $\Delta = \{(t, \mathsf{x}) : 0 \le \mathsf{x} \le t \le T\}$.

2.6 Literature Review

Fractional differential equations generalize ordinary differential equations to non-integer orders, offering an alternative framework for representing nonlinear dynamics [48]. Their ability to model complex phenomena has garnered significant interest, as they provide a more accurate representation of real-life processes than integer-order models. These equations effectively capture nonlocal spatial and temporal relations with power-law memory kernels. Their extensive engineering and science applications, such as modeling nonlinear oscillations during earthquakes, seepage flow in porous media [49], and fluid traffic, highlight their growing importance. Notably, fractional derivatives can address limitations associated with the assumption of continuous traffic flow. Consequently, FDEs have gained significant attention and expanded rapidly over the past few decades. In 1992, El-Sayed [50] studied the following fractional order differential equations:

$$\begin{cases} D_t^{\eta}\mathsf{x}(t) = \mathsf{f}(t, \mathsf{x}(t)), \quad t > 0, \quad \eta \in \mathbb{R}^+, \\ \frac{d^i \mathsf{x}(t)}{dt^i} = 0, \ \text{for } i = 0, 1, 2, \dots, [\eta], \end{cases} \tag{2.6}$$

where $[\eta]$ is the largest integer less or equal to η. The author proved the existence, uniqueness, smoothness, and continuation of the solution of the initial value problem (2.6). Later, El-Sayed [51] demonstrated the existence of a nondecreasing solution to the fractional functional differential equation:

$$\begin{cases} {}^{c}D_t^{\eta}\mathsf{x}(t) = \mathsf{f}\big(t, \mathsf{x}(t), {}^{c}D_t^{\eta_1}(\mathsf{x}(t-r)), \dots, {}^{c}D_t^{\eta_n}(\mathsf{x}(t-nr)), \ t \in (0,1), \\ \frac{d^i \mathsf{x}(t)}{dt^i} = 0, \ \text{for } t \geq 0, i = 1, 2, \dots, n, \end{cases} \tag{2.7}$$

where the function $\mathsf{f}(t, Y)$ satisfies the Carathéodory conditions, $t \to \mathsf{f}(t, Y)$ is measurable for every $Y \in \mathbb{R}^{n+1}$, $t \to \mathsf{f}(t, Y)$ is continuous for all $t \in (0, 1)$, $\eta \in (n, n+1]$, and $\eta_j \in (j-1, j)$ for $j = 1, 2, \dots, n$, and $\eta_0 = 0$. El-Sayed [52] demonstrated the existence of a unique solution to the problem (2.7) when the parameter r is equal to 0. This result was established based on the assumption that the function $\mathsf{f}(t, Y)$ is continuous with respect to t and satisfies the Lipschitz condition with respect to $Y \in \mathbb{R}^{n+1}$.

In [53, 54], El-Sayed concentrates on the fundamental analysis of the abstract fractional order Cauchy problem that follows:

$$\begin{cases} {}_0^C D_t^\eta \mathsf{x}(t) = \mathcal{A}\mathsf{x}(t), \; t > 0, \\ \mathsf{x}(0) = \mathsf{x}_0, \end{cases} \tag{2.8}$$

where $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ is either a bounded or densely defined closed linear operator on Banach space X. The author demonstrated that if $\mathcal{A}$ is a sectorial operator, then for $\mathsf{x}_0 \in D(\mathcal{A}^2)$, the problem has a unique solution given by

$$\mathsf{x}(t) = \mathsf{x}_0 - \int_0^t r_\eta e^p \mathsf{x}_0 dp, \tag{2.9}$$

where r_η is the resolvent operator of the following integral equation:

$$\mathsf{x}_\eta(t) = \mathsf{x}_0 - \frac{1}{\Gamma(\eta)} \int_0^t (t-p)^{\eta-1} \mathcal{A}\mathsf{x}_\eta(p) dp. \tag{2.10}$$

In [55], Daftardar-Gejji and Babakhani consider the following FDEs:

$$\begin{cases} {}_0^C D_t^\eta \big(\mathsf{x}(t) - \mathsf{x}(0)\big) = \mathcal{A}\mathsf{x}(t), \quad 0 < \eta \leq 1, \\ \mathsf{x}(0) = \mathsf{x}_0, \end{cases} \tag{2.11}$$

where $\mathsf{x} \in \mathbb{R}^n, \mathcal{A} \in \mathbb{R}^{n \times n}$. They studied the existence, uniqueness, and stability of (2.11).

In Banach space X, El-Borai [56] extended this results to a nonautonomous Cauchy problem of the fractional order of the form:

$$\begin{cases} {}_0^C D_t^{\eta} \mathsf{x}(t) + \mathcal{A}(t)\mathsf{x}(t) = \mathsf{f}(t), \quad t \in J = [0, T], \quad T < \infty, \\ \mathsf{x}(0) = \mathsf{x}_0, \end{cases} \tag{2.12}$$

where $0 < \eta \leq 1$, x and f are X-valued functions on J. Note that $D(\mathcal{A})$ is dense in a Banach space U and $\mathcal{A}(t)$ depends on t, $\{\mathcal{A}(t) : t \in J\}$ is a family of closed linear operators defined on the dense domain $D(\mathcal{A})$, $\mathsf{x}_0 \in D(\mathcal{A})$. Under certain assumptions on $\mathcal{A}$ and f, the author developed the existence, uniqueness, and continuous dependence on the initial data of the solution to (2.12).

Apart from this, many researchers have worked in this area, and numerous studies [57, 58, 59, 60, 61, 62, 63, 64, 65, 66] looked into the existence of solutions and the existence of mild solutions for various kinds of FDEs in the literature.

Another important area of FC is the study of fractional control problems, which has garnered significant attention in recent decades, underscoring its relevance and effectiveness in understanding the dynamics of complex systems. These problems involve using FC to analyze and design control systems for processes exhibiting fractional-order dynamics. While traditional control theory primarily focuses on integer-order systems governed by differential equations with integer-order derivatives, many real-world processes exhibit nonlocal and memory-dependent behaviors that integer-order models fail to capture adequately. FC offers a powerful framework to address these challenges by incorporating fractional-order derivatives and integrals.

FC enable control engineers and researchers to more effectively model and analyze systems with memory-dependent dynamics, long-term dependencies, and nonlocal interactions. Fractional-order models provide enhanced flexibility in capturing system characteristics such as anomalous diffusion, hereditary properties, and power-law behaviors commonly observed in complex systems.

The application of FC to control problems has driven advancements across diverse fields, including robotics, aerospace engineering, chemical processes, and biomedical systems. Fractional control techniques offer innovative tools and methods for system identification, controller design, stability analysis, optimal control, and robust control. This research has led to developing novel strategies and algorithms tailored to systems with fractional dynamics. These approaches often incorporate fractional-order controllers, fractional observers, fractional adaptive control, and fractional optimization techniques to enhance system performance, stability, and robustness.

Consider the following fractional differential control system:

$$\begin{cases} {}^{C}_{0}D^{\eta}_{t}\mathsf{x}(t) = \mathcal{A}(\mathsf{x})(\mathsf{x}) + \mathcal{G}(t)u(\mathsf{x}) + \mathsf{f}(t, \mathsf{x}(t)), \quad t \in J = [t_0, T], \\ \mathsf{x}(0) = \mathsf{x}_0, \end{cases} \tag{2.13}$$

where ${}^{c}D^{\eta}_{t}$ is the Caputo fractional derivative of order $\eta \in (0, 1]$, the state vector $\mathsf{x}(t) \in \mathbb{R}^n$, the control vector $\mathsf{u}(t) \in \mathbb{R}^m$, the matrices $\mathcal{A}$ and $\mathcal{G}$ are real constant matrices of dimensions $n \times n, n \times m$, respectively, and $\mathsf{f} : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function satisfying Carathéodory conditions.

When $\eta = 1$, the nonlinear fractional dynamical system (2.13) reduces to the following integer-order nonlinear dynamical system:

$$\begin{cases} \frac{d\mathsf{x}(t)}{dt} = \mathcal{A}(t)\mathsf{x}(t) + \mathcal{G}(t)\mathsf{u}(t) + \mathsf{f}(t, \mathsf{x}(t)), \quad t \in J = [t_0, T], \\ \mathsf{x}(0) = \mathsf{x}_0. \end{cases} \tag{2.14}$$

The controllability results of (2.14) have been established by using fixed-point theorems by many researchers (see, for example, [67, 68] and the references therein). When $\mathcal{A}(t) = \mathcal{A}$ and $\mathcal{G}(t) = \mathcal{G}$, Matignon and D'Andréa Novel in [69], established the controllability results for linear fractional dynamical systems (the absence of a nonlinear function $\mathsf{f}(t, \mathsf{x}(t))$ in (2.13).

2.6.1 Literature Review for Nonautonomous Fractional Integro-Differential System with Nonlocal Condition

In Section 2.6, we provided a brief literature review related to our works. Now, we focus on a problem-specific literature review, starting with the literature on nonautonomous fractional integro-differential systems. In 1991, Byszewski was the first to investigate the existence and uniqueness results for the evolution equation with nonlocal conditions in the context of Banach space [70, 71]. Further, [72], the authors investigated the existence of mild solutions and the controllability of a nonlinear fractional control systems with damping in the context of Hilbert space. In 2009, Balacharan and Park investigated the controllability of a fractional nonlinear autonomous integrodifferential equation with both local and nonlocal conditions using fixed point theorems and semi-group theory [73]. Again, Qin et el. [74], examined the controllability of a fractional impulsive integro-differential system with nonlocal conditions by means of a piecewise continuous control function and fixed point theory. Further in [6], authors studied the existence of mild solutions and the controllability of a class of fractional nonlinear and non-autonomous integro-differential equations with delay using resolvent operators, fixed point theory, and the Kuratowski measure of noncompactness. Building on the previous findings, we combined delay functions and nonlocal conditions in our problem to expand the scope of our results and progress further in this direction.

In 2023, Abuasbeh et al. [75], consider the following fractional integro-differential equations with non-instantaneous impulses:

$$\begin{cases} D^{\eta}_{\mathsf{s}_n,t}\mathsf{x}(t) = \mathcal{A}\mathsf{x}(t) + \mathcal{G}\mathsf{u}(t) + \mathsf{f}\big(t,\mathsf{x}(t)\big) + \displaystyle\int_{\mathsf{s}_n}^{t} \mathsf{h}\big(p,\mathsf{x}(p)\big)d\mathsf{w}(p), \\ \mathsf{s}_n \leq t \leq t_{n+1}, \\ \mathsf{x}(t) = \mathsf{g}_n\big(t,\mathsf{x}(t_n^-)\big), \quad t_n < t < \mathsf{s}_n, \quad n = 1,2,\ldots,N, \\ \mathsf{x}\big(\mathsf{s}_n^+\big) = \mathsf{x}\big(\mathsf{s}_n^-\big), \quad n = 1,2,\ldots,N \\ \mathsf{x}(0) = \mathsf{x}_0, \end{cases} \tag{2.15}$$

in Hilbert space, where $D^{\eta}_{\mathsf{s}_n,t}$ is a Caputo derivative of with lower limit s_n, and $1/2 < \eta < 1$. They examined the existence, uniqueness, and approximate controllability by using theories of FC, nonlinear analysis, and fixed-point theory. In addition to this, we will present further literature reviews on nonautonomous fractional integro-differential systems with nonlocal conditions in Chapters 3 and 4.

2.6.2 Literature Review for Nonautonomous Fractional Control System with Nonlocal Condition

In Section 2.6 and Subsection 2.6.1, we reviewed literature related to our studies. Continuing with the literature review, this section provides a brief overview of the existing works on nonautonomous fractional control systems, where achieving exact controllability is often challenging, making approximate controllability a valuable alternative for bringing the system state arbitrarily close to the desired state, particularly in real-world applications. Approximate controllability ensures that the system state can be brought arbitrarily close to the desired state, offering significant advantages in real-world applications. Unlike exact controllability, it is particularly useful for systems that cannot be controlled exactly, such as those with infinite-dimensional state spaces or those affected by modeling errors, noise, or uncertainties. Approximate controllability provides greater flexibility, robustness, and efficiency in control design, often requiring fewer resources and proving more practical for complex systems like fractional-order and DDEs. Its broader applicability makes it a valuable concept in scenarios where achieving the exact state is not essential.

In 1994, George [76] studied the approximate controllability of the following nonautonomous semilinear system:

$$\begin{cases} \dfrac{d\mathsf{x}(t)}{dt} = \mathcal{A}(t)\mathsf{x}(t) + \mathcal{G}\mathsf{u}\big(t\big) + \mathsf{f}(t, \mathsf{x}(t)), \quad 0 \leq t_0 < t \leq t_1 < \infty, \\ \mathsf{x}(t_0) = \mathsf{x}_0. \end{cases} \tag{2.16}$$

in Hilbert spaces by reducing the approximate controllability problem into a solvability problem, where $\mathcal{A}(t) : H \to H$ is a linear operator that may or may not be bounded. Their proof is based on the assumption that the corresponding linear system:

$$\begin{cases} \dfrac{d\mathsf{x}(t)}{dt} = \mathcal{A}(t)\mathsf{x}(t) + \mathcal{G}\mathsf{u}(t), \quad 0 \leq t_0 < t \leq t_1 < \infty, \\ \mathsf{x}(t_0) = \mathsf{x}_0. \end{cases} \tag{2.17}$$

is approximately controllable. In 2011, Sukavanam and Kumar [77] consider the following fractional functional semilinear differential equations:

$$\begin{cases} \dfrac{d^{\eta}\mathsf{x}(t)}{dt^{\eta}} = \mathcal{A}(t)\mathsf{x}(t) + \mathcal{G}\mathsf{u}(t) + \mathsf{f}(t, \mathsf{x}_t, \mathsf{u}(t)), \quad t \in (0, T], \\ \mathsf{x}_0(\tau) = \mathsf{g}(\tau), \quad \tau \in [-l, 0], \end{cases}$$

in Banach spaces, where $\mathsf{x} : [-l, T] \to X$ is a continuous function and $\mathsf{x}_t : [-l.0] \to X$ is defined $x_t(\tau) = \mathsf{x}(t + \tau)$, X is Banach space, and f is nonlinear and continuous. They studied the approximate controllability under the assumption that the corresponding linear system is approximately controllable.

In [78], the authors investigate the following fractional stochastic differential equations:

$$\begin{cases} {}_0^C D_t^{\eta}\big(\mathsf{x}(t) + \mathsf{f}(t, \mathsf{x}(t))\big) = \mathcal{A}(\mathsf{x})(t) + \mathsf{g}(t, \mathsf{x}(t))\dfrac{d\theta(t)}{dt} + \mathsf{h}(t, \mathsf{x}(t))\dfrac{d\sigma^{\mathcal{H}}}{dt}, \\ \zeta\mathsf{x}(t) = \sigma_1\mathsf{x}_t, \ \ t \in J = [t_0, T] \\ \mathsf{x}(0) = \mathsf{x}_0, \end{cases} \tag{2.18}$$

where $\mathcal{A}$ is an operator with a domain $D(\mathcal{A})$, which is a subset of $C\left(J, L^2\left(\Omega, X\right)\right)$, $\zeta : D\left(\zeta\right) \to R\left(\zeta\right)$ is a linear operator, and σ_1 is a linear and continuous mapping from U to X. Additionally, $\left\{\sigma^{\mathcal{H}}\right\}_{t\geq 0}$ represents fractional Brownian motion. They explored approximate boundary controllability in Hilbert spaces, proving the existence of mild solutions for neutral fractional stochastic systems with fractional Brownian motion. Sufficient conditions are derived using fractional calculus, semigroups, and the Schauder fixed-point theorem, with an example demonstrating the results.

In [79], the authors studied the interior approximate controllability of a semilinear second-order system by reformulating it as an equivalent first-order system. The controllability results are derived using the Leray-Schauder alternative theorem and the principle of contraction. Hakkar et al. [80] discussed a class of impulsive fractional stochastic differential equations driven by mixed fractional Brownian motions with infinite delay and Hurst parameter $\tilde{H} \in (1/2, 1)$. They established the existence of a piecewise continuous mild solution using fixed-point techniques, η-resolvent family, and fractional calculus. The approximate controllability of the system is also investigated, with results demonstrated through an example. Similarly, [81] studied the approximate controllability of neutral hemivariational inequality with impulses. Firstly, they derived the mild solution for the given system and proved existence and approximate controllability results using semigroup theory, properties of generalized Clarke's sub anddifferentials, and some fixed point theorem. Apart from this, we will present further literature reviews on nonautonomous fractional control systems with nonlocal conditions in Chapters 3 and 4.

2.6.3 Literature Review for Fractional Delay Differential Equations with Riesz-Caputo Derivative

Previously, in Section 2.6 and Subsections 2.6.1 and 2.6.2, we reviewed several articles on FDEs. Continuing with the study of FDEs, we briefly overview the literature for FDDEs with Riesz-Caputo derivatives in this subsection. As part of this study, we examined the research article [14], where the authors established several results for solutions of a fractional BVP involving the first-order Riesz-Caputo derivative. These results were derived using Gronwall's inequality and some fixed-point theorems. Further, as we have gone through [82], the authors discussed some existence results of the same FDEs subject to anti-periodic boundary conditions involving second-order Riesz derivative in the Caputo sense by employing fixed-point theorems and new fractional Gronwall's inequalities under the Lipschitz condition. Additionally, they investigated the existence and controllability of fractional evolution equations with nonlocal conditions in their work [83]. Furthermore, in [84], they delved into the Cauchy problem associated with stochastic non-autonomous evolution equations. In addition to this, we will present further literature reviews on nonautonomous fractional differential systems in Chapters 5 and 6.

2.6.4 Literature Review for Fractional Delay Integro-Differential Equations with Riesz-Caputo Derivative

In Section 2.6 and Subsections 2.6.1-2.6.3, a brief literature survey has been done on the FDEs. In this subsection, we extend our study in this direction by providing a concise literature review for fractional delay integro-differential equations involving the Riesz-Caputo derivative. In [85], the authors investigated a class of nonlinear multi-point boundary value problems involving the Riesz-Caputo derivative. They established the existence and uniqueness of solutions using Krasnoselskii's fixed-point theorem, Schauder's fixed-point theorem, and the Banach contraction principle. Similarly, Nass et al. [86], focused on the existence and uniqueness of solutions for a nonlinear Langevin equation incorporating Riesz-Caputo fractional derivatives with anti-periodic boundary conditions, employing various fixed-point theorems to support their findings. For further studies on problems involving Riesz-Caputo derivatives, refer to [82, 87, 88, 89] and cited therein. Apart from this, we will present further literature reviews on nonautonomous fractional differential systems in Chapters 5 and 6.

2.7 Objectives

In this thesis, we focus on exploring various qualitative properties, including existence, uniqueness, controllability, and approximate controllability, of different types of fractional DDEs involving Caputo and Riesz-Caputo fractional derivatives. The core approach of the proofs relies on the application of semi-group theory, FC, and fixed-point theory. By employing these mathematical frameworks, we transform complex problems related to existence, uniqueness, and controllability into more manageable fixed-point problems. In conclusion, the primary objectives of this thesis can be summarized as follows:

- To investigate the existence and controllability of solutions for a specific class of nonlinear fractional functional nonautonomous integro-differential equations involving Caputo fractional derivative with a nonlocal condition and $\eta \in (0, 1)$, within the framework of Banach space.
- To study the existence of mild solutions and approximate controllability of a nonlinear nonautonomous fractional control system with delay and nonlocal initial conditions for $\eta \in (0, 1)$, involving Caputo derivatives.
- To study the existence results for a family of fractional functional differential equations involving the Riesz-Caputo fractional derivative in a Banach space.
- To investigate the existence, uniqueness, and the existence of extremal solutions for a particular category of fractional delay integro-differential equations in the context of Banach space, incorporating the Riesz-Caputo fractional derivative.

2.8 Outline of the Thesis

This thesis is organized into seven chapters, each designed to systematically present the research and findings. The chapters are structured to ensure a logical progression, and an overview of the content covered in each chapter is provided below, offering insight into the key aspects of the work.

Chapter 1: **Preliminaries**

This chapter introduces the essential mathematical tools used throughout the thesis. It covers key notations, definitions, and lemmas, along with the basics of semigroup theory and fractional calculus, specifically the Riemann-Liouville operators, as well as the Caputo and Riesz-Caputo derivatives. Additionally, important fixed-point theorems are discussed, providing the foundation for proving the existence and uniqueness results in the subsequent chapters.

Chapter 2: **Introduction**

Chapter 2 introduces the fundamental concepts and context of the study, including its historical background. It provides an overview of key topics such as fractional calculus, DDEs, and evolution equations. Additionally, the chapter offers a concise introduction to the problems addressed in the thesis. Further, we provided a detailed literature survey along with the objective of the thesis.

Chapter 3: **Existence and Controllability Results of Fractional Integro-Differential Equations**

This chapter specifically examines the existence of mild solutions and controllability results for a class of fractional functional integro-differential equations subject to nonlocal conditions of the form $\mathsf{x}(0) + g(\mathsf{x}) = \mathsf{x}_0$, where g is a nonlinear function mapping from $C(J, X)$ to itself. Moreover, by incorporating a control operator $\mathcal{G}$ and a control function $\mathsf{u}(t)$ in the considered system, we extend the analysis to investigate the controllability of the fractional control system. Finally, we demonstrate a few examples supporting our discussed results.

Chapter 4: **Existence and Approximate Controllability Results for Fractional Integro-Differential Equations**

In this chapter, we extend the work done in Chapter 3 and studied the existence of mild solutions and the approximate controllability of a class of time-fractional, nonlinear, nonautonomous fractional control systems with delay. We provide two existence results using various fixed-point theorems. Further, we concluded a few approximate controllability results.

Chapter 5: **Study of Existence Results for Fractional Functional Differential Equations Involving Riesz-Caputo Derivative**

In this chapter, we explore the existence result of a fractional function differential equation by incorporating the Riesz-Caputo derivative. We provide a few existence results for the considered problem and gave a few examples for the demonstration of the utility of the proposed results.

Chapter 6: **Analysis of a Class of Fractional Delay Integro-Differential Equations with Riesz-Caputo Derivative**

In this chapter, we build upon the work presented in Chapter 5 by incorporating an integral term into the function argument. We introduce a few existence and uniqueness results, discuss the existence of extremal solutions, and provide illustrative examples to highlight the practical applications of the proposed results.

Chapter 7: **Conclusions and Future Plan**

Finally, Chapter 7 offers a comprehensive summary of the key findings, highlighting the significant results and conclusions drawn from the research conducted throughout the thesis. In addition to this, the chapter outlines potential directions for future work, emphasizing areas for further exploration and development based on the outcomes of this study.

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