A Study on Nonmonotone Methods to Solve Multiobjective Optimization Problems

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by

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List of Symbols

Symbol	**Description**
$\mathbb{N}$	Set of natural numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{R}^m_+$	Nonnegative orthant in $\mathbb{R}^m$ (components ≥ 0)
$\mathbb{R}^m_{++}$	Strictly positive orthant in $\mathbb{R}^m$ (components > 0)
$\mathbb{R}^n$	Euclidean space of dimension n
$\mathscr{B}(x)$	The Hessian approximation matrix at x
$I_{n\times n}$	Identity matrix of dimension n
$y_1 \succeq y_2$ or $y_2 \preceq y_1$	$y_1 - y_2$ consist of nonnegative components i.e., $y_1 - y_2 \in \mathbb{R}^m_+$
$y_1 \succ y_2$ or $y_2 \prec y_1$	$y_1 - y_2$ consist of positive components, i.e., $y_1 - y_2 \in \mathbb{R}^m_{++}$
$JF(x)$	Jacobian of F at x
Image$JF(x)$	Image set of Jacobian of F at x
$A \underset{\infty}{\subset} B$	A is an infinite subset of B
$\lVert \cdot \rVert_2$	2-norm (Euclidean norm)

Abbreviations

Abbreviation	**Description**
MOP	Multiobjective Optimization Problem
BFGS	Broyden-Fletcher-Goldfarb-Shanno
NMPRP	Nonmonotone Polak-Ribière-Polyak
NMCG	Nonmonotone Conditional Gradient Method
HSDY	Hestenes-Stiefel Dai-Yuan
HSPRP	Hestenes-Stiefel Polak-Ribière-Polyak
DYPRP	Dai-Yuan Polak-Ribière-Polyak
ALM	Augmneted Lagrangian Method

CERTIFICATE

It is certified that the work contained in the thesis titled *"A Study on Nonmonotone Methods to Solve Multiobjective Optimization Problems"* by ***Ashutosh Upadhayay*** has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

It is further certified that the student has fulfilled all requirements of Comprehensive Examination, Candidacy, and SOTA for the award of Ph.D. Degree.

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DECLARATION BY THE CANDIDATE

I, ***Ashutosh Upadhayay***, certify that the work embodied in this thesis is my own bonafide work and carried out by me under the supervision of ***Dr. Debdas Ghosh*** from ***January 2019*** to ***August 2024*** at the ***Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi***. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not willfully copied any other's work, paragraphs, text, data, results, *etc.*, reported in journals, books, magazines, reports, dissertations, theses, *etc.*, or available at websites and have not included them in this thesis and have not cited as my own work.

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PREFACE

Optimization is a crucial and versatile field that applies to many scientific disciplines, such as operations research, management science, economics and finance, and all engineering domains that involve some kind of extrema. Optimization process aims to achieve the best possible outcome among all feasible alternatives even under certain constraints. In essence, optimization is used to identify, describe and calculate the maximum or minimum of a function for a set of admissible points and specific conditions.

Optimization is a broad field that encompasses various methods and applications. It can be used to find the best solutions for different types of problems, such as maximizing efficiency, minimizing costs, or increasing profits. These problems may have different characteristics, such as convexity, linearity, nonlinearity, quadraticity, semidefiniteness, dynamism, integrality, or stochasticity. Optimization has a rich theoretical background and sophisticated algorithms. In general, optimization involves finding the optimal values of one or more decision variables that minimize or maximize a function or a set of functions subject to some constraints. Depending on the number of objectives, optimization problems can be classified into single or MOPs.

This thesis is devoted to the extensions of various classical methods for singleobjective optimization problems to MOPs. MOPs involve finding a set of solutions that are optimal with respect to two or more conflicting objectives. The main challenges of multiobjective optimization are to handle the trade-offs between the objectives, to deal with the possible existence of multiple Pareto optimal solutions, and to generate a representative and diverse approximation of the Pareto front.

The classical methods for multiobjective optimization can be broadly classified into three categories: scalarization methods, nonscalarization methods, and population-based methods. Scalarization methods transform the original problem into a singleobjective problem by using a scalar function that combines multiple objectives. Nonscalarization methods are techniques that do not transform a multiobjective problem into a singleobjective one by using a scalaring function. Instead, they try to find a set of solutions that represent the trade-offs between the conflicting objectives. Population-based methods maintain a set of solutions that are updated iteratively by using some selection and variation operators. All the above categories have their advantages and disadvantages, and the choice of the method depends on the characteristics of the problem and the preferences of the decision maker.

The aim of this thesis is to present some theoretical and practical aspects of classical methods for multiobjective optimization and to propose some novel extensions and improvements. The thesis consists of six chapters.

The first chapter introduces the basic concepts and definitions of multiobjective optimization and provides an overview of the existing methods and challenges. The second chapter introduces a nonmonotone Polak-Ribière-Polyak conjugate gradient method for MOPs and demonstrates its effectiveness on various test problems. The third chapter introduces projection-type hybrid conjugate gradient methods for MOPs with an application to an optimal control problem and demonstrates its effectiveness on various test problems. The fourth chapter introduces a nonmonotone quasi-Newton methods for unconstrained strongly convex MOPs. We explore two well-known types of nonmonotone line searches: one that considers the maximum of recent function values and the other that calculates their average. Moreover, we demonstrates its effectiveness on various test problems. The fifth chapter introduces a nonmonotone condition gradient method for MOPs and demonstrates its effectiveness on various test problems. The sixth chapter proposes a new scalarization-based method that incorporates the augmented Lagrangian technique for MOPs with an application to an optimal control problem. The seventh chapter summarizes the main contributions and findings of this thesis and suggests some directions for future research.