Chapter 3

Existence and Controllability Results of Fractional Integro-Differential Equations

3.1 Introduction

After the explanation of the significance of the fractional derivative in Chapters 1 and 2, we begin with our first problem under consideration. In particular, this chapter looks into the existence of mild solutions and controllability results for a family of fractional functional integro-differential equations along with nonlocal conditions $\mathsf{x}(0) + g(\mathsf{x}) = \mathsf{x}_0$, where g is a function from some function space to X. In general, g could be nonlinear or an integral. It reduces to the local condition $\mathsf{x}(0) = \mathsf{x}_0$ if $g := 0$. However, the increasing utilization of FC underscores the necessity for a robust theory akin to ordinary calculus. The last few decades have witnessed substantial advancements in understanding the existence, uniqueness, stability, regularity, controllability, and other qualitative behaviors of various FDEs. This progress, accompanied by the development of new analytical techniques modified for fractional operators, has extended classical criteria and enhanced our understanding of FC applications across diverse fields. Some researchers worked in this field and discussed the existence and uniqueness of mild solutions of FDEs [2, 6, 90] and cited theirs in.

On the other hand, some physical models require the behavior of a system at a point to depend on the values of the function over a range of points rather than just at a single point. This broader context is necessary to accurately describe phenomena such as long-range interactions, memory effects, and fractal or anomalous behavior. In such cases, nonlocal conditions are essential for capturing the complex dynamics of the system and providing a more realistic representation of its behavior. In 1991, Byszewski was the first to investigate the existence and uniqueness results for the evolution equation with nonlocal conditions in the context of Banach space [70, 71]. Consequently, the nonlocal conditions are significantly useful for the systems of FC, as they often require information from a wider range of points, driving their incorporation and application in such systems [91, 92, 93]. These conditions refer to initial or boundary conditions in a mathematical or physical system that depend on the entire history or behavior of the system rather than just its immediate surroundings or current state. They can be found in various scientific and engineering fields, and they are used when factors outside its immediate vicinity influence the behavior of a system. Further, they have diverse applications in fields like peridynamics, population dynamics, fluid mechanics, quantum mechanics, petroleum exploitation, thermodynamics, elasticity, wave propagation, etc. [94, 95, 96, 97, 98]. Nonlocal conditions play a crucial role in peridynamics, a theory describing material behavior, including deformation, fracture, and damage. Unlike local models, peridynamics considers influences beyond immediate neighbors, essential for accurately representing material behavior, especially in severe deformation or damage scenarios [99].

Inspired by the above-discussed results, in the chapter, we investigate the existence and uniqueness of mild solutions for the following fractional integro-differential equations:

$$\begin{cases} {}^{C}D_t^{\eta}\mathsf{x}(t) = \mathcal{A}(t)\mathsf{x}(t) \\ \qquad\qquad + \mathsf{f}\big(t, x(\mu_1(t)), \mathsf{x}(\mu_2(t)), \dots, \mathsf{x}(\mu_n(t)), \mathcal{H}\mathsf{x}(\mu_{n+1}(t))\big), \ t \in J, \\ \mathsf{x}(0) + \mathsf{g}(\mathsf{x}) = \mathsf{x}_0, \ \ \mathsf{x}_0 \in X, \end{cases} \tag{3.1}$$

where ${}_0^C D_t^{\eta}$ stands for the Caputo operator of order $\eta \in (0,1)$, $J = [0,T]$, and X is a Banach space. The delay terms $\mu_i : J \to J$ are continuous functions satisfying the conditions $0 \leq \mu_i(t) \leq t$, $(i = 1, 2, \dots, n, n+1)$, $n \in \mathbb{N}$. Moreover, the function $\mathsf{f} : J \times X^{n+1} \to X$ is a continuous function. Further, $\{\mathcal{A}(t)\}_{t\geq 0}$ is a family of closed linear operators defined on the domain $D(\mathcal{A})$, where $D(\mathcal{A})$ is dense in X. Additionally, the integral operator $\mathcal{H}$ is defined by

$$\mathcal{H}\mathsf{x}(t) = \int_0^t \mathsf{h}\big(t, p, \mathsf{x}(\mu_{n+1}(p))\big)dp,$$

where $\mathsf{h} : \Delta \times X \to X$ is nonlinear continuous function such that $\Delta = \{(t,p) : 0 \leq \mathsf{x} \leq t \leq T\}$.

We also discuss the concept of controllability, which is a fundamental concept in mathematical control theory. Controllability analysis for a dynamical system involves investigating whether it is possible to manipulate the system's state to reach any desired state within a given time frame using an admissible control. Incorporating fractional derivatives with control systems offers enhanced flexibility and versatility compared to ordinary control systems by capturing memory and hereditary effects, enabling better modeling of complex dynamics and improving system robustness. Additionally, fractional controllers exhibit superior performance in handling non-integer order behaviors, leading to improved control accuracy and stability in various applications [100, 101]. In recent years numerous researchers have studied controllability problems for various kinds of fractional dynamical systems [102, 103, 104, 105]. In 2009, Balacharan and Park investigated the controllability of a fractional nonlinear autonomous integro-differential equation with both local and nonlocal conditions using fixed-point theorems and semi-group theory [73]. Later, Qin et al. [74] examined the controllability of a fractional impulsive integro-differential system with nonlocal conditions by means of a piecewise continuous control function and fixed-point theory. Further in [6], authors studied the existence of mild solutions and the controllability of a class of fractional nonlinear and non-autonomous integro-differential equations with delay using resolvent operators, fixed-point theory, and the Kuratowski measure of noncompactness. Building on the previous findings, we combined delay functions and nonlocal conditions in our problem to expand the scope of our results and progress further in this direction.

We consider the following fractional control system:

$$\begin{cases} {}_0^C D_t^{\eta} \mathsf{x}(t) = \mathcal{A}(t)\mathsf{x}(t) + \mathcal{G}\mathsf{u}(t) \\ \qquad\qquad + \mathsf{f}\big(t, x(\mu_1(t)), \mathsf{x}(\mu_2(t)), \ldots, \mathsf{x}(\mu_n(t)), \mathcal{H}\mathsf{x}(\mu_{n+1}(t))\big),\ t \in J, \\ \mathsf{x}(0) + \mathsf{g}(\mathsf{x}) = \mathsf{x}_0, \quad \mathsf{x}_0 \in X, \end{cases} \tag{3.2}$$

where $\mathcal{G} \in BL(U, X)$. We note that U is a Banach space and $\mathsf{u}(\cdot)$ is a control function in a Banach space of admissible control functions $L^2(J, U)$.

We aim to go deeper into the study of nonautonomous systems and establish additional results beyond what has been achieved thus far. First, we extend the autonomous integro-differential equations of fractional order to nonautonomous fractional integro-differential equations. We prove the existence and uniqueness of a mild solution for the problem (3.1) under suitable assumptions on the operator f. Additionally, we investigate the controllability result of the problem (3.2). We use the FC technique, theory of nonlinear analysis, resolvent operator theory, and fixed-point techniques to prove these conclusions.

The structure of this chapter is in the following manner: To begin, in Section 3.2, we discuss fundamental definitions that are mandatory to establish the main results of our study. Section 3.3 establishes the existence and uniqueness of mild solutions of the problem (3.1). We explore the controllability result for the control system (3.2) in Section 3.4. In Section 3.5, we discuss a few examples illustrating the applicability of the proposed results. At the end, the conclusion is provided in Section 3.6.

3.2 Preliminary Results

In this section, we discuss the definition of mild solution for the considered problem (3.1) and the Hölder's inequality.

Lemma 3.1. ***(Hölder's Inequality)*** *[12, 106] Let $p, q \geq 1$ with $1 \leq p, q \leq \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then, for $\mathsf{f}_1 \in L^p(J, U)$ and $\mathsf{f}_2 \in L^q(J, X)$, we have $\mathsf{f}_1\mathsf{f}_2 \in L^1(J, X)$ and the following inequality holds*

$$\|\mathsf{f}_1\mathsf{f}_2\|_{L^1} \leq \|\mathsf{f}_1\|_{L^p}\|\mathsf{f}_2\|_{L^q},$$

Definition 3.2. (Mild Solution) [6, 73] A function $\mathsf{x} \in C(J, X)$ is said to be a mild solution of the system (3.1) for every $\mathsf{u} \in L^2(J, U)$ if it satisfies the following integral equation:

$$\mathsf{x}(t) = \mathcal{R}_\eta(t,0)\big(\mathsf{x}(0) - \mathsf{g}(\mathsf{x})\big) + \frac{1}{\Gamma(\eta)}\int_0^t \mathcal{R}_\eta(t,p)(t-p)^{\eta-1}$$
$$\times \big[\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \dots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{x}(\mu_{n+1}(p))\big)\big]\,dp. \tag{3.3}$$

3.3 Existence and Uniqueness Results

In this section, we discuss the existence and uniqueness of mild solutions for the system (3.1) under different set of assumptions on f.

Theorem 3.3. *Suppose that the following conditions hold:*

(H_1) *The resolvent operator $\mathcal{R}_\eta(t,p)$ is compact for $t, p > 0$.*

(H_2) *The function $\mathsf{f} : J \times X^{n+1} \to X$ is continuous and there exist non-negative Lebesgue integrable function $\mathsf{k}_i \in L^{1/\beta}(J, \mathbb{R}_+)$, $(i = 1, 2 \dots, n, n+1)$, $0 < \beta < \eta$ such that for any $\mathsf{x}_i \in X$, $(1, 2, \dots, n, n+1)$,*

$$\big\|\mathsf{f}\big(t, \mathsf{x}_1, \mathsf{x}_2, \dots, \mathsf{x}_n, \mathsf{x}_{n+1}\big)\big\| \le \sum_{i=1}^{n+1} \mathsf{k}_i(t)\|\mathsf{x}_i\|^\gamma, \quad 0 < \gamma < 1.$$

(H_3) *The function $\mathsf{h} : \Delta \times X \to X$ is continuous, and $\|\mathsf{h}(t,p,\mathsf{x})\| \le \|\mathsf{x}\|^\gamma$.*

(H_4) *The function g is continuous, and there exists a constant $\mathsf{c} > 0$ such that $\big\|\mathsf{g}(\mathsf{x})\big\| \le \mathsf{c}$.*

Then, the system (3.1) has a mild solution.

Proof. Define an operator $\Phi : C\,(J, X) \to C\,(J, X)$

$$\Phi \mathsf{x}(t) = \mathcal{R}_\eta(t,0)\big(\mathsf{x}_0 - \mathsf{g}(\mathsf{x})\big) + \frac{1}{\Gamma(\eta)}\int_0^t \mathcal{R}_\eta(t,p)(t-p)^{\eta-1} \times \Big[\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \dots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{x}(\mu_{n+1}(p))\big)\Big]dp. \tag{3.4}$$

Our aim is to prove that the system (3.1) has a mild solution. For this, we have to show that the operator (3.4) has a fixed-point. For the sake of clarity, the proof is divided into the following steps.

Step 1: First, we aim to demonstrate the continuity of the mapping $\Phi : C\,(J, X) \to C\,(J, X)$. For this, consider a sequence $\{\mathsf{x}_m\}_{m\in\mathbb{N}} \subseteq C\,(J, X)$ with x_m converging to $\mathsf{x} \in C\,(J, X)$. There exists a positive integer r ($\mathsf{r} \geq 1$) such that $\|\mathsf{x}_m\| \leq \mathsf{r}$ for all m. Define the set $\mathcal{B}_\mathsf{r}$ as follows:

$$\mathcal{B}_\mathsf{r} = \big\{\mathsf{x} \in C\,(J, X) : \|\mathsf{x}\| \leq \mathsf{r}\big\}.$$

Since for each $t \in J$, we have $0 \leq \mu_i(t) \leq t$; $(i = 1, 2, \dots, n, n+1)$. Thus, for $\mathsf{x}_m, \mathsf{x} \in \mathcal{B}_\mathsf{r}$, we get

$$\big\|\mathsf{x}_m\big(\mu_i(t)\big) - \mathsf{x}\big(\mu_i(t)\big)\big\| \leq \|\mathsf{x}_m - \mathsf{x}\| \to 0 \;\; \text{as } m \to \infty.$$

So, by the continuity of h, we have

$$\mathsf{h}\big(t, p, \mathsf{x}_m(\mu_{n+1}(t))\big) \to \mathsf{h}\big(t, p, \mathsf{x}(\mu_{n+1}(t))\big) \;\; \text{as } m \to \infty,$$

Again, by the continuity of f, we obtain

$$\mathsf{f}\big(t, \mathsf{x}_m(\mu_1(t)), \mathsf{x}_m(\mu_2(t)), \dots, \mathsf{x}_m(\mu_n(t)), \mathcal{H}x_m(\mu_n(t))\big)$$

$$\to \mathsf{f}\big(t, \mathsf{x}(\mu_1(t)), \mathsf{x}(\mu_2(t)), \ldots, \mathsf{x}(\mu_n(t)), \mathcal{H}\mathsf{x}(\mu_{n+1}(t))\big) \quad \text{as } m \to \infty,$$

for any $t \in J$ uniformly. That is, for any $\epsilon > 0$, there exists a natural number $n_0 \in \mathbb{N}$ such that for any $m > n_0, t \in J$, we have

$$\begin{aligned}
&\big\|\mathsf{f}\big(t, \mathsf{x}_m(\mu_1(t)), \mathsf{x}_m(\mu_2(t)), \ldots, \mathsf{x}_m(\mu_n(t)), \mathcal{H}x_m(\mu_n(t))\big) \\
&\qquad - f\big(t, \mathsf{x}(\mu_1(t)), \mathsf{x}(\mu_2(t)), \ldots, \mathsf{x}(\mu_n(t)), \mathcal{H}\mathsf{x}(\mu_{n+1}(t))\big)\big\| \\
&\qquad \le \frac{\Gamma(\eta+1)}{\mathcal{P}^* T^\eta}\frac{\epsilon}{2},
\end{aligned}$$

where $\mathcal{P}^* = \max_{0 \le p \le t \le T} \big\|\mathcal{R}_\eta(t,p)\big\|$. Hence, for any $m > n_0,\ t \in \mathcal{J}$, we have

$$\begin{aligned}
\big\|\Phi\mathsf{x}_m - \Phi\mathsf{x}\big\| &= \big\|\mathcal{R}_\eta(t,0)\big(\mathsf{x}_0 - \mathsf{g}(\mathsf{x}_m)\big) - \mathcal{R}_\eta(t,0)\big(\mathsf{x}_0 - \mathsf{g}(\mathsf{x})\big)\big\| + \frac{1}{\Gamma(\eta)}\int_0^t (t-p)^{\eta-1} \\
&\quad \times \big\|\mathsf{f}\big(p, \mathsf{x}_m(\mu_1(p)), \mathsf{x}_m(\mu_2(p)), \ldots, \mathsf{x}_m(\mu_n(p)), \mathcal{H}\mathsf{x}_m(\mu_{n+1}(p))\big) \\
&\quad - \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{x}(\mu_{n+1}(p))\big)\big\|\big\|\mathcal{R}_\eta(t,p)\big\|dp \\
&\le \mathcal{P}^*\big\|\mathsf{g}(\mathsf{x}_m) - \mathsf{g}(\mathsf{x})\big\| + \frac{\mathcal{P}^*}{\Gamma(\eta)}\int_0^t (t-p)^{\eta-1} \\
&\quad \times \big\|\mathsf{f}\big(p, \mathsf{x}_m(\mu_1(p)), \mathsf{x}_m(\mu_2(p)), \ldots, \mathsf{x}_m(\mu_n(p)), \mathcal{H}\mathsf{x}_m(\mu_{n+1}(p))\big) \\
&\quad - \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{x}(\mu_{n+1}(p))\big)\big\|dp \\
&\le \mathcal{P}^*\big\|\mathsf{g}(\mathsf{x}_m) - \mathsf{g}(\mathsf{x})\big\| + \frac{\mathcal{P}^* T^\eta}{\Gamma(\eta+1)}\frac{\Gamma(\eta+1)}{\mathcal{P}^* T^\eta}\frac{\epsilon}{2} \\
&\le \mathcal{P}^*\big\|\mathsf{g}(\mathsf{x}_m) - \mathsf{g}(\mathsf{x})\big\| + \frac{\epsilon}{2}.
\end{aligned}$$

Since g is continuous, then for any $\epsilon > 0$ there exists a $m_0 \in \mathbb{N}$ such that

$$\big\|\mathsf{g}(\mathsf{x}_m) - \mathsf{g}(\mathsf{x})\big\| < \frac{\epsilon}{2\mathcal{P}^*}, \quad \forall\ m > m_0.$$

Now, late us take $m^* = \max\{m_0, n_0\}$. Then, for any $m > m^*$, we have $\big\|\Phi\mathsf{x}_m - \Phi\mathsf{x}\big\| < \epsilon$. Consequently, $\Phi : C\,(J, X) \to C\,(J, X)$ is continuous.

Step 2: Let us demonstrate that the operator $\Phi : C(J,X) \to C(J,X)$ is completely continuous. For this, first we show that the set $\mathcal{K} = \{\Phi \mathsf{x}(t) : \mathsf{x}(\cdot) \in \mathcal{B}_{\mathsf{r}}\}$ is relatively compact. Clearly, the set $\mathcal{K}$ is compact for $t = 0$. Now, for $t \in (0, T]$, we choose ζ such that $0 < \zeta < t,$ and $\mathsf{x} \in \mathcal{B}_{\mathsf{r}}$, we define the operator Φ_ζ as follows:

$$\begin{aligned}\Phi_\zeta \mathsf{x}(t) &= \mathcal{R}_\eta(t,0)\big(\mathsf{x}_0 - \mathsf{g}(\mathsf{x})\big) + \frac{\mathcal{R}_\eta(t,\zeta)}{\Gamma(\eta)} \int_0^{t-\zeta} (t-p)^{\eta-1} \mathcal{R}_\eta(\zeta,p) \\ &\quad \times \big[\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{x}(\mu_{n+1}(p))\big)\big]\, dp \\ &= \mathcal{R}_\eta(t,0)\big(\mathsf{x}_0 - \mathsf{g}(\mathsf{x})\big) + \frac{\mathcal{R}_\eta(t,\zeta)}{\Gamma(\eta)} \mathcal{Q}(t,\zeta),\end{aligned}$$

where

$$\begin{aligned}\mathcal{Q}(t,\zeta) &= \int_0^{t-\zeta} (t-p)^{\eta-1} \mathcal{R}_\eta(\zeta,p) \\ &\quad \times \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{x}(\mu_{n+1}(p))\big)\, dp.\end{aligned}$$

Clearly, $\mathcal{Q}(t,\zeta)$ is bounded in $\mathcal{B}_{\mathsf{r}}$ since $\mathcal{R}_\eta(t,\zeta)$ is compact and g is bounded in the set $\mathcal{B}_{\mathsf{r}}$. Therefore, the set $\mathcal{B}_{\mathsf{r}}^* = \{\mathcal{Q}(t,\zeta) : \mathsf{x}(\cdot) \in \mathcal{B}_{\mathsf{r}}\}$ is relatively compact. Moreover, for each $\mathsf{x} \in \mathcal{B}_{\mathsf{r}}$

$$\begin{aligned}\big\|\Phi \mathsf{x} - \Phi_\zeta \mathsf{x}\big\| &\le \frac{1}{\Gamma(\eta)} \int_{t-\zeta}^{t} (t-p)^{\eta-1} \big\|\mathcal{R}_\eta(t,p)\big\| \\ &\quad \times \big\|\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{x}(\mu_{n+1}(p))\big)\big\| dp \\ &\le \frac{\mathcal{P}^*}{\Gamma(\eta)} \int_{t-\zeta}^{t} (t-p)^{\eta-1} \sum_{i=1}^{n} \mathsf{k}_i(p)\|\mathsf{x}_i\|^\gamma + \mathsf{k}_{n+1}(p)\|\mathcal{H}\mathsf{x}_{n+1}\|^\gamma dp \\ &\le \frac{\mathcal{P}^*}{\Gamma(\eta)} \int_{t-\zeta}^{t} (t-p)^{\eta-1}\ \mathsf{r}^\gamma \left(\sum_{i=1}^{n} \mathsf{k}_i(p) + T\mathsf{k}_{n+1}(p)\right) dp \\ &\le \frac{\mathcal{P}^*}{\Gamma(\eta)} \left(\frac{1-\beta}{\eta-\beta}\right)^{1-\beta} \zeta^{\eta-\beta} \mathsf{r}^\gamma \left(\sum_{i=1}^{n} \|\mathsf{k}_i\|_{L^{\frac{1}{\beta}}} + T\|k_{n+1}\|_{L^{\frac{1}{\beta}}}\right) \\ &\to 0 \ \text{ as } \ \zeta \to 0 \ \text{ since } \ \eta - \beta > 0.\end{aligned}$$

For $0 \leq t \leq T$, the set $\mathcal{K} = \{\Phi \mathsf{x}(t) : \mathsf{x}(\cdot) \in \mathcal{B}_{\mathsf{r}}\}$ is arbitrarily close to a relatively compact set $\mathcal{B}_{\mathsf{r}}^* = \{\mathcal{Q}(t,\zeta) : \mathsf{x}(\cdot) \in \mathcal{B}_{\mathsf{r}}\}$. Therefore, for $t \in J$, the set $\mathcal{K} = \{\Phi \mathsf{x}(t) : \mathsf{x}(\cdot) \in \mathcal{B}_{\mathsf{r}}\}$ is relatively compact.

Step 3: Finally, we show that the set $\mathcal{K} = \{\Phi \mathsf{x}(t) : \mathsf{x}(\cdot) \in \mathcal{B}_{\mathsf{r}}\}$ is a family of equicontinuous functions on J. To prove this, let $t_1, t_2 \in J$ such that $t_1 < t_2$. Then, we get

$$
\begin{aligned}
&\big\|\Phi\mathsf{x}(t_2) - \Phi\mathsf{x}(t_1)\big\| \\
&\quad \leq \big\|\mathcal{R}_\eta(t_2,0)\big(\mathsf{x}_0 - \mathsf{g}(\mathsf{x}(t_2))\big) - \mathcal{R}_\eta(t_1,0)\big(\mathsf{x}_0 - \mathsf{g}(\mathsf{x}(t_1))\big)\big\| \\
&\qquad + \bigg\|\frac{1}{\Gamma(\eta)}\int_0^{t_2}(t_2-p)^{\eta-1}\mathcal{R}_\eta(t_2,p) \\
&\qquad \times \Big[\mathsf{f}\big(p,\mathsf{x}(\mu_1(p)),\mathsf{x}(\mu_2(p)),\ldots,\mathsf{x}(\mu_n(p)),\mathcal{H}\mathsf{x}(\mu_{n+1}(p))\big)\Big]dp \\
&\qquad - \frac{1}{\Gamma(\eta)}\int_0^{t_1}(t_1-p)^{\eta-1}\mathcal{R}_\eta(t_1,p) \\
&\qquad \times \Big[\mathsf{f}\big(p,\mathsf{x}(\mu_1(p)),\mathsf{x}(\mu_2(p)),\ldots,\mathsf{x}(\mu_n(p)),\mathcal{H}\mathsf{x}(\mu_{n+1}(p))\big)\Big]dp\bigg\| \\
&\quad \leq \big\|\mathcal{R}_\eta(t_2,0)\big(\mathsf{x}_0 - \mathsf{g}(\mathsf{x}(t_2))\big) - \mathcal{R}_\eta(t_1,0)\big(\mathsf{x}_0 - \mathsf{g}(\mathsf{x}(t_1))\big)\big\| \\
&\qquad + \bigg\|\frac{1}{\Gamma(\eta)}\int_0^{t_1}\Big[(t_2-p)^{\eta-1}\mathcal{R}_\eta(t_2,p) - (t_1-p)^{\eta-1}\mathcal{R}_\eta(t_1,p)\Big] \\
&\qquad \times \Big[\mathsf{f}\big(p,\mathsf{x}(\mu_1(p)),\mathsf{x}(\mu_2(p)),\ldots,\mathsf{x}(\mu_n(p)),\mathcal{H}\mathsf{x}(\mu_{n+1}(p))\big)\Big]dp\bigg\| \\
&\qquad + \bigg\|\frac{1}{\Gamma(\eta)}\int_{t_1}^{t_2}(t_2-p)^{\eta-1}\mathcal{R}_\eta(t_2,p) \\
&\qquad \times \Big[\mathsf{f}\big(p,\mathsf{x}(\mu_1(p)),\mathsf{x}(\mu_2(p)),\ldots,\mathsf{x}(\mu_n(p)),\mathcal{H}\mathsf{x}(\mu_{n+1}(p))\big)\Big]dp\bigg\| \\
&\quad = I_1 + I_2 + I_3.
\end{aligned}
$$

Now, by using the hypothesis (H_1), we have $I_1 \to 0$ as $t_1 \to t_2$. Further,

$$
I_2 = \bigg\|\frac{1}{\Gamma(\eta)}\int_0^{t_1}\Big[(t_2-p)^{\eta-1}\mathcal{R}_\eta(t_2,p) - (t_1-p)^{\eta-1}\mathcal{R}_\eta(t_1,p)\Big]
$$

$$\times \left[\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{x}(\mu_{n+1}(p))\big)\right] dp\Bigg\|$$

$$\leq \frac{1}{\Gamma(\eta)} \int_0^{t_1} \Big[(t_2 - p)^{\eta-1}\mathcal{R}_\eta(t_2, p) - (t_1 - p)^{\eta-1}\mathcal{R}_\eta(t_1, p)\Big]$$

$$\times \left(\sum_{i=1}^{n} \mathsf{k}_i(p)\|\mathsf{x}_i\|^\gamma + \mathsf{k}_{n+1}(p)\|\mathcal{H}\mathsf{x}_i\|^\gamma\right) dp$$

$$\leq \frac{\mathcal{P}^*}{\Gamma(\eta)} \left(\frac{1-\beta}{\eta-\beta}\right)^{1-\beta} \mathsf{r}^\gamma \left[t_2^{\eta-\beta} - t_1^{\eta-\beta} - \big(t_2 - t_1\big)^{\eta-\beta}\right] \left(\sum_{i=1}^{n} \|\mathsf{k}_i\|_{L^{\frac{1}{\beta}}} + T\|k_{n+1}\|_{L^{\frac{1}{\beta}}}\right)$$

$$\to 0 \quad \text{as} \quad t_1 \to t_2.$$

Clearly, $I_3 \to 0$ as $t_1 \to t_2$. Consequently, $\big\|\Phi\mathsf{x}(t_2) - \Phi\mathsf{x}(t_1)\big\| \to 0$ as $t_1 \to t_2$. Hence, the set $\mathcal{K}$ is equicontinuous. Therefore, $\Phi : C(J, X) \to C(J, X)$ is completely continuous as an application of the Arzelá-Ascoli theorem.

Step 4: For $0 < \lambda < 1$, let $\mathsf{x} = \lambda\Phi\mathsf{x}$. Then, we have

$$\|\mathsf{x}(t)\| = \big\|\mathcal{R}_\eta(t, 0)\big(\mathsf{x}_0 - \mathsf{g}(\mathsf{x})\big)\big\| + \frac{1}{\Gamma(\eta)} \int_0^t (t-p)^{\eta-1}\big\|\mathcal{R}_\eta(t, p)\big\|$$

$$\times \big\|\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{x}(\mu_{n+1}(p))\big)\big\| dp$$

$$\leq \mathcal{P}^*\big(\|\mathsf{x}_0\| + \mathsf{c}\big) + \frac{\mathcal{P}^*}{\Gamma(\eta)} \left(\frac{1-\beta}{\eta-\beta}\right)^{1-\beta} T^{\eta-\beta}\mathsf{r}^\gamma$$

$$\times \left(\sum_{i=1}^{n} \|\mathsf{k}_i\|_{L^{\frac{1}{\beta}}} + T\|k_{n+1}\|_{L^{\frac{1}{\beta}}}\right) = \bar{M} \text{ say.}$$

Hence, there exists a $\bar{M}_* > \bar{M}$ such that $\|\mathsf{x}\| \neq \bar{M}_*$. Let $\mathcal{K}_* = \{\mathsf{x} \in C(J, X) : \|\mathsf{x}\| \leq \bar{M}_*\}$. Then, there is no point $\mathsf{x} \in \partial\mathcal{K}_*$ such that $\mathsf{x} = \lambda\Phi\mathsf{x}$ for $\lambda \in (0, 1)$. It follows from Lemma 1.3 that Φ has a fixed-point $\mathsf{x} \in \bar{\mathcal{K}}_*$, which is the mild solution of the problem (3.1). □

Theorem 3.4. *Suppose that the following conditions hold:*

(K_1) *The function* $\mathsf{f} : J \times X^{n+1} \to X$ *is continuous and there exist non-negative Lebesgue integrable function* $\mathsf{l}_i(t) \in L^{1/\beta}(J, \mathbb{R}_+)$, $(i = 1, 2, \ldots, n, n+1)$, $0 <$

$\beta < \eta$ *such that*

$$\left\|\mathsf{f}\big(t, \mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n, \mathsf{x}_{n+1}\big) - \mathsf{f}\big(t, \mathsf{y}_1, \mathsf{y}_2, \ldots, \mathsf{y}_n, \mathsf{y}_{n+1}\big)\right\| \\ \leq \sum_{i=1}^{n+1} \mathsf{l}_i(t)\|\mathsf{x}_i - \mathsf{y}_i\|, \ \forall \ \mathsf{x}_i, \mathsf{y}_i \in X.$$

(K_2) *There exists a non-negative Lebesgue integrable function* $\mathsf{l}^*(t) \in L^{1/\beta}(J, \mathbb{R}_+)$ *such that*

$$\|\mathsf{h}(t, p, \mathsf{x}) - \mathsf{h}(t, p, \mathsf{y})\| = \mathsf{l}^*(t)\|\mathsf{x} - \mathsf{y}\| \ \forall \ \mathsf{x}_i, \mathsf{y}_i \in X.$$

(K_3) *There exists a positive constant* $\mathcal{L}$ *such that*

$$\|\mathsf{g}(\mathsf{x}) - \mathsf{g}(\mathsf{y})\| \leq \mathcal{L}\|\mathsf{x} - \mathsf{y}\|, \ \forall \ \mathsf{x}_i, \ \mathsf{y}_i \in X,$$

and let $\mathcal{M} = \sup_{t\in J} \int_0^t \mathsf{l}^*(s)ds$ *such that*

$$\left[\mathcal{P}^*\mathcal{L} + \frac{\mathcal{P}^*}{\Gamma(\eta)}\left(\frac{1-\beta}{\eta-\beta}\right)^{1-\beta} T^{\eta-\beta}\left\{\left(\sum_{i=1}^{n} \|\mathsf{l}_i\|_{L^{\frac{1}{\beta}}}\right) + \mathcal{M}\|\mathsf{l}_{n+1}\|_{L^{\frac{1}{\beta}}}\right\}\right] = \rho < 1.$$

Then, the system (3.1) *has a unique mild solution.*

Proof. Let Φ be the function defined in Theorem 3.3. Then, for all $\mathsf{x}, \mathsf{y} \in X$ and $t \in J$, we obtain:

$$\begin{aligned}\left\|\Phi\mathsf{x} - \Phi\mathsf{y}\right\| &\leq \left\|\mathcal{R}_\eta(t, 0)\big(\mathsf{x}_0 - \mathsf{g}(\mathsf{x}) - \mathcal{R}_\eta(t, 0)\big(\mathsf{x}_0 - \mathsf{g}(\mathsf{y})\right\| \\ &\quad + \frac{1}{\Gamma(\eta)} \int_0^t (t-p)^{\eta-1}\|\mathcal{R}_\eta(t, p)\| \\ &\quad \times \left\|\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{x}(\mu_{n+1}(p))\big)\right.\end{aligned}$$

$$
\begin{aligned}
&- \mathsf{f}\big(p, \mathsf{y}(\mu_1(p)), \mathsf{y}(\mu_2(p)), \ldots, \mathsf{y}(\mu_n(p)), \mathcal{H}\mathsf{y}(\mu_{n+1}(p))\big)\big\| dp \\
\leq\ & \mathcal{P}^*\|\mathsf{g}(\mathsf{x}) - \mathsf{g}(\mathsf{y})\| + \frac{\mathcal{P}^*}{\Gamma(\eta)} \int_0^t (t-p)^{\eta-1} \sum_{i=1}^{n} \mathsf{l}_i(p)\|\mathsf{x}_i - \mathsf{y}_i\| \\
&+ \mathsf{l}_{n+1}(p)\big\|\mathcal{H}\mathsf{x}(\mu_{n+1}(p)) - \mathcal{H}\mathsf{y}(\mu_{n+1}(p))dp\big\| \\
\leq\ & \mathcal{P}^*\mathcal{L}\|\mathsf{x} - \mathsf{y}\| + \frac{\mathcal{P}^*}{\Gamma(\eta)}\left(\frac{1-\beta}{\eta-\beta}\right)^{1-\beta} T^{\eta-\beta}\Bigg[\bigg(\sum_{i=1}^{n} \|\mathsf{l}_i\|_{L^{\frac{1}{\beta}}}\bigg)\|\mathsf{x}_i - \mathsf{y}_i\| \\
&+ \|\mathsf{l}_{n+1}\|_{L^{\frac{1}{\beta}}}\big\|\mathcal{H}\mathsf{x}(\mu_{n+1}(p)) - \mathcal{H}\mathsf{y}(\mu_{n+1}(p))\big\|\Bigg]
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\big\|\Phi\mathsf{x} - \Phi\mathsf{y}\big\| \leq\ & \mathcal{P}^*\mathcal{L}\|\mathsf{x} - \mathsf{y}\| + \frac{\mathcal{P}^*}{\Gamma(\eta)}\left(\frac{1-\beta}{\eta-\beta}\right)^{1-\beta} T^{\eta-\beta}\Bigg[\bigg(\sum_{i=1}^{n} \|\mathsf{l}_i\|_{L^{\frac{1}{\beta}}}\bigg)\|\mathsf{x}_i - \mathsf{y}_i\| \\
&+ \|\mathsf{l}_{n+1}\|_{L^{\frac{1}{\beta}}}\big\|\mathcal{H}\mathsf{x}(\mu_{n+1}(p)) - \mathcal{H}\mathsf{y}(\mu_{n+1}(p))\big\|\Bigg]. \qquad (3.5)
\end{aligned}
$$

Again,

$$
\begin{aligned}
&\big\|\mathcal{H}\mathsf{x}(\mu_{n+1}(p)) - \mathcal{H}\mathsf{y}(\mu_{n+1}(p))\big\| \\
&\qquad\leq \int_0^t \|\mathsf{h}\big(t, s, x(\mu_{n+1}(s))\big) - \mathsf{h}\big(t, s, y(\mu_{n+1}(s))\big)\| ds \\
&\qquad\leq \int_0^t \mathsf{l}^*(s)\|x - y\| ds \\
&\qquad\leq \mathcal{M}\|\mathsf{x} - \mathsf{y}\|.
\end{aligned}
$$

This implies

$$
\big\|\mathcal{H}\mathsf{x}(\mu_{n+1}(t)) - \mathcal{H}\mathsf{y}(\mu_{n+1}(t))\big\| \leq \mathcal{M}\|\mathsf{x} - \mathsf{y}\|. \qquad (3.6)
$$

So, by the equations (3.5) and (3.6), we get

$$\begin{aligned}
\|\Phi \mathsf{x} - \Phi \mathsf{y}\| &\leq \mathcal{P}^*\mathcal{L}\|\mathsf{x}-\mathsf{y}\| + \frac{\mathcal{P}^*}{\Gamma(\eta)}\left(\frac{1-\beta}{\eta-\beta}\right)^{1-\beta} T^{\eta-\beta} \\
&\quad \times \left[\left(\sum_{i=1}^{n} \|\mathsf{l}_i\|_{L^{\frac{1}{\beta}}}\right)\|\mathsf{x}-\mathsf{y}\| + \mathcal{M}\|\mathsf{l}_{n+1}\|_{L^{\frac{1}{\beta}}}\|\mathsf{x}-\mathsf{y}\|\right] \\
&\leq \left[\mathcal{P}^*\mathcal{L} + \frac{\mathcal{P}^*}{\Gamma(\eta)}\left(\frac{1-\beta}{\eta-\beta}\right)^{1-\beta} T^{\eta-\beta}\right. \\
&\quad \left. \times \left\{\left(\sum_{i=1}^{n} \|\mathsf{l}_i\|_{L^{\frac{1}{\beta}}}\right) + \mathcal{M}\|\mathsf{l}_{n+1}\|_{L^{\frac{1}{\beta}}}\right\}\right]\|\mathsf{x}-\mathsf{y}\| \\
&\leq \rho\|\mathsf{x}-\mathsf{y}\|
\end{aligned}$$

Therefore, the operator Φ is a contraction mapping. Consequently, Φ has a unique fixed-point by the Banach contraction mapping principle, leading to the conclusion that system (3.1) has a unique mild solution. □

3.4 Controllability Result

We established the existence and uniqueness of mild solutions for the equation (3.1) in the preceding Section 3.3. In this section, we introduce another crucial concept known as controllability. The proof of the controllability results for the control system (3.2) is presented here utilizing the theory of FC, semigroup theory, and the generalized contraction theorem. Before proceeding further, let's define the following definitions:

Definition 3.5. (Mild Solution) [6, 73] A function $\mathsf{x} \in C(J, X)$ is said to be a mild solution of the control system (3.2) for every $\mathsf{u} \in L^2(J, U)$ if it satisfies the following integral equation:

$$\mathsf{x}(t) = \mathcal{R}_\eta(t,0)\big(\mathsf{x}(0) - \mathsf{g}(\mathsf{x})\big) + \frac{1}{\Gamma(\eta)} \int_0^t R_\eta(t,p)(t-p)^{\eta-1}$$
$$\times \Big[\mathcal{G}\mathsf{u}(p) + \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{x}(\mu_{n+1}(p))\big)\Big] dp. \quad (3.7)$$

Definition 3.6. [6, 73] The control system (3.2) is said to be controllable over the interval J if, $\forall\, \mathsf{x}_0, \mathsf{x}_1 \in X$, there exists a control function $u \in L^2(J, U)$ such that the solution $\mathsf{x}(\cdot)$ of (3.2) satisfies $\mathsf{x}(0) + \mathsf{g}(\mathsf{x}) = \mathsf{x}_0$ and $\mathsf{x}(T) = \mathsf{x}_1$.

Theorem 3.7. *Consider the following conditions hold:*

(R_1) *Suppose* $\mathsf{f} : J \times X^{n+1} \to X$ *is continuous with* $\max_{t,\in J} \big\|\mathsf{f}(t, 0, \ldots, 0)\big\| = \mathcal{P}$ *for* $\mathcal{P} > 0$*, and there exist non-negative constants* $\mathcal{P}_i s$ $(i = 1, 2, \ldots, n, n+1)$ *such that*

$$\big\|\mathsf{f}\big(t, \mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n, \mathsf{x}_{n+1}\big) - \mathsf{f}\big(t, \mathsf{y}_1, \mathsf{y}_2, \ldots, \mathsf{y}_n, \mathsf{y}_{n+1}\big)\big\|$$
$$\leq \sum_{i=1}^{n+1} \mathcal{P}_i \big\|\mathsf{x}_i - \mathsf{y}_i\big\|, \ \forall\, \mathsf{x}_i, \mathsf{y}_i \in X.$$

(R_2) *The bounded linear operator* $\mathcal{Z} : L^2(J, U) \to X$ *given by*

$$\mathcal{Z}\mathsf{u} = \frac{1}{\Gamma(\eta)} \int_0^t (t-p)^{\eta-1} \mathcal{R}_\eta(T, p)\ \mathcal{G}\mathsf{u}(p)\ dp,$$

has an induced operator $\tilde{\mathcal{Z}}^{-1}$ *with values in* $L^2(J, U)/KerZ$*. Further, there exist positive constants* $\mathsf{c}_1, \mathsf{c}_2$*, such that* $\|\mathcal{G}\| \leq \mathsf{c}_1$ *and* $\|\tilde{\mathcal{Z}}^{-1}\| \leq \mathsf{c}_2$*.*

(R_3) *For all* $\mathsf{x}_i, \mathsf{y}_i \in X$ $(i = 1, 2, \ldots, n, n+1)$, $\exists\ \mathcal{Q}_1, \mathcal{Q}_2 > 0$ *such that*

$$\big\|\mathsf{h}(t, p, \mathsf{x}) - \mathsf{h}(t, p, \mathsf{y})\big\| \leq \mathcal{Q}_1 \|\mathsf{x} - \mathsf{y}\|, \ \text{and} \ \max_{(t,p) \in \Delta} \big\|\mathsf{h}(t, p, 0)\big\| = \mathcal{Q}_2.$$

(R_4) *The mapping* g *is continuous and there exist positive constants* $\mathsf{m}_1, \mathsf{m}_2 > 0$, *such that*

$$\mathsf{m}_1 = \max_{\mathsf{x} \in C(J,X)} \|\mathsf{g}(\mathsf{x})\|, \ \textit{and} \ \big\|\mathsf{g}(\mathsf{x}) - \mathsf{g}(y)\big\| \le \mathsf{m}_2 \|\mathsf{x} - \mathsf{y}\|.$$

(R_5) *Set* $\mathcal{P}^* = \max_{0 \le p \le t \le T} \big\|\mathcal{R}_\eta(t,p)\big\|$ *and*

$$\mathcal{P}^*\big(\|\mathsf{x}_0\| + \mathsf{m}_1\big) + \gamma \mathcal{P}^* \mathsf{c}_1 \mathsf{c}_2 \big[\|\mathsf{x}_1\| + \mathcal{P}^*\|\mathsf{x}_0\| + \gamma \mathcal{P}^* \mathcal{N}_2\big] + \gamma \mathcal{P}^* \mathcal{N}_2 \le \zeta,$$

for some $\zeta > 0$, *where*

$$\mathcal{N}_2 = \zeta \sum_{i=1}^{n} \mathcal{P}_i + \mathcal{P}_{n+1} T(\mathcal{Q}_1 \zeta + \mathcal{Q}_2) + \mathcal{P}, \quad \gamma = \frac{T^\eta}{\Gamma(\eta+1)}.$$

(R_6) *Set* $\mathsf{q} = \mathcal{P}^* \mathsf{m}_2 + \gamma \mathcal{P}^* \left(\sum_{i=1}^{n} \mathcal{P}_i + \mathcal{P}_{n+1} T \mathcal{Q}_1 \right) \left(\mathsf{c}_1 \mathsf{c}_2 \gamma \mathcal{P}^* + 1 \right)$ *be such that* $0 \le \mathsf{q} < 1$.

Then, the control system (3.2) *is controllable over the interval* J.

Proof. Suppose that $\mathcal{S} = C\,(J, X)$, and $\mathcal{S}_\zeta = \big\{\mathsf{x} \in \mathcal{S} : \ \mathsf{x}(0) + \mathsf{g}(\mathsf{x}) = \mathsf{x}_0, \text{ and } \ \|\mathsf{x}(t)\| \le \zeta \text{ for } t \in J\big\}$.

Let $\Phi : \mathcal{S}_\zeta \to \mathcal{S}_\zeta$ be an operator defined by

$$\begin{aligned}
\Phi \mathsf{x}(t) = {} & \mathcal{R}_\eta(t,0)\big(\mathsf{x}_0 - \mathsf{g}(\mathsf{x})\big) + \frac{1}{\Gamma(\eta)} \int_0^t \mathcal{R}_\eta(t,\eta)(t-\eta)^{\eta-1} \mathcal{G} \tilde{\mathcal{Z}}^{-1} \\
& \times \left[\mathsf{x}_1 - \mathcal{R}_\eta(T,0)\mathsf{x}_0 - \frac{1}{\Gamma(\eta)} \int_0^t \mathcal{R}_\eta(T,p)(t-p)^{\eta-1} \right. \\
& \times \left. \Big(\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{x}(\mu_{n+1}(p))\big) \Big) dp \right] (\eta) d\eta \\
& + \frac{1}{\Gamma(\eta)} \int_0^t \mathcal{R}_\eta(t,p)(t-p)^{\eta-1} \\
& \times \Big(\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{x}(\mu_{n+1}(p))\big) \Big) dp.
\end{aligned}$$

According to the hypothesis $(\mathcal{H}_2)$, for an arbitrary function $\mathsf{x}(\cdot)$ we choose the control

$$\begin{aligned}\mathsf{u}(t) =&\tilde{\mathcal{Z}}^{-1}\bigg[\mathsf{x}_1 - \mathcal{R}_\eta(T,0)\mathsf{x}_0 - \frac{1}{\Gamma(\eta)}\int_0^t \mathcal{R}_\eta(T,p)(t-p)^{\eta-1} \\ &\times \big[\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{x}(\mu_{n+1}(p))\big)\big]dp\bigg].\end{aligned}$$

Using the above control, we prove the operator Φ has a fixed-point. This fixed-point is then a solution of the control system (3.2). Clearly, $\big(\Phi\mathsf{x}\big)(T) = \mathsf{x}_1$, which means that the control u steeps the system (3.2) from the initial state x_0 to x_1 in time T provided we can obtain a fixed-point of the operator Φ. First, we show that the operator Φ maps $\mathcal{S}_\zeta$ into itself.

$$\begin{aligned}\big\|\Phi\mathsf{u}(t)\big\| =& \big\|\mathcal{R}_\eta(t,0)\big(\mathsf{x}(0)-\mathsf{g}(\mathsf{x})\big)\big\| \\ &+ \frac{1}{\Gamma(\eta)}\int_0^t (t-\eta)^{\eta-1}\big\|\mathcal{R}_\eta(t,\eta)\big\|\big\|\mathcal{G}\big\|\big\|\tilde{\mathcal{Z}}^{-1}\big\|\bigg[\big\|\mathsf{x}_1\big\| + \big\|\mathcal{R}_\eta(T,0)\mathsf{x}_0\big\| \\ &+ \frac{1}{\Gamma(\eta)}\int_0^t (t-p)^{\eta-1}\big\|\mathcal{R}_\eta(T,p)\big\| \\ &\times \Big\{\big\|\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{x}(\mu_{n+1}(p))\big) \\ &- \mathsf{f}\big(p,0,0,\ldots,0\big)\big\| + \big\|\mathsf{f}\big(p,0,0,\ldots,0\big)\big\|\Big\}dp\bigg]d\eta \\ &+ \frac{1}{\Gamma(\eta)}\int_0^t (t-p)^{\eta-1}\big\|\mathcal{R}_\eta(t,p)\big\| \\ &\times \Big\{\big\|\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{x}(\mu_{n+1}(p))\big) \\ &- \mathsf{f}\big(p,0,0,\ldots,0\big)\big\| + \big\|\mathsf{f}\big(p,0,0,\ldots,0\big)\big\|\Big\}dp \\ \leq& \mathcal{P}^*\big\|\mathsf{x}_0 - \mathsf{g}(\mathsf{x})\big)\big\| + \frac{T^\eta}{\Gamma(\eta+1)}\mathcal{P}^*\mathsf{c}_1\mathsf{c}_2\bigg[\big\|\mathsf{x}_1\big\| + \mathcal{P}^*\big\|\mathsf{x}_0\big\| \\ &+ \frac{T^\eta}{\Gamma(\eta+1)}\mathcal{P}^*\Big\{\zeta\sum_{i=0}^n \mathcal{P}_i + \mathcal{P}_{n+1}T\big(\mathcal{Q}_1\zeta + \mathcal{Q}_2\big) + \mathcal{P}\Big\}\bigg] \\ &+ \frac{T^\eta}{\Gamma(\eta+1)}\mathcal{P}^*\bigg[\zeta\sum_{i=0}^n \mathcal{P}_i + \mathcal{P}_{n+1}T\big(\mathcal{Q}_1\zeta + \mathcal{Q}_2\big) + \mathcal{P}\bigg]\end{aligned}$$

$$\leq \mathcal{P}^*\big(\|\mathsf{x}_0\| + \mathsf{m}_1\big) + \gamma\mathcal{P}^*\mathsf{c}_1\mathsf{c}_2\Big[\|\mathsf{x}_1\| + \mathcal{P}^*\|\mathsf{x}_0\| + \gamma\mathcal{P}^*\mathcal{N}_2\Big] + \gamma\mathcal{P}^*\mathcal{N}_2$$
$$\leq \zeta.$$

Therefore, Φ maps $\mathcal{S}_\zeta$ to $\mathcal{S}_\zeta$. Next, we show that Φ is a contractive function on $\mathcal{S}_\zeta$. For this, let $\mathsf{x}, \mathsf{y} \in \mathcal{S}_\zeta$, we have,

$$\begin{aligned}
&\big\|\Phi\mathsf{x}(t) - \Phi\mathsf{y}(t)\big\| \\
&= \big\|\mathcal{R}_\eta(t,0)\big(\mathsf{x}(0) - \mathsf{g}(\mathsf{x})\big) - \mathcal{R}_\eta(t,0)\big(\mathsf{x}(0) - \mathsf{g}(\mathsf{y})\big)\big\| \\
&\quad + \frac{1}{\Gamma(\eta)}\int_0^t (t-\eta)^{\eta-1}\big\|\mathcal{R}_\eta(t,\eta)\mathcal{G}\tilde{\mathcal{Z}}^{-1}\big\| \times \frac{1}{\Gamma(\eta)}\int_0^T (T-p)^{\eta-1}\big\|\mathcal{R}_\eta(T,p)\big\| \\
&\quad \times \bigg\|\bigg[\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{x}(\mu_{n+1}(p))\big) \\
&\quad - \mathsf{f}\big(p, \mathsf{y}(\mu_1(p)), \mathsf{y}(\mu_2(p)), \ldots, \mathsf{y}(\mu_n(p)), \mathcal{H}\mathsf{y}(\mu_{n+1}(p))\big)\bigg]\bigg\| dp d\eta \\
&\quad + \frac{1}{\Gamma(\eta)}\int_0^t (t-p)^{\eta-1}\big\|\mathcal{R}_\eta(t,p)\big\| \\
&\quad \times \bigg\|\bigg[\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), \mathcal{H}\mathsf{x}(\mu_{n+1}(p))\big) \\
&\quad - \mathsf{f}\big(p, \mathsf{y}(\mu_1(p)), \mathsf{y}(\mu_2(p)), \ldots, \mathsf{y}(\mu_n(p)), \mathcal{H}\mathsf{y}(\mu_{n+1}(p))\big)\bigg]\bigg\| dp \\
&\leq \mathcal{P}^*\mathsf{m}_2\big\|\mathsf{x} - \mathsf{y}\big\| \\
&\quad + \frac{T^\eta}{\Gamma(\eta+1)}\mathcal{P}^*\mathsf{c}_1\mathsf{c}_2\frac{T^\eta}{\Gamma(\eta+1)}\mathcal{P}^*\bigg[\sum_{i=0}^{n}\mathcal{P}_i + \mathcal{P}_{n+1}T\mathcal{Q}_1\bigg]\big\|\mathsf{x} - \mathsf{y}\big\| \\
&\quad + \frac{T^\eta}{\Gamma(\eta+1)}\mathcal{P}^*\bigg[\sum_{i=1}^{n}\mathcal{P}_i + \mathcal{P}_{n+1}T\mathcal{Q}_1\bigg]\big\|\mathsf{x} - \mathsf{y}\big\| \\
&\leq \bigg[\mathcal{P}^*\mathsf{m}_2 + \gamma\mathcal{P}^*\bigg(\sum_{i=1}^{n}\mathcal{P}_i + \mathcal{P}_{n+1}T\mathcal{Q}_1\bigg)\bigg(\gamma\mathcal{P}^*\mathsf{c}_1\mathsf{c}_2 + 1\bigg)\bigg]\big\|\mathsf{x} - \mathsf{y}\big\| \\
&\leq \mathsf{q}\big\|\mathsf{x} - \mathsf{y}\big\|.
\end{aligned}$$

This implies that

$$\left\|\Phi\mathsf{x}(t)-\Phi\mathsf{y}(t)\right\| \leq \left\|\mathsf{x}-\mathsf{y}\right\|.$$

Therefore, the operator Φ is a contraction mapping on $\mathcal{S}_\varsigma$. By the generalized Banach contraction principle, we conclude that the control system (3.2) has a mild solution that satisfies $\mathsf{x}(T)=\mathsf{x}_1$. Thus, the control system (3.2) is controllable on J.

□

3.5 Applications

In this section, we give two illustrative examples of our proposed results.

Example 3.1. *Let's consider the following fractional differential equations:*

$$\begin{cases} {}_0^C D_t^\eta \mathsf{x}(\upsilon,t) = \mathsf{a}(\upsilon,t)\mathsf{x}_{\upsilon\upsilon}(\upsilon,t) + \dfrac{e^t \sin t}{t}\ \dfrac{e^{x(\upsilon,\mu_1(t))}-e^{-x(\upsilon,\mu_1(t))}}{e^{x(\upsilon,\mu_1(t))}+e^{-x(\upsilon,\mu_1(t))}} \\ \qquad\qquad\qquad + \dfrac{e^t \cos t}{t} \tanh\big(\mathsf{x}(\upsilon,\mu_2(t))\big) \\ \qquad\qquad\qquad + \dfrac{e^t}{t}\displaystyle\int_0^t pt\sin\big(\mathsf{x}(\upsilon,\mu_{n+1}(p))\big)dp, \\ \mathsf{x}\big(\upsilon,0\big) + \mathsf{g}(\mathsf{x}(\upsilon,t)) = \varphi_0(\upsilon), \quad \varphi\in L^2[0,1] \\ \mathsf{x}(0,t)=\mathsf{x}(1,t)=0, \quad (\upsilon,t)\in[0,1]\times[0,1]. \end{cases} \tag{3.8}$$

Consider the Banach space $X=L^2[0,1]$, consisting of square integrable functions. Let $\mathsf{a}:[0,1]\times[0,1]\to[0,1]$ be a continuous function.

Define a linear operator $\mathcal{A}(t)$ by $\mathcal{A}(t)x=\mathsf{a}(\upsilon,t)\mathsf{x}_{\upsilon\upsilon}$ with the domain $D(\mathcal{A}) = H^2(0,1)\cap H_0^1(0,1)$. Then, the family of bounded linear operators $\{\mathcal{A}(t)\}_{t\geq 0}$ generates an evolution system $\mathcal{R}_\eta(t,p)\leq \mathsf{k}$ for $\mathsf{k}>0$ and $p<t$, as discussed in the references

herein [107, 2]. Let $\mathbf{x}(t) = \mathbf{x}(\cdot, t)$. Then, the system (3.8) can be transformed into the following abstract form:

$$\begin{cases} {}_0^C D_t^\eta \mathsf{x}(t) = \mathcal{A}(t)\mathsf{x}(t) + \dfrac{e^t \sin t}{t} \, \dfrac{e^{x(\mu_1(t))} - e^{-x(\mu_1(t))}}{e^{x(\mu_1(t))} + e^{-x(\mu_1(t))}} \\ \qquad\qquad + \dfrac{e^t \cos t}{t} \tanh\big(\mathsf{x}(\mu_2(t))\big) \\ \qquad\qquad + \dfrac{e^t}{t} \displaystyle\int_0^t t \ p \ \sin\big(\mathsf{x}(\mu_{n+1}(p))\big) dp, \ \ t \in [0,1], \\ \mathsf{x}(0) + \mathsf{g}(\mathsf{x}) = \mathsf{x}_0, \quad \mathsf{x}_0 \in X. \end{cases} \tag{3.9}$$

Let $\mathcal{R}_\eta(t,p)$ be a compact resolvent operator for $t, p > 0$, and let g be a continuous function satisfying $\|\mathsf{g}(\mathsf{x})\| \leq \mathsf{c}$ for $\mathsf{c} > 0$. It is evident that $\mathsf{f} : J \times X^{n+1} \to X$ and $\mathsf{h} : \Delta \times X \to X$ are continuous functions, and

$$\big\|\mathsf{h}\big(t, p, \mathsf{x}(\mu_{n+1}(p))\big)\big\| \leq \big\|\mathsf{x}(\mu_{n+1}(t))\big\|^{1/2}.$$

Further, for any $\mathsf{x}_i, \mathsf{y}_i \in X$, $(i = 1, 2, \ldots, n, n+1)$, we get

$$\begin{aligned} &\frac{e^t \sin t}{t} \, \frac{e^{\mathsf{x}(\mu_1(t))} - e^{-\mathsf{x}(\mu_1(t))}}{e^{x(\mu_1(t))} + e^{-\mathsf{x}(\mu_1(t))}} + \frac{e^t \cos t}{t} \tanh\big(\mathsf{x}(\mu_2(t))\big) + \frac{e^t}{t} \int_0^t t \ p \ \sin\big(\mathsf{x}(\mu_{n+1}(p))\big) dp \\ &\quad \leq \frac{e^t \sin t}{t} \big\|\mathsf{x}(\mu_1(t))\big\|^{1/2} + \frac{e^t \cos t}{t} \big\|\mathsf{x}(\mu_2(t))\big\|^{1/2} + \frac{e^t}{t} \big\|\mathsf{x}(\mu_{n+1}(t))\big\|^{1/2}, \ \text{for } \gamma = \frac{1}{2}. \end{aligned}$$

Thus, we have $\mathsf{k}_1 = \dfrac{e^t \sin t}{t}, \mathsf{k}_2 = \dfrac{e^t \cos t}{t}, \mathsf{k}_{n+1} = \dfrac{e^t}{t}$, and $\mathsf{k}_i = 0$, for $i = 3, 4 \ldots, n$. Clearly, $\mathsf{k}_i \in L^{1/\beta}(J, \mathbb{R}_+)$ where $0 < \beta < \eta$.

So, all the hypotheses of Theorem 3.3 are satisfied, and it follows that there exists a mild solution for the system (3.9), and consequently for (3.8).

Example 3.2. *Let's examine the subsequent fractional functional differential equations:*

$$\begin{cases} {}_0^C D_t^\eta \mathsf{x}(t) = \mathcal{A}(t)\mathsf{x}(t) + \mathcal{G}\mathsf{u}(t) + \dfrac{t^2}{1+t^2}\,\mathsf{x}(\mu_1(t)) + \dfrac{\cos t}{100}\mathsf{x}(\mu_2(t)) \\ \qquad\qquad + \dfrac{1}{1+e^{-t}}\displaystyle\int_0^t t\ p\ \mathsf{x}(\mu_{n+1}(p))dp, \quad t \in [0,1], \\ \mathsf{x}(0) + \sin \mathsf{x} = 0. \end{cases} \tag{3.10}$$

Let $X = U = L^2[0,1]$. Suppose the linear operator $\mathcal{Z} : L^2(J,U) \to X$ defined by

$$\mathcal{Z}\mathsf{u} = \frac{1}{\Gamma(\eta)} \int_0^T (t-p)^{\eta-1} \mathcal{R}_\eta(T,p)\ \mathcal{G}\mathsf{u}(p)\ dp,$$

is bounded, and it has an induced operator $\tilde{\mathcal{Z}}^{-1}$ that takes values in $L^2(J,U)/\mathrm{Ker}\mathcal{Z}$ and there exist constants $\mathsf{c}_1, \mathsf{c}_2 > 0$ such that $\|\mathcal{G}\| \leq \mathsf{c}_1,$ and $\|\tilde{\mathcal{Z}}^{-1}\| \leq \mathsf{c}_2$.

If we compare equation (3.10) with the equation (3.2), we get

$$\begin{aligned} &\mathsf{f}\big(t, x(\mu_1(t)), \mathsf{x}(\mu_2(t)), \ldots, \mathsf{x}(\mu_n(t)), \mathcal{H}\mathsf{x}(\mu_{n+1}(t))\big) \\ &\quad = \frac{t^2}{1+t^2}\,\mathsf{x}(\mu_1(t)) + \frac{\cos t}{100}\mathsf{x}(\mu_2(t)) + \frac{1}{1+e^{-t}}\int_0^t t\ p\ \mathsf{x}(\mu_{n+1}(p))dp, \end{aligned}$$

where

$$\mathcal{H}\mathsf{x}(\mu_{n+1}(t)) = \frac{1}{1+e^{-t}}\int_0^t t\ p\ \mathsf{x}(\mu_{n+1}(p))dp.$$

For all $\mathsf{x}, \mathsf{y} \in X$, we get

$$\big\|\mathsf{h}\big(t, p, \mathsf{x}(\mu_{n+1}(t))\big) - \mathsf{h}\big(t, p, \mathsf{y}(\mu_{n+1}(t))\big)\big\| \leq \big\|\mathsf{x}(\mu_{n+1}(t)) - \mathsf{y}(\mu_{n+1}(t))\big\|,$$

and $\max\limits_{(t,p)\in\Delta} \big\|\mathsf{h}(t,p,0)\big\| = 0$. Thus, we have $\mathcal{Q}_1 = 1,$ and $\mathcal{Q}_2 = 0$.

Clearly, f is continuous, and for all $\mathsf{x}_i, \mathsf{y}_i \in X, (i = 1, 2, \ldots, n, n+1),\ t \in J$,we have

$$\begin{aligned}\big\|\mathsf{f}\big(\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_{n+1}\big) - \mathsf{f}\big(\mathsf{y}_1, \mathsf{y}_2, \mathsf{y}_{n+1}\big)\big\| \\ &\leq \big\|\mathsf{x}(\mu_1(t)) - \mathsf{y}(\mu_1(t))\big\| + \frac{1}{100}\big\|\mathsf{x}(\mu_2(t)) - \mathsf{y}(\mu_2(t))\big\| \\ &+ \big\|\mathsf{x}(\mu_{n+1}(t)) - \mathsf{y}(\mu_{n+1}(t))\big\|,\end{aligned}$$

and $\max_{t,\in J}\big\|\mathsf{f}(t, 0, \ldots, 0)\big\| = 0$. So, we obtain $\mathcal{P}_1 = \mathcal{P}_{n+1} = 1, \mathcal{P}_2 = \dfrac{1}{100}$, and $\mathcal{P} = 0$. Let $\mathsf{g}(\mathsf{x}) = \sin\mathsf{x}$, then it is clear that $\mathsf{g}(\mathsf{x})$ is a continuous function from $C\,(J, X)$ to itself with $\max_{\mathsf{x}\in C(J,X)}\|\sin\mathsf{x}\| = \mathsf{m}_1 = 1$, and $\big\|\sin\mathsf{x} - \sin\mathsf{y}\big\| \leq \|\mathsf{x} - \mathsf{y}\|$, this implies that $\mathsf{m}_2 = 1$.

Let for some $\zeta > 0$

$$\mathcal{P}^* + \frac{\mathcal{P}^*\mathsf{c}_1\mathsf{c}_2}{\Gamma(\eta+1)}\left[\|\mathsf{x}_1\| + \frac{201\mathcal{P}^*\zeta}{100\Gamma(\eta+1)}\right] + \frac{201\mathcal{P}^*\zeta}{100\Gamma(\eta+1)} \leq \zeta,$$

and

$$\mathsf{q} = \mathcal{P}^* + \frac{201\mathcal{P}^*}{100\Gamma(\eta+1)}\left(\frac{\mathsf{c}_1\mathsf{c}_2\gamma\mathcal{P}^*}{100\Gamma(\eta+1)} + 1\right) \text{ for some } \quad 0 \leq \mathsf{q} < 1.$$

Thus, all the conditions of Theorem 3.7 are satisfies; consequently, the system (3.10) is controllable.

3.6 Conclusion

In this chapter, we investigate a class of nonautonomous fractional integro-differential equations with delay. The existence and uniqueness of mild solutions to the fractional integro-differential equations (3.1) are established using the theory of resolvent operators and fixed-point theorems under the various set of assumptions on f. To achieve the controllability of the control system (3.2), the theory of resolvent operators and the Banach fixed-point theorem are employed. Finally, two instances are given to show the efficacy of the proposed outcomes.

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