Chapter 1

Introduction

Integral transform played an important role to solve the problems of partial differential equations, wavelets, signal processing, physics, electronics, mathematical sciences and other areas of engineering. With the help of integral transform and its inversion formula, many applications are possible in mathematical sciences and other areas. Solutions of partial-differential equations can be obtained in the form of integral representation by exploiting integral transform theory. In each integral transform, there are specific types of kernel, which played the leading role to know the nature of the solution of a given problem.

Fourier transform is an integral representation of the absolutely integrable function and complex exponential type kernel. The Hankel transform which integral representation is the product of absolutely integrable function and the Bessel function of the first kind. The aforesaid transform is useful to find the solution of infinite cylindrical boundary value problems. Exploiting the Fourier transform technique the theory of pseudo-differential operators was developed and obtained many observations by many mathematicians.

Pseudo-differential operators are the generalization of partial differential operators and it played a significant role in the problems of partial differential equations, numerical analysis and quantum physics. In 1965, Kohn and Nirenberg [32] developed the calculus of pseudo-differential operators and its various properties. Hörmander [26] extended the theory of pseudo-differential operators in elliptic partial-differential operators. Later on, Grushin [22], Zaidman [82], Fefferman [17], Illner [30], Kato [31], Nagase [43], Treves [72], Kumano-go [34], Taylor [70], Wong [81] and others made a proper foundation of pseudo-differential operators and studied various properties by using the Fourier transform technique.

For the Hankel transform concern, many authors studied various important properties of pseudo-differential operators associated with a certain class of symbols involving the Bessel operator. In 1995, the pseudo-differential operators associated with the Bessel operator were developed by Pathak and Pandey [52] and found its various properties. Later on, Pathak and Pandey [53] defined Sobolev-type spaces associated with the Bessel operator and studied many important properties of the Hankel potential operators. In the same year, Pathak and Upadhyay [56] investigated product and commutators for the pseudo-differential operators associated with a homogeneous symbol class by taking the Hankel transform technique. Further, Pathak and Pandey [54] defined the minimal and maximal pseudo-differential operators associated with the Bessel operator and studied its various properties. In 2001, Pathak and Upadhyay [57] proved the L^p_μ-boundedness of pseudo-differential operators involving the Bessel operator and applied this theory in Sobolev-type spaces. In 2006, Upadhyay [78] derived the L^p-norm inequality of the product of pseudo-differential operators involving the Hankel transform.

The Weinstein transform is the generalization of Fourier and Hankel transforms which kernel is the product of complex exponential function and normalized Bessel function of the first kind. The Weinstein transform has a rich calculus and is applicable in many areas of mathematical sciences. Many authors exploited the theory of the Weinstein transform and found many interesting results. Othmani and Trimeche [49] established real Paley-Wiener Theorems for the Weinstein transform associated with the Weinstein operator, Mohammed and Ghribi [42] discussed the properties of the Sobolev space of exponential type, Mejjaoli and Salem [40] found new results on the continuous Weinstein wavelet transform by taking Weinstein transform technique, Salem [64] established Hardy-Littlewood-Sobolev-type inequalities associated with the Weinstein operator, Saoudi and Nefzi [68] considered the boundedness and compactness of localization operators associated with the Weinstein-Winger transform and found many important observations.

Therefore, our main objective in this thesis is to study various properties of pseudo-differential operators associated with the Weinstein transform.

1.1 The Fourier Transform

From [62, 81], definitions and properties of the Fourier transform are discussed.

Definition 1.1.1. The Fourier transform of $\phi \in L^1(\mathbb{R}^n)$ is defined by

$$(\mathcal{F}\phi)(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}\phi(x)dx, \qquad \xi \in \mathbb{R}^n, \tag{1.1.1}$$

where $\langle x, \xi\rangle = x_1\xi_1 + \cdots + x_n\xi_n$.

If $\phi \in L^1(\mathbb{R}^n)$ and $\mathcal{F}\phi \in L^1(\mathbb{R}^n)$, then the inversion formula of the Fourier transform is defined by

$$\phi(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x', \xi' \rangle} (\mathcal{F}\phi)(\xi) d\xi, \quad a.e. \;\; \xi \in \mathbb{R}^n. \tag{1.1.2}$$

Definition 1.1.2. Let $\phi \in L^1(\mathbb{R}^n)$ and $\psi \in L^1(\mathbb{R}^n)$. Then the convolution of ϕ, ψ is given by

$$(\phi * \psi)(x) = \int_{\mathbb{R}^n} \phi(x - y)\psi(y) dy, \qquad x \in \mathbb{R}^n. \tag{1.1.3}$$

Properties of the Fourier transform

1. If $\phi \in L^1(\mathbb{R}^n)$, then the function $\mathcal{F}$ is continuous on $\mathbb{R}^n$ and

$$\|\mathcal{F}\phi\|_{L^\infty(\mathbb{R}^n)} \leq \|\phi\|_{L^1(\mathbb{R}^n)}.$$

2. **Riemann-Lebesgue Lemma:** Let $\phi \in L^1(\mathbb{R}^n)$. Then

   (i) $\mathcal{F}\phi$ is a continuous function on $\mathbb{R}^n$,

   (ii) $lim_{|\xi| \to \infty} (\mathcal{F}\phi)(\xi) = 0$,

   (iii) $\phi_k \to \phi$ in $L^1(\mathbb{R}^n)$ implies $\mathcal{F}\phi_k \to \mathcal{F}\phi$ uniformly on $\mathbb{R}^n$.

3. Let $\phi, \psi \in L^1(\mathbb{R}^n)$, then $\phi * \psi \in L^1(\mathbb{R}^n)$ and we have

$$\mathcal{F}(\phi * \psi) = (2\pi)^{-\frac{n}{2}} \mathcal{F}(\phi)\mathcal{F}(\psi). \tag{1.1.4}$$

4. Let $p, q, r \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. Then for all $\phi \in L^p(\mathbb{R}^n)$ and $\psi \in L^q(\mathbb{R}^n)$, the function $\phi * \psi$ belongs to $L^r(\mathbb{R}^n)$ and

$$\|\phi *_w \psi\|_{L^r(\mathbb{R}^n)} \leq \|\phi\|_{L^p(\mathbb{R}^n)} \|\psi\|_{L^q(\mathbb{R}^n)}. \tag{1.1.5}$$

1.2 The Hankel Transform

From Haimo [23], Pathak and Dixit [51], definitions and properties of the Hankel transform are discussed.

For fixed $\mu > 0$ we define

$$d\sigma(x) = \frac{x^{2\mu}}{2^{\mu-1/2}\Gamma(\mu+1/2)}dx,$$

and

$$j(x) = 2^{\mu-1/2}\Gamma(\mu+1/2)x^{1/2-\mu}J_{\mu-1/2}(x), \tag{1.2.1}$$

where $J_{\mu-1/2}(x)$ is the Bessel function of order $\mu - 1/2$.

We denote by $L^p_\sigma(0,\infty)$, the space of all real-valued measurable functions ϕ defined on $(0,\infty)$ with norm

$$\|\phi\|_{L^p_\sigma(0,\infty)} = \left(\int_0^\infty |\phi(x)|^p d\sigma(x)\right)^{1/p}, \quad 1 \leq p < \infty \tag{1.2.2}$$

$$\|\phi\|_{L^\infty_\sigma(0,\infty)} = esssup_{x\in(0,\infty)}|\phi(x)|. \tag{1.2.3}$$

Definition 1.2.1. The Hankel transform of $\phi \in L^1_\sigma(0,\infty)$ is defined by

$$(h_\mu\phi)(\xi) = \int_0^\infty j(x\xi)\phi(x)d\sigma(x), \quad \xi \in (0,\infty) \tag{1.2.4}$$

where the Bessel function j is defined in (1.2.1).

If $\phi \in L^1_\sigma(0,\infty)$ and $h_\mu\phi \in L^1_\sigma(0,\infty)$, then the inversion formula of the Hankel transform is

$$\phi(x) = \int_0^\infty j(x\xi)(h_\mu\phi)(\xi)d\sigma(\xi). \tag{1.2.5}$$

Let $\Delta(x,y,z)$ be the area of triangle with sides x,y,z if such triangle exists. Set

$$D(x,y,z)=\frac{2^{3\mu-5/2}\big[\Gamma(\mu+1/2)\big]^2}{\Gamma(\mu)\pi^{1/2}}(xyz)^{-2\mu+1}\big[\Delta(x,y,z)\big]^{2\mu-2}, \tag{1.2.6}$$

if Δ exists and zero otherwise.

In form of integral representation of the basic function is given by

$$D(x,y,z)=\int_0^\infty j(xt)j(yt)j(zt)d\sigma(t). \tag{1.2.7}$$

Therefore, we have

$$\int_0^\infty j(zt)D(x,y,z)d\sigma(z)=j(xt)j(yt), \tag{1.2.8}$$

for $D(x,y,z)\geq 0$ and $D(x,y,z)$ is symmetric in x,y,z.

Setting $t=0$ in (1.2.8), then we obtain

$$\int_0^\infty D(x,y,z)d\sigma(z)=1. \tag{1.2.9}$$

Definition 1.2.2. Let $\phi\in L^1_\sigma(0,\infty)$. Then translation of $\phi(x)$ is defined by

$$(\tau_x\phi)(y)=\int_0^\infty \phi(z)D(x,y,z)d\sigma(z),\quad 0<x,y<\infty. \tag{1.2.10}$$

Definition 1.2.3. Let $\phi,\psi\in L^1_\sigma(0,\infty)$. Then the Hankel convolution of ϕ,ψ is

$$(\phi\#\psi)(x)=\int_0^\infty (\tau_x\phi)(y)\psi(y)d\sigma(y),\quad 0<x<\infty. \tag{1.2.11}$$

Properties of the Hankel transform

1. Let $\phi, \psi \in L^1_\sigma(0,\infty)$. Then

$$h_\mu(\phi\#\psi)(x) = (h_\mu\phi)(x)(h_\mu\psi)(x), \quad 0 < x < \infty. \tag{1.2.12}$$

2. Let $p, q, r \in [1,\infty]$ such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. Then for all $\phi \in L^p_\sigma(0,\infty)$ and $\psi \in L^q_\sigma(0,\infty)$, the function $\phi\#\psi$ belongs to $L^r_\sigma(0,\infty)$ and

$$\|\phi\#\psi\|_{L^r_\sigma(0,\infty)} \leq \|\phi\|_{L^p_\sigma(0,\infty)}\|\psi\|_{L^q_\sigma(0,\infty)}. \tag{1.2.13}$$

1.3 The Schwartz Space and Weinstein Operator

In this section, the definition of Schwartz space and other related results are discussed by using the theory of Weinstein transform. Later on, from [10, 42, 46, 68, 73], definitions and the properties of the Weinstein operator are given.

Definition 1.3.1. By $S_*(\mathbb{R}^{n+1}_+)$, we denote the set of all rapidly decreasing functions $\phi \in C^\infty_*(\mathbb{R}^{n+1}_+)$ such that

$$\sup_{x\in\mathbb{R}^{n+1}_+} \Big|(1 + \|x\|^2)^q D^\beta_x \phi(x)\Big| < \infty, \quad \forall\, q \in \mathbb{N}_0, \beta \in \mathbb{N}^{n+1}_0.$$

$S_*(\mathbb{R}^{n+1}_+)$ is called the Schwartz space and it forms a vector space over the field of complex numbers.

The topology on $S_*(\mathbb{R}^{n+1}_+)$ is defined by the seminorms $\tau_{q,l}$, $(q,l) \in \mathbb{N}\times\mathbb{N}$, given by

$$\tau_{q,l}(\phi) = \sup_{\substack{|\beta|\leq l \\ x\in\mathbb{R}^{n+1}_+}} \Big|(1 + \|x\|^2)^q D^\beta_x \phi(x)\Big|. \tag{1.3.1}$$

The $L^p_\alpha(\mathbb{R}^{n+1}_+)$ - norm of a Lebesgue measurable function ϕ is defined by

$$\|\phi\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)} = \left(\int_{\mathbb{R}^{n+1}_+} |\phi(x)|^p d\mu_\alpha(x)\right)^{1/p} < \infty, \quad 1 \leq p < \infty \tag{1.3.2}$$

$$\|\phi\|_{L^\infty_\alpha(\mathbb{R}^{n+1}_+)} = esssup_{x\in\mathbb{R}^{n+1}_+}|\phi(x)| < \infty, \tag{1.3.3}$$

and

$$d\mu_\alpha(x) = A_\alpha x_{n+1}^{2\alpha+1}dx, \tag{1.3.4}$$

where

$$A_\alpha = \frac{1}{(2\pi)^{\frac{n}{2}}2^\alpha\Gamma(\alpha+1)}, \tag{1.3.5}$$

and dx is the Lebesgue measure on $\mathbb{R}^{n+1}_+$.

Lemma 1.3.2. *The Schwartz space $S_*(\mathbb{R}^{n+1}_+)$ is a subspace of $L^p_\alpha(\mathbb{R}^{n+1}_+)$, for all $p \in [1,\infty]$ and $\alpha > -\frac{1}{2}$.*

Theorem 1.3.3. *Let $p \in [1,\infty)$ and $\alpha > -\frac{1}{2}$. Then the Schwartz space $S_*(\mathbb{R}^{n+1}_+)$ is dense in $L^p_\alpha(\mathbb{R}^{n+1}_+)$.*

The Weinstein operator on $\mathbb{R}^{n+1}_+$ for $(n+1)$ variables is defined by

$$\Delta_{\alpha,n,x} = \sum_{j=1}^{n+1}\frac{\partial^2}{\partial x_j^2} + \frac{2\alpha+1}{x_{n+1}}\frac{\partial}{\partial x_{n+1}} = \Delta_n + L_\alpha, \quad \alpha > -\frac{1}{2}, \tag{1.3.6}$$

where

$$\Delta_n = \sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}, \tag{1.3.7}$$

is the Laplacian operator on $\mathbb{R}^n$ of the first n variables and

$$L_\alpha = \frac{\partial^2}{\partial x_{n+1}^2} + \frac{2\alpha+1}{x_{n+1}}\frac{\partial}{\partial x_{n+1}}, \tag{1.3.8}$$

is the Bessel operator on $(0,\infty)$ for the last variable.

For all $x \in \mathbb{C}_{+}^{n+1}$, the system

$$\begin{cases} \frac{\partial^2 u}{\partial x_k^2}(x) = -\xi_k^2 u(x) \quad \text{for} \quad 1 \leq k \leq n \\ L_\alpha u(x) = -\xi_{n+1}^2 u(x), \quad \alpha > -1/2 \\ u(0) = 1, \ \frac{\partial u}{\partial x_{n+1}}(0) = 0, \ \frac{\partial u}{\partial x_k}(0) = -i\xi_k \quad \text{for} \quad 1 \leq k \leq n \end{cases} \tag{1.3.9}$$

has a unique solution which is denoted by $W_\alpha(x,\xi)$ and

$$W_\alpha(x,\xi) = e^{-i\langle x^{'},\xi^{'}\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1}), \quad \forall \xi \in \mathbb{C}_{+}^{n+1} \tag{1.3.10}$$

where $x = (x^{'}, x_{n+1})$, $\xi = (\xi^{'}, \xi_{n+1})$ and $\hat{J}_\alpha$ the normalized Bessel function of index α is

$$\hat{J}_\alpha(x) = \Gamma(\alpha+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \ \Gamma(\alpha+k+1)} \Big(\frac{x}{2}\Big)^{2k}, \quad \forall x \in \mathbb{C}.$$

The system (1.3.9) is equivalent to the system

$$\begin{cases} \Delta_{\alpha,n,x} u(x) = \big(\Delta_n + L_\alpha\big) u(x) = -\|\xi\|^2 u(x), \quad \alpha > -1/2 \\ u(0) = 1, \ \frac{\partial u}{\partial x_{n+1}}(0) = 0, \ \frac{\partial u}{\partial x_k}(0) = -i\xi_k \quad \text{for} \quad 1 \leq k \leq n \end{cases}$$

The integral representation of $\hat{J}_\alpha$ is

$$\hat{J}_\alpha(x\xi) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} \ \cos(x\xi t) dt, \quad \alpha > -\frac{1}{2}. \tag{1.3.11}$$

Using the Euler's formula and properties of the definite integral (1.3.11) can be written as

$$\hat{J}_\alpha(x\xi) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_{-1}^1 (1-t^2)^{\alpha-\frac{1}{2}} \ e^{ix\xi t} dt, \quad \alpha > -\frac{1}{2}. \tag{1.3.12}$$

Properties of $\hat{J}_\alpha(x\xi)$

1. $\forall\ x \in \mathbb{R}, \xi \in \mathbb{C}, \quad \left|\hat{J}_\alpha(x\xi)\right| \leq e^{|Im(\xi)||x|}$.
2. $\forall\ x \in \mathbb{R}, \xi \in \mathbb{C}$ *and* $\nu \in \mathbb{N}, \quad \left|\frac{d^\nu}{d\xi^\nu}\hat{J}_\alpha(x\xi)\right| \leq |x|^\nu e^{|Im(\xi)||x|}$.
3. $\forall\ x \in \mathbb{R}, \quad \left|\hat{J}_\alpha(x\xi)\right| \leq 1$, if and only if $\xi \in \mathbb{R}$.

Proposition 1.3.4. *For all* $(x, \xi) \in \mathbb{C}_+^{n+1} \times \mathbb{C}_+^{n+1}$, *we have*

1. $W_\alpha(\xi, x) = W_\alpha(x, \xi) = e^{-i\langle x', \xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})$.
2. $W_\alpha(-x, \xi) = W_\alpha(x, -\xi) = e^{i\langle x', \xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})$.
3. $W_\alpha(0, \xi) = 1$.
4. $\forall\ \nu \in \mathbb{N}_0^{n+1}, \left|D_\xi^\nu W_\alpha(x, \xi)\right| \leq \|x\|^{|\nu|}\ e^{\|x\|\mathrm{Im}(\|\xi\|)}$.
5. $\forall\ (x, \xi) \in \mathbb{R}_+^{n+1} \times \mathbb{R}_+^{n+1}, \left|W_\alpha(x, \xi)\right| \leq 1$.

1.4 The Weinstein Transform

In this section, from [10, 42, 68, 76, 77], definitions and properties of the Weinstein transform are discussed. Later on, definition and various properties of the Weinstein convolution are given.

Definition 1.4.1. The Weinstein transform of $\phi \in L_\alpha^1(\mathbb{R}_+^{n+1})$ is defined by

$$(\mathcal{F}_\alpha\phi)(\xi) = \int_{\mathbb{R}_+^{n+1}} e^{-i\langle x', \xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\phi(x)d\mu_\alpha(x), \qquad \xi \in \mathbb{R}_+^{n+1} \tag{1.4.1}$$

where $d\mu_\alpha(x)$ is the measure on $\mathbb{R}_+^{n+1}$ given by (1.3.4).

Let $\phi \in L^1_\alpha(\mathbb{R}^{n+1}_+)$ such that $\mathcal{F}_\alpha\phi \in L^1_\alpha(\mathbb{R}^{n+1}_+)$, then inversion formula of the Weinstein transform is

$$\phi(x) = \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})(\mathcal{F}_\alpha\phi)(\xi)d\mu_\alpha(\xi), \quad a.e. \;\; \xi \in \mathbb{R}^{n+1}_+. \tag{1.4.2}$$

Properties of the Weinstein transform

Proposition 1.4.2. *1. If $\phi \in L^1_\alpha(\mathbb{R}^{n+1}_+)$, then the function $\mathcal{F}_\alpha\phi$ is continuous on $\mathbb{R}^{n+1}_+$ and*

$$\|\mathcal{F}_\alpha\phi\|_{L^\infty_\alpha(\mathbb{R}^{n+1}_+)} \leq \|\phi\|_{L^1_\alpha(\mathbb{R}^{n+1}_+)}.$$

2. Let $\phi \in S_*(\mathbb{R}^{n+1}_+)$ and $k \in \mathbb{N}_0$, then

$$[\mathcal{F}_\alpha(\Delta^k_{\alpha,n,x}\phi)](\xi) = (-1)^k\|\xi\|^{2k}(\mathcal{F}_\alpha\phi)(\xi), \tag{1.4.3}$$

and

$$[\Delta^k_{\alpha,n,\xi}(\mathcal{F}_\alpha\phi)](\xi) = \mathcal{F}_\alpha[(-1)^k\|x\|^{2k}\phi](\xi). \tag{1.4.4}$$

Proposition 1.4.3. *Let ϕ and ψ be in $L^1_\alpha(\mathbb{R}^{n+1}_+)$. Then we have*

$$\int_{\mathbb{R}^{n+1}_+} (\mathcal{F}_\alpha\phi)(\xi)\psi(\xi)d\mu_\alpha(\xi) = \int_{\mathbb{R}^{n+1}_+} \phi(\xi)(\mathcal{F}_\alpha\psi)(\xi)d\mu_\alpha(\xi). \tag{1.4.5}$$

Theorem 1.4.4. *The Weinstein transform $\mathcal{F}_\alpha$ is a topological isomorphism from $S_*(\mathbb{R}^{n+1}_+)$ onto itself and*

$$\mathcal{F}^{-1}_\alpha\phi(\xi) = \mathcal{F}_\alpha\phi(-\xi), \quad \textit{for all} \;\; \xi \in \mathbb{R}^{n+1}_+. \tag{1.4.6}$$

Proposition 1.4.5. *1. If $\phi, \psi \in S_*(\mathbb{R}^{n+1}_+)$, then we have*

$$\int_{\mathbb{R}^{n+1}_+} \phi(x)\overline{\psi(x)}d\mu_\alpha(x) = \int_{\mathbb{R}^{n+1}_+} (\mathcal{F}_\alpha\phi)(\xi)\overline{(\mathcal{F}_\alpha\psi)(\xi)}d\mu_\alpha(\xi). \tag{1.4.7}$$

2. The Weinstein transform $\mathcal{F}_\alpha$ *extends uniquely to an isometric isomorphism on* $L^2_\alpha(\mathbb{R}^{n+1}_+)$*. Now, we have*

$$\int_{\mathbb{R}^{n+1}_+} |\phi(x)|^2 d\mu_\alpha(x) = \int_{\mathbb{R}^{n+1}_+} |(\mathcal{F}_\alpha \phi)(\xi)|^2 d\mu_\alpha(\xi). \tag{1.4.8}$$

Definition 1.4.6. The continuous linear functionals on $S_*(\mathbb{R}^{n+1}_+)$ are called tempered distributions and are denoted by $S'_*(\mathbb{R}^{n+1}_+)$.

If $f \in L^p_\alpha(\mathbb{R}^{n+1}_+)$, the tempered distribution associated with f is defined by

$$\langle f, \phi \rangle = \int_{\mathbb{R}^{n+1}_+} f(x)\phi(x) d\mu_\alpha(x), \quad \phi \in S_*(\mathbb{R}^{n+1}_+). \tag{1.4.9}$$

Definition 1.4.7. The Weinstein transform $\mathcal{F}_\alpha f$ of $f \in S'_*(\mathbb{R}^{n+1}_+)$ is defined by

$$\langle \mathcal{F}_\alpha f, \phi \rangle = \langle f, \mathcal{F}_\alpha \phi \rangle, \quad \phi \in S_*(\mathbb{R}^{n+1}_+). \tag{1.4.10}$$

Theorem 1.4.8. *The Weinstein transform is a topological isomorphism from* $S'_*(\mathbb{R}^{n+1}_+)$ *onto itself.*

Definition 1.4.9. Let $\phi, \psi \in L^1_\alpha(\mathbb{R}^{n+1}_+)$. Then the Weinstein convolution of ϕ, ψ is given by

$$(\phi *_w \psi)(x) = \int_{\mathbb{R}^{n+1}_+} \tau^\alpha_x \phi(y) \psi(y) d\mu_\alpha(y), \tag{1.4.11}$$

where

$$(\tau^\alpha_x \phi)(y) = \int_{\mathbb{R}^{n+1}_+} \mathcal{D}_\alpha(x, y, z) \phi(z) d\mu_\alpha(z), \tag{1.4.12}$$

and

$$\mathcal{D}_\alpha(x,y,z) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',t'\rangle}\hat{J}_\alpha(x_{n+1}t_{n+1})e^{-i\langle y',t'\rangle}\hat{J}_\alpha(y_{n+1}t_{n+1}) \times e^{-i\langle z',t'\rangle}\hat{J}_\alpha(z_{n+1}t_{n+1})d\mu_\alpha(t). \tag{1.4.13}$$

From Chapter 2, $\mathcal{D}_\alpha(x,y,z)$ is well defined.

Setting $t=0$ in (2.3.21), then we obtain

$$\int_{\mathbb{R}_+^{n+1}} \mathcal{D}_\alpha(x,y,z)d\mu_\alpha(z) = 1. \tag{1.4.14}$$

Proposition 1.4.10. *The following properties associated with the Weinstein convolution are given.*

1 . *Let* $\phi \in L^1_\alpha(\mathbb{R}_+^{n+1})$ *and* $\psi \in L^1_\alpha(\mathbb{R}_+^{n+1})$*, then*

$$\|\phi *_w \psi\|_{L^1_\alpha(\mathbb{R}_+^{n+1})} \le \|\phi\|_{L^1_\alpha(\mathbb{R}_+^{n+1})}\|\psi\|_{L^1_\alpha(\mathbb{R}_+^{n+1})}. \tag{1.4.15}$$

2. *Let* $\phi \in L^1_\alpha(\mathbb{R}_+^{n+1})$ *and* $\psi \in L^p_\alpha(\mathbb{R}_+^{n+1})$*,* $1 \le p \le \infty$*. Then*

$$\|\phi *_w \psi\|_{L^p_\alpha(\mathbb{R}_+^{n+1})} \le \|\phi\|_{L^1_\alpha(\mathbb{R}_+^{n+1})}\|\psi\|_{L^p_\alpha(\mathbb{R}_+^{n+1})}. \tag{1.4.16}$$

3. *Let* $p,q,r \in [1,\infty]$ *such that* $\frac{1}{p}+\frac{1}{q}-\frac{1}{r}=1$*. Then for all* $\phi \in L^p_\alpha(\mathbb{R}_+^{n+1})$ *and* $\psi \in L^q_\alpha(\mathbb{R}_+^{n+1})$*, we have*

$$\|\phi *_w \psi\|_{L^r_\alpha(\mathbb{R}_+^{n+1})} \le \|\phi\|_{L^p_\alpha(\mathbb{R}_+^{n+1})}\|\psi\|_{L^q_\alpha(\mathbb{R}_+^{n+1})}. \tag{1.4.17}$$

4. If $\phi \in L^1_\alpha(\mathbb{R}^{n+1}_+)$ and $\psi \in L^1_\alpha(\mathbb{R}^{n+1}_+)$, then

$$\mathcal{F}_\alpha(\phi *_w \psi) = \mathcal{F}_\alpha(\phi)\mathcal{F}_\alpha(\psi). \tag{1.4.18}$$

From [81, p. 3], throughout other chapters Leibniz's formula and general Leibniz's formula for derivatives are used which are given below:

1. Let f and g be smooth functions. Then

$$D^\beta(fg) = \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \big(D^\gamma f\big)\big(D^{\beta-\gamma} g\big). \tag{1.4.19}$$

2. Let $P(D) = \sum_{|\gamma| \leq m} C_\gamma D^\gamma$ be a linear partial-differential operator with constant coefficients and $P(\xi)$ its symbol. Then

$$P(D)(fg) = \sum_{|\rho| \leq m} \frac{1}{\rho!} \big(P^{(\rho)}(D) f\big)\big(D^\rho g\big), \tag{1.4.20}$$

where $P^{(\rho)}(D)$ is the linear partial-differential operator with symbol $P^{(\rho)}(\xi)$ given by

$$P^{(\rho)}(D) = \partial^\rho P(D).$$

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