Chapter 1

Introduction and Preliminaries

1.1 A Fractal Journey

The study of fractals, a captivating branch of mathematics, has its roots in the groundbreaking work of Benoit Mandelbrot and John Hutchinson. Their contributions paved the way for understanding the intricate world of fractal geometry and its applications in various fields.

Benoit Mandelbrot, a mathematician and visionary, coined the term "fractal" and introduced the concept of self-similarity in the 1970s. He demonstrated how complex patterns in nature, such as coastlines and clouds, could be described and analyzed using fractal geometry. Mandelbrot's exploration of fractals led to the development of a new geometric language that allowed the understanding of irregular and fragmented shapes, which traditional Euclidean geometry could not adequately represent. In the aftermath of the publication of Mandelbrot's seminal work, "The Fractal Geometry of Nature" [1], there has been a progressively mounting inclination towards leveraging fractals as a means of modeling a broad spectrum of natural entities and processes.

John Hutchinson [2] expanded upon Mandelbrot's work by introducing a powerful mathematical tool called the iterated function system (IFS). Hutchinson's IFS provided a framework for generating self-similar fractal structures through the repeated application of mathematical transformations. This breakthrough allowed the construction of a wide range of visually captivating fractals and further deepened our understanding of their underlying principles.

For the discussion of this section, let (X, d) be a complete metric space, and let $H(X)$ be the family of all non-empty compact subsets of X, endowed with the Hausdorff metric

$$h_d(B, C) := \max \left\{ \max_{b \in B} \min_{c \in C} d(b, c), \max_{c \in C} \min_{b \in B} d(b, c) \right\}.$$

It is well known [2] that $(H(X), h_d)$ is a complete metric space. Let $k > 1$ be a positive integer, and let, for $i = 1, 2, \ldots k$, w_i be a contractive self-map on X, i.e., there exists real number $m_i \in [0, 1)$ such that

$$d\left(w_i(x), w_i(y)\right) \leq m_i d(x, y) \quad \forall \quad x, y \in X.$$

Definition 1.1.1. The system $\{X; w_1, w_2, \ldots, w_k\}$ is called (hyperbolic) IFS.

The IFS defined above generates a mapping W from $H(X)$ into $H(X)$, referred to as the Hutchinson-Barnsley map

$$W(F) = \cup_{i=1}^{k} w_i(F).$$

It is well-known [2, 3] that the map W is then a contraction mapping, with respect to the Hausdorff metric h_d, and the contractivity factor m of W is equal to $\max\{m_1, m_2, \ldots, m_k\}$. By the Banach contraction principle, there exists a unique non-empty compact subset F^* such that $F^* = \cup_{i=1}^{k} w_i(F^*)$, termed the attractor corresponding to the IFS.

1.2 Fractal Dimension

One key aspect of fractal geometry is the concept of dimensionality. Fractals possess dimensions that are often non-integer values, unlike the whole numbers found in classical geometry. Two commonly used dimensions in fractal analysis are the topological dimension and the Hausdorff dimension.

The topological dimension refers to the number of coordinates needed to describe a fractal object. It focuses on the connectivity and neighborhood relationships within the structure. However, the topological dimension alone does not provide a comprehensive measure of the intricate details and self-similarity exhibited by fractals.

In contrast, the Hausdorff dimension offers a more refined measure of a fractal's complexity. It captures the scaling properties and spatial occupancy of a fractal within the underlying metric space. The Hausdorff dimension allows us to quantify the visual intricacy and detail of a fractal by considering how densely it fills its space at different scales. In comparison to topological dimension, fractal dimension can assign non-integer values to a set's dimension. Together, the topological dimension and the Hausdorff dimension provide valuable insights into the nature of fractal objects. They offer tools for characterizing, quantifying, and comparing different types of fractals, enabling researchers to unravel the hidden beauty and mathematical foundations of these complex structures.

Kolomogorov [4] introduced the concept of fractal dimension, which Mandelbrot [1] subsequently applied to quantify irregular patterns. Additional information about various dimensions can be read in [5].

The study of fractals and their dimensions has found applications in diverse fields, including computer graphics, physics, biology, and finance. Fractals have proven to be invaluable for modeling natural phenomena, analyzing complex data sets, generating realistic visual effects, and understanding intricate patterns in various domains. Different types of dimension exist for different mathematical spaces.

The objective of this part is to familiarize the reader with two different types of fractal dimension, namely the Hausdorff dimension and the box (Minkowski) dimension. Readers can find more information on these types and other fractal dimensions in reference [5]. It should be emphasized that the information provided here is just an overview of the subject.

Suppose that F is a subset of $\mathbb{R}^n$ and s is a non-negative real number. For any $\delta > 0$, we define

$$\mathcal{H}^s_\delta(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : F \subset \cup_{i=1}^{\infty} U_i, 0 < |U_i| < \delta \right\},$$

where $|U_i| = \sup\{\|x-y\| : x, y \in U_i\}$ known as the diameter of U_i. We write

$$\mathcal{H}^s(F) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(F).$$

$\mathcal{H}^s(F)$ is called the s-dimensional Hausdorff Measure of F.

Definition 1.2.1. For any subset A of $\mathbb{R}^n$, the Hausdorff dimension of A is given by the value:

$$\dim_H(F) = \inf \{ s : \mathcal{H}^s(F) = 0 \} = \sup\{ s : \mathcal{H}^s(F) = \infty \}.$$

Definition 1.2.2. Let $A \subseteq \mathbb{R}^n$ be bounded and non-empty and $N_\delta(A)$ be lowest number of sets at most δ diameter with covering A. If

$$\lim_{\delta \to 0} \frac{\log N_\delta(A)}{-\log \delta} = \liminf_{\delta \to 0} \frac{\log N_\delta(A)}{-\log \delta} = \limsup_{\delta \to 0} \frac{\log N_\delta(A)}{-\log \delta},$$

then $\lim_{\delta \to 0} \frac{\log N_\delta(A)}{-\log \delta}$ is known as box dimension and it is denoted as $\dim_B(A)$. $\liminf_{\delta \to 0} \frac{\log N_\delta(A)}{-\log \delta}$ and $\limsup_{\delta \to 0} \frac{\log N_\delta(A)}{-\log \delta}$ are known as lower box dimension and upper box dimension respectively. The following relation is established between these fractal dimensions. (see [5, 6]):

$$\dim_H A \leq \underline{\dim}_B A \leq \overline{\dim}_B A.$$

Lemma 1.2.3 ([7], Lemma 3.1)**.** *Let $g : X \to Y$ be a continuous map between metric spaces (X, d_X) and (Y, d_Y). For a fixed Lipschitz map $f : X \to Y$, we get*

$$\dim_H(Gr(f+g)) = \dim_H(Gr(g)) \quad \textit{and} \quad \dim_B(Gr(f+g)) = \dim_B(Gr(g)).$$

1.3 Univariate Fractal Interpolation Function

The notion of fractal interpolation functions (FIFs) was introduced by Barnsley [8] using IFS. Following Barnsley's work, there has been steadily increasing interest in the use of FIFs to approximate functions. Navascués and her group [9, 10, 11] have defined and studied a family of non-affine fractal functions f^α known as α-fractal functions corresponding to a continuous function f on closed and bounded interval of $\mathbb{R}$. Here the function f^α approximates and interpolates f simultaneously in a compact subset of $\mathbb{R}$.

In this section, we provide details pertaining to Barnsley's definition of the univariate FIF.

Consider a collection of interpolation points denoted as $\{(x_i, y_i) : i = 0, 1, 2, \dots, N\}$ with the condition that $x_0 < x_1 < x_2 < \cdots < x_N$. Set $\Sigma_N = \{1, 2, \dots, N\}, I = [x_0, x_N]$ and for each $i \in \Sigma_N$, we define $I_i = [x_{i-1}, x_i]$. Furthermore, $L_i : I \to I_i$ defined as a contraction homeomorphism for each $i \in \Sigma_N$, satisfies:

$$L_i(x_0) = x_{i-1}, \quad L_i(x_N) = x_i.$$

Now let $K = I \times \mathbb{R}$ be equipped with the Euclidean metric. For $i \in \Sigma_N$, define mappings $F_i : K \to \mathbb{R}$ subject to the following conditions:

$$|F_i(x, y) - F_i(x, y_*)| \leq c_i |y - y_*|,$$

$$F_i(x_0, y_0) = y_{i-1}, F_i(x_N, y_N) = y_i, i \in \Sigma_N, \tag{1.3.1}$$

where $(x, y), (x, y_*) \in K$ and $0 \leq c_i < 1$ for all $i \in \Sigma_N$. We shall take

$$L_i(x) = a_i x + b_i, \quad F_i(x, y) = \alpha_i(x) y + q_i(x),$$

where $\alpha_i : I \to \mathbb{R}$ is referred to as the scaling factor, is a continuous function with $\|\alpha_i\|_\infty < 1$ for every $i \in \Sigma_N$. In the above expressions, a_i and b_i are determined so that the conditions $L_i(x_0) = x_{i-1}, L_i(x_N) = x_i$ are satisfied. The function $q_i : I \to \mathbb{R}$ is a suitable continuous function satisfying the join-up conditions imposed on the bivariate map F_i. That is, $q_i(x_0) = y_{i-1} - \alpha_i(x_0) y_0$ and $q_i(x_N) = y_i - \alpha_i(x_N) y_N$ for all $i \in \Sigma_N$. Subsequently, for $i \in \Sigma_N$, we define

$$W_i : K \to K, \quad W_i(x, y) = (L_i(x), F_i(x, y)).$$

Theorem 1.3.1 ([8], Theorem 1)**.** *Consider the IFS $\mathcal{I} := \{K; W_1, W_2, \ldots, W_N\}$ defined above. Let $\mathcal{G} := \{h^* : I \to \mathbb{R} \mid h^*$ be continuous on $I, h^*(x_0) = y_0$ and $h^*(x_N) = y_N\}$ be endowed with the uniform metric $d(h^*, h_*) = \max\{|h^*(x) - h_*(x)| : x \in I\}$. Define an operator T on $(\mathcal{G}, d)$ as*

$$\begin{aligned}(Th^*)(x) &= F_i\left(L_i^{-1}(x), h^* \circ L_i^{-1}(x)\right)\\ &= \alpha_i\left(L_i^{-1}(x)\right) h^*\left(L_i^{-1}(x)\right) + q_i\left(L_i^{-1}(x)\right), \quad x \in I_i, \quad i \in \Sigma_N.\end{aligned}$$

1. *$\mathcal{I}$ possesses a unique attractor denoted as G.*

2. *G is the graph of a continuous function $g : I \to \mathbb{R}$ that satisfies $g(x_i) = y_i$ for $i \in \{0\} \cup \Sigma_N$.*

3. *The operator T is known as the Read-Bajraktarević operator on the metric space $(\mathcal{G}, d)$, acts as a contraction map with a contraction factor $\|\alpha\|_\infty = \max\{\|\alpha_i\|_\infty : i \in \Sigma_N\}$. This operator has a unique fixed point represented by the function g. Consequently, g fulfills the following functional equation reflecting self-referentiality:*

$$g(x) = \alpha_i\left(L_i^{-1}(x)\right) g\left(L_i^{-1}(x)\right) + q_i\left(L_i^{-1}(x)\right), \quad x \in I_i, \quad i \in \Sigma_N. \tag{1.3.2}$$

Definition 1.3.2. The function g in the previous theorem is a FIF because it signifies that the graph of the interpolation function behaves like a fractal, specifically in Hutchinson's sense, where it acts as the attractor of an IFS.

Remark 1.3.1. *It is important to note that the fundamental distinctions between a FIF and conventional interpolation methods. These distinctions can be summarized as follows:*

(i) FIFs are constructed using the concept of IFS theory, which provides a functional equation for the interpolant and implies a recurring pattern at smaller scales.

(ii) FIFs are generated through iterative process due to their implicit formulation.

(iii) FIFs include scaling factors that are closely associated to the fractal dimension of the interpolant's graph. This characteristic grants flexibility in choosing an interpolant, in contrast to the uniqueness of specific traditional interpolation techniques.

Remark 1.3.2. *In general, the FIF established in Theorem 1.3.1 is not smooth or differentiable. To illustrate this, [12] has demonstrated that for an FIF g originating from a specialized category of IFS, the collection of points where g is not smooth is dense on interpolation interval I. However, when dealing with problems of a differential nature, it is possible to select the parameters of the generating IFS in a specific manner that enables the construction of a fractal interpolant that exhibits $\mathcal{C}^k$-continuity. This concept is further elaborated upon in references [13, 14, 15, 16].*

Main features of FIFs are that their graphs are self-referential in the sense of Equation (1.3.2) and that they usually have non-integral box or Hausdorff dimension. For a special choice of mappings F_i, namely, $F_i(x, y) := c_i x + d_i + \alpha_i y$, where the coefficients c_i and d_i are determined by Equation (1.3.1), and the $\alpha_i \in (-1, 1)$ are free parameters, the resulting FIF is called affine. Fractal interpolation has experienced significant progress in its theoretical foundations and practical implementations, with various authors contributing to its substantial impact on research. It is worthwhile to explore prominent publications such as references [17, 18], as well as the references cited within those works.

FIFs provide a wide variety of applications in several fields. Păcurar et al. [19] have recreated epidemic curves using the fractal interpolation. Considering epidemic curves as fractal structures, it might be an effective technique to recover missing bits of information owing to inadequate testing and thereby anticipate disease progression.

A fractal approach to the epidemic curve may help to analyse and simulate different epidemics. Authors in [20] have introduced a single-image super-resolution technique based on the rational fractal interpolation model, which is more suited for defining image structures. A novel wrinkle generating approach based on fractal interpolation and the extraction of partition wrinkle lines is provided in [21]. Authors in [22] have suggested a novel way for financial analysis of an enterprise, using fractal interpolation to compute missing observations in a financial database. Readers are referred to [18, 23, 24] for more applications of FIFs and fractal dimensions.

1.4 α-Fractal Functions

As previously discussed, interpolation and approximation are tightly interwoven disciplines in mathematics, where insights and findings in one often carry implications for the other. This relationship is particularly evident in the realm of FIFs when considering the concept of α-fractal functions, introduced by Navascués [9]. α-fractal functions form a parameterized family of self-referential functions associated with a specified continuous function defined over closed and bounded interval. These self-referential functions are generated through perturbations applied to an initial continuous function, referred to as the germ function. These perturbations are accomplished using an IFS , characterized by several significant parameters:

(*i*) Partitioning the domain over which the germ function operates. (*ii*) Incorporating a vector parameter known as the scale vector, which is associated with the domain partition. (*iii*) Employing a function known as the base function, which relies on the values of the germ function at its endpoints.

Define an interval $I = [x_0, x_N]$. Furthermore, equip the space $\mathcal{C}(I)$ with the uniform norm. Let $f \in \mathcal{C}(I)$ be a function we call the germ function. To construct an IFS, consider the following components:

1. Partition I into Δ with $\Delta = x_0, x_1, \ldots, x_N$ such that $x_0 < x_1 < \cdots < x_N$.

2. For each $i \in \Sigma_N$, define a continuous function $\alpha_i : I \to \mathbb{R}$ such that $\|\alpha_i\|_\infty < 1$. We refer to these functions as scaling functions. The collection of these functions forms a vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in (\mathcal{C}(I))^N$, which we call scaling vector.

3. Define a continuous function $s : I \to \mathbb{R}$ with $s\left(x_0\right) = f\left(x_0\right), s\left(x_N\right) = f\left(x_N\right)$ and $s \neq f$. This function is termed as base function.

Now consider a dataset represented as $\{(x_i, f\left(x_i\right)) : i = 0, 1, \ldots, N\}$. Define two maps as follows:

(i) $L_i(x) = a_i x + b_i$.

(ii) $F_i(x, y) = \alpha_i(x)y + f\left(L_i(x)\right) - \alpha_i(x)s(x)$ is a function that combines the scaling function, the linear map, and the base function for each i in the range Σ_N.

According to Theorem 1.3.1, there exists an associated IFS denoted as $\mathcal{I} := \{K; W_1, W_2, \ldots, W_N\}$, where $W_i(x, y) = (L_i(x), F_i(x, y))$ has a unique attractor, which is the graph of a continuous function $f^\alpha_{\Delta,s} : I \to \mathbb{R}$. It satisfies the condition that

$$f^\alpha_{\Delta,s}\left(x_i\right) = f\left(x_i\right), \quad i \in \{0\} \cup \Sigma_N.$$

Furthermore, $f^\alpha_{\Delta,s}$ follows a self-referential equation, which is given by:

$$f^\alpha_{\Delta,s}(x) = f(x) + \alpha_i\left(L_i^{-1}(x)\right)\left(f^\alpha_{\Delta,s} - s\right)\left(L_i^{-1}(x)\right) \quad \forall\, x \in I_i, \quad i \in \Sigma_N.$$

In most cases, the function $f^\alpha_{\Delta,s}$ is not smooth (i.e., it lacks differentiability), especially when we relax the constraints on α and s. Additionally, the graph of $f^\alpha_{\Delta,s}$ has dimensions (box dimension and Hausdorff dimension) that are not whole numbers, as seen in [25]. We can think of $f^\alpha_{\Delta,s}$ as a kind of fractal perturbation to the original function f. For simplicity in notation, we often denote it as f^α.

Definition 1.4.1. The function $f^\alpha_{\Delta,s}$ is called the α-fractal function associated to f with respect to the parameters α, Δ and s.

From the functional equation of the α-fractal function, it is immediate (see, for instance, [9, 10]) that

$$\left\|f^\alpha_{\Delta,s} - f\right\|_\infty \leq \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty}\|f - s\|_\infty.$$

Remark 1.4.1. *Notice that function $f^\alpha_{\Delta,s}$ gives us a collection of self-referential functions associated with a given germ function f. These self-referential functions can vary depending on the specific values chosen for the parameters α, s, and Δ. It's worth highlighting that, for appropriate selections of α or s, each function in this class can simultaneously serve as both an interpolator and an approximation of the original function f.*

Remark 1.4.2. *The parameter vector α plays a crucial role in determining the fractal nature of the resulting function. We can construct functions belonging to the parameterized class, denoted as $f^\alpha_{\Delta,s}$, to either maintain or change specific characteristics of the original function f, such as its level of smoothness or roughness [12, 16]. More recently, Viswanathan and his colleagues have found appropriate parameter values for α, ensuring that the α-fractal function $f^\alpha_{\Delta,s}$ retains the same shape properties as the source function f [26]. This discovery brings together two* *separate fields of study: the theory of FIFs and univariate constrained approximation. These fields were previously developing independently and in parallel.*

1.5 Fractal Operator

We now turn our attention to the discussion of the fractal operator associated with the concept of the α-fractal function. When we establish fixed values for the parameters α, Δ, and s employed in the construction of the α-fractal function, we can introduce an operator denoted as $\mathcal{F}^{\alpha}_{\Delta,s}$. This operator is responsible for transforming each given germ function within the space $\mathcal{C}(I)$ into its corresponding fractal representation. That is,

$$\mathcal{F}^{\alpha}_{\Delta,s} : \mathcal{C}(I) \to \mathcal{C}(I), \quad \mathcal{F}^{\alpha}_{\Delta,s}(f) = f^{\alpha}_{\Delta,s} \quad \forall f \in \mathcal{C}(I).$$

Furthermore, If $s = Lf$ holds, where L is a bounded linear operator from the function space $\mathcal{C}(I)$ to itself, we can define a bounded linear operator known as fractal operator. For the sake of brevity, we often refer to this as $\mathcal{F}^{\alpha}$.

Theorem 1.5.1. *[[27], Theorem 2.2]. Let Id be the identity operator on $\mathcal{C}(I)$.*

1. *The fractal operator $\mathcal{F}^{\alpha}_{\Delta,L} : \mathcal{C}(I) \to \mathcal{C}(I)$ is a bounded linear map. Further, the operator norms satisfy the following inequalities*

$$\left\|Id - \mathcal{F}^{\alpha}_{\Delta,L}\right\| \le \frac{|\alpha|_\infty}{1 - |\alpha|_\infty}\|Id - L\|, \quad \left\|\mathcal{F}^{\alpha}_{\Delta,L}\right\| \le 1 + \frac{|\alpha|_\infty}{1 - |\alpha|_\infty}\|Id - L\|.$$

2. *For $|\alpha|_\infty < \|L\|^{-1}, \mathcal{F}^{\alpha}_{\Delta,L}$ is bounded below. In particular, $\mathcal{F}^{\alpha}_{\Delta,L}$ is an injective map.*

3. *For* $|\alpha|_\infty < \|L\|^{-1}$, *the fractal operator* $\mathcal{F}^\alpha_{\Delta,L}$ *is not a compact operator.*

4. *If* $|\alpha|_\infty < (1 + \|Id - L\|)^{-1}$, *then* $\mathcal{F}^\alpha_{\Delta,L}$ *is a topological isomorphism (i.e., a bijective bounded linear map with a bounded inverse). Moreover,*

$$\left\| \left(\mathcal{F}^\alpha_{\Delta,L}\right)^{-1} \right\| \leq \frac{1 + |\alpha|_\infty}{1 - |\alpha|_\infty \|L\|}.$$

The representation of fractal functions as operators, although not immediately evident in the construction of FIFs, allows for their integration with established mathematical disciplines such as operator theory, complex analysis, harmonic analysis, and approximation theory, as documented in references [6, 9, 10, 15, 28, 29, 30, 31, 32].

Barnsley and Harrington [13] selected parameters in such a way that the corresponding fractal function $f^\alpha \in \mathcal{C}^k(I)$. Viswanathan et al. [26] demonstrated that through a well-chosen IFS, the corresponding univariate α- fractal functions preserve shape properties such as positivity, monotonicity and convexity of the germ function f.

Theorem 1.5.2 ([26],Theorem 3.2)**.** *(Positive fractal polynomial approximation). Let* f *be a continuous function defined on* I *satisfying* $f(x) \geq 0$ *for all* $x \in I$. *To each* $\epsilon > 0$, *there corresponds a fractal polynomial* p^α *such that* $p^\alpha(x) \geq 0$ *for all* $x \in I$ *and* $\|f - p^\alpha\|_\infty < \epsilon$.

1.6 Construction of Sierpiński Gasket (SG)

Let $V_0 = \{p_1, p_2, p_3\}$ be the vertices of an equilateral triangle on the plane $\mathbb{R}^2$ and $L_i(t) = \frac{1}{2}(t + p_i)$, where $i = 1, 2, 3$, be three contractions of the plane that constitute an IFS. For $N \in \mathbb{N}$, we denote the collection of all words with length N by $\{1, 2, 3\}^N$, that is, if $w \in \{1, 2, 3\}^N$, then $w = w_1, w_2, \ldots, w_N$, where $w_i \in \{1, 2, 3\}$. We define, for $w \in \{1,2,3\}^N$,

$$L_w = L_{w_1} \circ L_{w_2} \circ \cdots \circ L_{w_N}.$$

SG is the attractor of this system:

$$SG = L_1(SG) \cup L_2(SG) \cup L_3(SG).$$

Define V_1 by $V_1 = \{p_1, p_2, p_3, L_1(p_2), L_2(p_3), L_3(p_1)\}$. Define V_N by

$$V_N = \{p_1, p_2, p_3, L_w(p_2), L_w(p_3), L_w(p_1) : w \in \{1,2,3\}^N\},$$

where V_N is the set of N-th stage vertices of SG. Consider $V_* = \bigcup_{N=1}^{\infty} V_N$.

Definition 1.6.1. Consider a closed and bounded subset $X \subset \mathbb{R}^2$. We denote the oscillation of $g : X \to \mathbb{R}$ over X by $\mathrm{OSC}_g[X]$ and define as

$$\mathrm{OSC}_g[X] = \sup_{t,z \in X} |g(t) - g(z)|.$$

Lemma 1.6.2 ([33], Lemma 3.2)**.** *Suppose that $f \in \mathcal{C}(SG)$. Let $\delta = \frac{1}{2^n}$ for some $n \in \mathbb{N}$ and $N_\delta(Gr(f))$ denote the number of δ-cubes intersecting $Gr(f)$, then*

$$2^n \sum_{\omega \in I^n} OSC_f[L_\omega(SG)] \leq N_\delta(Gr(f)) \leq 2.3^n + 2^n \sum_{\omega \in I^n} OSC_f[L_\omega(SG)],$$

where $OSC_f(A) = \sup_{x,y \in A} |f(x) - f(y)|$ and $I = \{1,2,3\}$.

Definition 1.6.3. A function $f : SG \to \mathbb{R}$ can be described as Hölder continuous if $\exists$ a positive constant K and $0 < \sigma \leq 1$ such that

$$|f(x) - f(y)| \leq K\|x - y\|_2^{\sigma} \ \text{ for all } \ x, y \in SG.$$

Here, the function f is called Lipschitz continuous, if $\sigma = 1$.

Let f be a Hölder continuous function with exponent σ, then σ-th Hölder seminorm can be defined as

$$[f]_\sigma = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|_2^\sigma}$$

and now consider the Hölder space

$$\mathcal{H}^\sigma(SG) := \{g : SG \to \mathbb{R} : \ g \text{ is Hölder continuous with exponent } \sigma\}.$$

Furthermore, it can be observed that the space $\mathcal{H}^\sigma(SG)$, when endowed with the norm $\|g\| := \|g\|_\infty + [g]_\sigma$, is a Banach space.

1.7 α- Fractal Function on SG

Following the work of Barnsley [8], Özdemir, Kocak and Celik [34] introduced the idea of FIFs on SG. Further, Ruan [35] constructed FIFs on a more general fractal domain. Recently in [36], Navascués, Verma and Viswanathan have discussed the smoothness property of vector-valued FIFs on SG as well as other elementary properties. Verma and Viswanathan [37] have recently studied the box dimension of the graph of the α-fractal function defined on a closed and bounded interval of $\mathbb{R}$. This literature study motivated us to calculate the box dimension of the graph of the α-fractal function defined on SG. Recently, in [7], authors have introduced a new notion of constrained approximation termed as dimension perserving approximation and found some applications of α-fractal functions therein. In [38], some authors have studied the dimensional analysis of α-fractal functions.

1.8 Self-similar Measure Defined on SG

Definition 1.8.1. Let (X, d) be a compact metric space. Let μ_p be a Borel Measure on X, if $\mu_p(X) = 1$, then μ_p is said to $\mathcal{B}orel$ Probability Measure.

Definition 1.8.2. Let (X, d) be a compact metric space. Let $\mathcal{P}$ denote the set of $\mathcal{B}orel$ probability measures on X. The Hutchinson metric d_H on $\mathcal{P}$ is defined by

$$d_H(\mu_p, v) = \sup\left\{ \int_X f d\mu_p - \int_X f dv : f : X \to \mathbb{R}, f \text{ is continuous}, \\ |f(x) - f(y)| \leq d(x, y)\ \forall x, y \in X \right\}$$

for all $\mu_p, v \in \mathcal{P}$.

Definition 1.8.3 ([3], Definition 6.1)**.** Let (X, d) be a compact metric space and let $\mathcal{P}$ denote the space of $\mathcal{B}orel$ probability measures on X. Let

$$\{X; W_1, W_2, \ldots, W_N; p_1, p_2, \ldots, p_N\}$$

be a hyperbolic IFS with probability vectors. The Markov operator associated with the IFS is the function $M : \mathcal{P} \to \mathcal{P}$ defined by

$$M(v) = p_1 v \circ W_1^{-1} + p_2 v \circ W_2^{-1} + ... + p_N v \circ W_N^{-1}$$

for all $v \in \mathcal{P}$.

Lemma 1.8.4 ([3], Lemma 6.1)**.** *Let M denote the Markov operator associated with a hyperbolic IFS, as given in Definition 1.8.3. Suppose $f : X \to \mathbb{R}$ is either a simple function or a continuous function. Let $v \in \mathcal{P}$, then we have*

$$\int_X f d(M(v)) = \sum_{i=1}^{N} p_i \int_X f \circ W_i \ dv.$$

Theorem 1.8.5 ([3], Theorem 6.1)**.** *Let (X, d) be a compact metric space. Let*

$$\{X; W_1, W_2, \ldots, W_N; p_1, p_2, \ldots, p_N\}$$

be a hyperbolic IFS with probabilities. Let $s \in (0, 1)$ be a contractivity factor for the IFS. Let $M : \mathcal{P} \to \mathcal{P}$ be the associated Markov operator. Then M is a contraction mapping with contractivity factor s with respect to the Hutchinson metric on $\mathcal{P}$. That is,

$$d_H(M(v), M(\mu_p)) \leq s d_H(v, \mu_p) \quad \forall \quad v, \mu_p \in \mathcal{P}.$$

In particular, there is a unique measure $\mu_p \in \mathcal{P}$ such that

$$M\mu_p = \mu_p.$$

Theorem 1.8.6. *[3] Let (X, d) be a compact metric space. Let*

$$\{X; W_1, W_2, ..., W_N; p_1, p_2.....p_N\}$$

be a hyperbolic IFS with probabilities. Let μ_p be the associated invariant measure. Then the support of μ_p is the attractor of the IFS $\{X; W_1, W_2, ..., W_N\}$.

The self-similar measure μ_p associated with the IFS $\{\mathbb{R}^2, L_1, L_2, L_3\}$ can be expressed as follows:

$$\mu_p = \frac{1}{3} \sum_{i=1}^{3} \mu_p \circ L_i^{-1},$$

$$d\mu_p = \frac{1}{3} \sum_{i=1}^{3} d(\mu_p \circ L_i^{-1}).$$

Note that the measure μ_p is supported on SG and $\mu_p(SG) = 1$. Now using the definition of a self-similar measure and applying a change of variables, we have

$$\begin{aligned}\int_{L_w(SG)} g \circ L_w^{-1}(t) \;\; d\mu_p(t) &= \frac{1}{3}\sum_{w=1}^{3}\int_{L_w(SG)} g \circ L_w^{-1}(t) \;\; d(\mu_p \circ L_w^{-1})(t) \\ &= \frac{1}{3}\sum_{w=1}^{3}\int_{SG} g(\tilde{t}) d\mu_p(\tilde{t}) \\ &= \int_{SG} g(\tilde{t}) \;\; d\mu_p(\tilde{t}),\end{aligned} \tag{1.8.1}$$

for any continuous function $g : SG \to \mathbb{R}$.

1.9 Energy and Laplacian on SG

FIFs were defined by Celik et al. on SG in [34] and by Ruan [35] on the basis of post-critically finite (p.c.f.) self-similar sets, presented by Kigami [39] to investigate fractal analysis. On SG, Ri and Ruan [40] have determined the conditions for the uniform FIFs to have finite energy. Recently, Ri [41] has generated the graphs of FIFs on SG.

R.S. Strichartz's [42] contributions to mathematics are marked by his pioneering work in extending the theory of differential equations to fractal domains. His rigorous definitions of fundamental operators like the Laplacian are specifically applicable in non-traditional settings, such as fractal domains where traditional smoothness assumptions do not apply, such as on fractal geometries. This groundbreaking approach deepens our understanding of mathematical structures and enhances differential equations' applicability across diverse and challenging domains.

Laplacian on SG can be defined in two ways. The first approach is based on the probability theory and the second approach based on calculus, was conceived by

Kigami [39]. However, we follow the second approach and develop the approximation theory on the well-known fractal domain of SG. Strichartz [42] gave a notion of polynomials using Laplacian on SG. Here we use the theory of fractal interpolation to construct a class of new functions, so-called fractal polynomials [9, 43, 44].

For the analysis on fractals, we need the following basic concepts and properties in this section. We refer to [42] for further information.
We begin with the construction of a complete graph Γ_0 via vertex set V_0. The following procedure is used to recursively construct the graph Γ_m at m^{th} level. First we construct the graph Γ_{m-1} using the vertex set V_{m-1} for some $m \geq 1$. The graph Γ_m on V_m is defined as follows: The edge relation $u \sim_m v$ exists for any $u, v \in V_m$ if and only if $u = L_i(u^{'}),\ v = L_i(v^{'})$ with $u^{'} \sim_{m-1} v^{'}$ and $i \in I$. Equivalently, $u \sim_m v$ if and only if one gets $\boldsymbol{\omega} \in I^N$ with $u, v \in L_{\boldsymbol{\omega}}(V_0)$.

Definition 1.9.1. Consider $m \in \mathbb{N} \cup \{0\}$ and the graph energy at m^{th} level on Γ_m is denoted by E_m and defined as follows:

$$E_m(g) = \left(\frac{5}{3}\right)^m \sum_{u \sim_m v} \big(g(u) - g(v)\big)^2,$$

which satisfies $E_{m-1}(g) = \min E_m(\tilde{g})$, where the minimum is computed through every $\tilde{g}$ such that $\tilde{g}|_{V_{m-1}} = g$ for each $g : V_* \to \mathbb{R}$ with $m \geq 1$. It is worth noting that $\big(E_m(g)\big)_{m=0}^{\infty}$ is an increasing sequence for all g.

The limit

$$E(g) := \lim_{m \to \infty} E_m(g)$$

is referred to as the energy of g on V_*. If g has finite energy, then $E(g) < \infty$.
Note that every g on V_* whose energy is finite are uniformly continuous. Since V_* is dense in SG, which immediately implies that there is a unique extension of g as a

continuous function on SG. Let $\mathrm{dom}(\mathcal{E})$ be the set of continuous functions on SG with finite energy. We now define for each $f, g \in \mathrm{dom}(\mathcal{E})$

$$\mathcal{E}(f,g) = \lim_{m\to\infty} \left(\frac{5}{3}\right)^m \sum_{x\sim_m y} (f(x)-f(y))(g(x)-g(y)).$$

Definition 1.9.2. Let $g \in \mathcal{C}(SG)$ be such that $E_{m-1}(g) = E_m(g)$ for $m = 1, 2, \ldots$. Then we say that g is a harmonic function.

Definition 1.9.3. Let $f \in \mathrm{dom}(\mathcal{E})$ and $g \in \mathcal{C}(SG)$. Then $f \in \mathrm{dom}\Delta$ with $\Delta f = g$ if

$$\mathcal{E}(f,h) = -\int_{SG} gh \, d\mu_p \ \text{ for all } \ h \in \mathrm{dom}_0(\mathcal{E}),$$

where $\mathrm{dom}_0(\mathcal{E}) = \{f \in \mathrm{dom}(\mathcal{E}) : f|_{V_0} = 0\}$.

The Laplacian of f can also be determined using a pointwise formula. Define a graph Laplacian Δ_m on Γ_m by

$$\Delta_m f(x) := \sum_{y\sim_m x} \big(f(y)-f(x)\big), \ x \in V_m \backslash V_0.$$

Using Theorems 2.2.1 and 2.2.12 of [42], we get the following lemma.

Lemma 1.9.4. *Let $f \in \mathrm{dom}(\Delta)$, then the pointwise formula*

$$\Delta f(x) = \frac{3}{2}\lim_{m\to\infty} 5^m \Delta_m f(x) \tag{1.9.1}$$

holds with the limit uniform across $V_* \backslash V_0$. Conversely, suppose f is a continuous function and the right side of Equation (1.9.1) converges uniformly to a continuous function on $V_* \backslash V_0$. Then $f \in \mathrm{dom}(\Delta)$ and Equation (1.9.1) holds.

It can be seen that the Laplacian is a linear operator from the above lemma. Let h be a harmonic function. Now using the well-known rule $\frac{1}{5} - \frac{2}{5}$, we have $\Delta_m h(x) = 0$

for any $m \in \mathbb{N}$ and $x \in V_m \backslash V_0$ so that $\Delta h = 0$.

The space of polynomials on the unit interval may be represented as the space of solutions to $\Delta^k = 0$ for some k. So one can define a polynomial on SG with standard Laplacian which is also discussed in [42] as the solution of the same equation. We define

$$\mathcal{H}_k(SG) = \{f : \Delta^{k+1} f = 0\}.$$

These functions are referred as a multi-harmonic. In the case of $\mathcal{H}_0$, the class of harmonic functions is three-dimensional and a function in $\mathcal{H}_0$ is obtained uniquely by boundary values. For $\mathcal{H}_1$, the class of bi-harmonic functions is six- dimensional. For $\mathcal{H}_2$, the class of tri-harmonic functions is nine-dimensional. For $\mathcal{H}_k$, the class of multi-harmonic functions is $3(k+1)$ dimensional.

The upcoming theorem concerning the denseness of real-valued piecewise harmonic functions is simple to prove by following [42, Theorem 1.4.4].

Theorem 1.9.5. *Let $f \in \mathcal{C}(SG)$, then there exists a sequence $(h_m) \in \mathcal{S}(H_0, V_m, \mathbb{R})$ satisfying $h_m|_{V_m} = f|_{V_m}$ such that $h_m \to f$ uniformly.*

1.10 Bivariate α-Fractal Function

The majority of the researchers have focused on FIFs just for the univariate case, and very little attempt is made to build the α-fractal functions for the bivariate case. To deal with the more unpredictable calculation covered up inside the development, fractal surface procedure is required. Massopust [45] was the first to give the development of fractal surfaces through IFSs. He thought about the case in which the area is three-sided, and the interpolants on the limit of the space are coplanar. A more general development was demonstrated by Geronimo and Hardin [46]. Xie and Sun [47] developed fractal interpolation surfaces on rectangular domains with the assistance of scaling factors and without utilizing limit conditions, However, this led to attractors which are not a graphs of continuous functions. Bouboulis and Dalla [48] used coplanar data points and demonstrated that the attractor is a continuous surface. We encourage the reader to read [6, 49, 50, 51, 52, 53] for some other constructions and dimensional results. These constructions are different from our approach considered here. Ruan and Xu [54] have recently introduced a general framework to construct fractal interpolation surfaces.

Computing the fractal dimension of the graph of a function has been an interesting and integral part of the fractal geometry. We refer the non-fractalist reader to [55] for a better introduction to the area of the fractal dimension of bivariate functions. Malysz [56] studied the box dimension of fractal interpolation surfaces. Recently, Kong et al. [57] have calculated the box dimension of bilinear fractal interpolation surfaces by estimating the oscillation of functions. To the best of my knowledge, more recently, Verma and Viswanathan [43] have introduced the bivariate α-fractal function by borrowing the construction of Ruan and Xu [54]. In [58], the same authors have also added approximation theoretic results of bivariate α-fractal functions along with their fractal dimensions. Recently, Jha et al. [11] have estimated the box dimension of α-fractal function. In the same paper, they have put some effort to study some approximation and smoothness properties of α-fractal function.

The following construction of bivariate α-fractal function is given by Ruan and Xu [54].

1.11 Construction of Bivariate α-Fractal Function

Let $I = [a, b]$ and $J = [c, d]$. Let a continuous function $f : \square = I \times J \to \mathbb{R}$ be given. A net Δ is defined as follows: $\Delta := \big\{(x_i, y_j) : i = 0, 1, \dots, N; j = 0, 1, \dots, M\big\}$ such that

$$x_0 < x_1 < \cdots < x_N \quad \text{and} \quad y_0 < y_1 < \cdots < y_M,$$

which forms a partition of $\square$. We use $\Sigma_N = \{1, 2, \dots, N\}$, $\Sigma_{N,0} = \{0, 1, \dots N\}$, $\partial\Sigma_{N,0} = \{0, N\}$, $\partial\Delta = \{(x_i, y_j), \quad \forall\ (i, j) \in \partial\Sigma_{N,0} \times \partial\Sigma_{M,0}\}$, $\Sigma = \Sigma_N \times \Sigma_M$ and $\text{int}\Sigma_{N,0} = \{1, 2, \dots, N-1\}$.

Let $s \in \mathcal{C}(\square)$ be a function satisfying $s \neq f$,

$$s(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \forall\ \boldsymbol{x} \in \partial\Delta.$$

Let $\alpha : I \times J \to \mathbb{R}$ be a continuous function such that

$$\|\alpha\|_\infty := \sup\big\{|\alpha(\boldsymbol{x})| : \boldsymbol{x} \in \square\big\} < 1.$$

We define linear functions $u_i : I \to I_i := [x_{i-1}, x_i]$ and $v_j : J \to J_j := [y_{j-1}, y_j]$ as follows:

$$u_i(x) = a_i x + b_i, \quad v_j(y) = c_j y + d_j,$$

where constants involved are suitably determined by the following set of equations:

$$u_i(x_0) = x_{i-1}, \quad u_i(x_N) = x_i, \quad \text{if } i \text{ is odd},$$

$$u_i(x_0) = x_i, \quad u_i(x_N) = x_{i-1}, \quad \text{if } i \text{ is even},$$

$$v_j(y_0) = y_{j-1}, \quad v_j(y_M) = y_j, \ \text{if j is odd}, \quad \text{and}$$

$$v_j(y_0) = y_j, \quad v_j(y_N) = y_{j-1}, \quad \text{if j is even.}$$

Set $K = \square \times \mathbb{R}$ and define $F_{ij} : K \to \mathbb{R}$ by

$$F_{ij}(\boldsymbol{x}, z) = \alpha\big(P_{ij}(\boldsymbol{x})\big)z + f\big(P_{ij}(\boldsymbol{x})\big) - \alpha\big(P_{ij}(\boldsymbol{x})\big)s(\boldsymbol{x}),$$

where $\boldsymbol{x} = (x, y)$ and $P_{ij}(\boldsymbol{x}) := (u_i(x), v_j(y)))$. Let us define $\square_{ij} = I_i \times J_j$. For each $(i, j) \in \Sigma$, we define $W_{ij} : K \to \square_{ij} \times \mathbb{R}$ by

$$W_{ij}(x, y, z) = \big(P_{ij}(\boldsymbol{x}), F_{ij}(x, y, z)\big).$$

Let us mention two examples for such function $s \in \mathcal{C}(I \times J, \mathbb{R})$.

1. $s(\boldsymbol{x}) = f(\boldsymbol{x})h(\boldsymbol{x})$, where $h \in \mathcal{C}(\square)$ is a fixed non-constant function such that $h(\boldsymbol{x}) = 1,\ \forall\ \boldsymbol{x} \in \partial\Delta$.

2. $s(\boldsymbol{x}) = (f \circ h)(\boldsymbol{x})$, where $h \in \mathcal{C}(\square, \square)$ is a fixed map $h \neq Id$, the identity map and $h(\boldsymbol{x}) = \boldsymbol{x},\ \forall\ \boldsymbol{x} \in \partial\Delta$.

Theorem 1.11.1 ([43], Theorem 3.1)**.** *Let $\big\{K, W_{ij} : (i, j) \in \Sigma\big\}$ be the IFS defined as above. Then there exists a unique continuous function $f^{\alpha}_{\Delta,s} : \square \to \mathbb{R}$ such that $f^{\alpha}_{\Delta,s}(\boldsymbol{x}) = f(\boldsymbol{x})$ for all $\boldsymbol{x} \in \partial\Delta$ and $Gr(f^{\alpha}_{\Delta,s}) = \cup_{(i,j)\in\Sigma_{N,0}\times\Sigma_{M,0}} W_{ij}(Gr(f^{\alpha}_{\Delta,s}))$, where $Gr(f^{\alpha}_{\Delta,s}) = \big\{(x, y, f^{\alpha}_{\Delta,s}(\boldsymbol{x})) : \boldsymbol{x} \in \square\big\}$ is the graph of $f^{\alpha}_{\Delta,s}$.*

Remark 1.11.1. *Being the fixed point of Read-Bajraktarević (RB)-operator [43], $f^{\alpha}_{\triangle,s}$ satisfies the functional equation:*

$$f^{\alpha}_{\Delta,s}(\boldsymbol{x}) = F_{ij}\Big(Q_{ij}(\boldsymbol{x}), f^{\alpha}_{\Delta,s}(Q_{ij}(\boldsymbol{x}))\Big) \quad \forall\ \ \boldsymbol{x} \in \square_{ij},$$

where $\boldsymbol{x} = (x, y)$ *and* $Q_{ij}(\boldsymbol{x}) := (u_i^{-1}(x), v_j^{-1}(y))$. *That is, for all* $(i, j) \in \Sigma$ *and* $\boldsymbol{x} \in \square_{ij}$*, we have*

$$f^{\alpha}_{\Delta,s}(\boldsymbol{x}) = f(\boldsymbol{x}) + \alpha(\boldsymbol{x}) f^{\alpha}_{\Delta,s}(Q_{ij}(\boldsymbol{x})) - \alpha(\boldsymbol{x}) s(Q_{ij}(\boldsymbol{x})). \tag{1.11.1}$$

The function $f^{\alpha}_{\Delta,s}$ appeared in the previous remark is important enough to be dignified with a name of its own.

Definition 1.11.2. [43] The function $f^{\alpha}_{\Delta,s}$ is said to be (bivariate) α-fractal function.

Remark 1.11.2. *From the self-referential functional equation, we have*

$$\|f^{\alpha}_{\Delta,s}\|_\infty \leq \|f\|_\infty + \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty} \|f - s\|_\infty.$$

This tells us that α*-fractal function is bounded by a fixed number which does not depend on net* $\triangle$*. It is enough to work with* $X = \square \times [-M, M]$*, where* $M = \|f\|_\infty + \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty} \|f - s\|_\infty$ *for any admissible net of* $\square$.

Note 1.11.1. In this note, we recall Theorem 5.16 in [58]. With the metric

$$d_{\square}(\boldsymbol{x}, \boldsymbol{y}) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}, \quad \text{where } \boldsymbol{x} = (x_1, x_2),\ \boldsymbol{y} = (y_1, y_2),$$

we consider f, s and α such that

$$\begin{aligned} |f(\boldsymbol{x}) - f(\boldsymbol{y})| &\leq K_f d_{\square}(\boldsymbol{x}, \boldsymbol{y})^\sigma, \\ |s(\boldsymbol{x}) - s(\boldsymbol{x})| &\leq K_s d_{\square}(\boldsymbol{x}, \boldsymbol{y})^\sigma, \\ |\alpha(\boldsymbol{x}) - \alpha(\boldsymbol{x})| &\leq K_\alpha d_{\square}(\boldsymbol{x}, \boldsymbol{y})^\sigma, \end{aligned} \tag{1.11.2}$$

for every $\boldsymbol{x}, \boldsymbol{y} \in \square$, and for fixed $K_f, K_s > 0$, and for fixed $0 < \sigma \leq 1$. Assume that for some $k_f > 0, \delta_0 > 0$ the following holds: for each $\boldsymbol{x} \in \square$ and $0 < \delta < \delta_0$, there exists $\boldsymbol{y}$ such that $d_{\square}(\boldsymbol{x}, \boldsymbol{y}) \leq \delta$ and

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| \geq k_f d_{\square}(\boldsymbol{x}, \boldsymbol{y})^{\sigma}. \tag{1.11.3}$$

Furthermore, we suppose $N = M,\ x_i - x_{i-1} = \frac{1}{N},\ y_j - y_{j-1} = \frac{1}{M}, \forall\ i \in \Sigma_N,\ j \in \Sigma_M$ and constant scaling function α. Now, let $K_{f^{\alpha}_{\Delta,s}}$ is the Hölder constant of $f^{\alpha}_{\Delta,s}$. If $\|\alpha\|_{\infty} < \min\left\{\frac{1}{M}, \frac{k_f - (\|s\|_{\infty}+M)K_{\alpha}}{(K_{f^{\alpha}_{\Delta,s}}+K_s)M^{\sigma}}\right\}$, then $\dim_B\left(Gr(f^{\alpha}_{\Delta,s})\right) = 3 - \sigma$.

Remark 1.11.3. *With the assumptions in the above note, one may construct dimension preserving approximants for a given function, see, for instance, [7, Theorem 3.16].*

For a bivariate function f, we denote the derivative of (k,l)-th order by $D^{(k,l)}f$, that is, $D^{(k,l)}f := \dfrac{\partial^{k+l} f}{\partial x^k \partial y^l}$. Let

$$\mathcal{C}^{m,n}(\square) = \{f : \square \to \mathbb{R};\ D^{(k,l)}f \in \mathcal{C}(\square),\ \ \forall\ \ 0 \leq k \leq m,\ 0 \leq l \leq n\}.$$

If $D^{(m,n)}f(\boldsymbol{x}) \geq 0,\ \forall\ \ \boldsymbol{x} \in \square$, then we say the function f is (m,n)-convex. Let $g \in \mathcal{C}((\square))$ such that $\dim(Gr(g)) > 2$. We may refer to [59] for the existence of such functions. The function $f : \square \to \mathbb{R}$ defined by $f(x,y) := \int\limits_a^x \int\limits_c^y g(t,s)dtds$ satisfies the following:

$$\dim(Gr(f)) = 2 \quad \text{and} \quad \dim Gr(D^{(1,1)}f) = \dim(Gr(g)) > 2,$$

where dim denotes a fractal dimension.

Recall that the tensor product Bernstein polynomial on $\square$ is defined as:

$$\begin{aligned} B_{m,n}(f)(\boldsymbol{x}) =& \frac{1}{(b-a)^m(d-c)^n}\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}(x-a)^i(b-x)^{m-i} \\ & (y-c)^j(d-y)^{n-j} f\Big(a+\frac{i(b-a)}{m}, c+\frac{j(d-c)}{n}\Big). \end{aligned} \tag{1.11.4}$$

Let us approximate a function $f \in \mathcal{C}^{k,l}(\square)$ by $B_{m,n}(f)$, then we have the following:

- $B_{m,n}(f) \to f$ uniformly on $\square$.
- $\Big(D^{(k,l)}(B_{m,n}(f))\Big) \to D^{(k,l)}f$ uniformly on $\square$.
- Since $B_{m,n}(f)$ and $D^{(k,l)}(B_{m,n}(f))$ are polynomials, then $\dim\Big(Gr\big(D^{(k,l)}(B_{m,n}(f))\big)\Big) = \dim(Gr(B_{m,n}(f))) = \dim(Gr(f)) = 2.$

We refer the reader to [60] for several properties of bivariate Bernstein polynomials. The above items may conclude that the approximation by Bernstein polynomials maintains the smoothness of a function but not (necessarily) the dimensions of its partial derivatives.

Navascués [10] developed the notion of (univariate) α-fractal function via so-called (univariate) fractal operator. Recently in [43, 58], her collaborators have extended some of her results in a bivariate setting. On putting $L = B_{m,n}$ in [43, Theorem 3.1], we have a unique function $f^{\alpha}_{\Delta,B_{m,n}} \in \mathcal{C}(\square)$ such that

$$f^{\alpha}_{\Delta,B_{m,n}}(\boldsymbol{x}) = f(\boldsymbol{x}) + \alpha(\boldsymbol{x})\ f^{\alpha}_{\Delta,B_{m,n}}\big(Q_{ij}(\boldsymbol{x})\big) - \alpha(\boldsymbol{x})\ B_{m,n}(f)\big(Q_{ij}(\boldsymbol{x})\big), \tag{1.11.5}$$

for $\boldsymbol{x} \in \square_{ij},\ (i,j) \in \Sigma_N \times \Sigma_M$.

Following the work of [43], we define a single-valued fractal operator $\mathcal{F}^{\alpha}_{m,n} : \mathcal{C}(\square) \to \mathcal{C}(\square)$ by

$$\mathcal{F}^{\alpha}_{m,n}(f) = f^{\alpha}_{\Delta,B_{m,n}}.$$

In [43], several operator theoretic results for fractal operator are obtained. We recall that $\mathcal{F}^{\alpha}_{m,n}$ is a bounded linear operator, see, for instance, [43, Theorem 3.2].

1.12 An Overview of Approximation and Operator Theoretic Results

Lemma 1.12.1 ([61], Lemma 1). *Let $(X, \|.\|)$ be a Banach space, $T : X \to X$ be a linear operator. Suppose there exist constants $\lambda_1, \lambda_2 \in [0, 1)$ such that*

$$\|Tx - x\| \leq \lambda_1 \|x\| + \lambda_2 \|Tx\|, \quad \forall \;\; x \in X.$$

Then T is a topological isomorphism, and

$$\frac{1 - \lambda_2}{1 + \lambda_1} \|x\| \leq \|T^{-1}x\| \leq \frac{1 + \lambda_2}{1 - \lambda_1} \|x\|, \; \forall \; x \in X.$$

Definition 1.12.2. ([62]). Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be normed linear spaces. For a multi-valued (set-valued) mapping $T : X \rightrightarrows Y$, the domain of T is defined by $\mathrm{Dom}(T) := \{x \in X : T(x) \neq \emptyset\}$. Then $T : X \rightrightarrows Y$ is

1. *Convex* if

$$\lambda T(x_1) + (1 - \lambda)T(x_2) \subseteq T\big(\lambda x_1 + (1 - \lambda)x_2\big), \; \forall \; x_1, x_2 \in \mathrm{Dom}(T), \quad \lambda \in [0, 1].$$

2. *Process* if

$$\lambda T(x) = T(\lambda x),\ \forall\ x \in X,\ \lambda > 0,\ \text{and } 0 \in T(0).$$

3. *Linear* if

$$\beta T(x_1) + \gamma T(x_2) \subseteq T\big(\beta x_1 + \gamma x_2\big),\ \forall\ x_1, x_2 \in \text{Dom}(T),\ \beta, \gamma \in \mathbb{R}.$$

4. *Closed* if the graph of T defined by $Gr(T) := \big\{(\boldsymbol{x}) \in X \times Y : y \in T(x)\big\}$ is closed.

5. *Lipschitz* if

$$T(x_1) \subseteq T(x_2) + l\|x_1 - x_2\|_X\, U_Y,\ \forall\ x_1, x_2 \in \text{Dom}(T),\ \text{for some constant } l > 0,$$

where $U_Y = \{y \in Y : \|y\|_Y \leq 1\}$.

6. *Lower semicontinuous* at $x \in X$ if there exists a $\delta > 0$ such that

$$U \cap T(x') \neq \emptyset \quad \text{whenever } \|x - x'\|_X < \delta$$

holds for a given open set U in Y satisfying $U \cap T(x) \neq \emptyset$.

Note that the above definitions are also applicable in metric spaces with obvious modifications, see, for instance, [62].

Theorem 1.12.3 ([63], Corollary 1.4)**.** *Let $T : Dom(T) = X \rightrightarrows Y$ be linear such that $T(0) = \{0\}$. Then T is single-valued.*

Theorem 1.12.4 ([63], Corollary 2.1)**.** *Let $T : Dom(T) = X \rightrightarrows Y$ be such that $T(x_0)$ is singleton for some $x_0 \in X$. Then the following are equivalent:*

- *T is single-valued and affine.*

- *T is convex.*

Proposition 1.12.5 ([5], Proposition 9.6)**.** *Let F be the attractor of an IFS consisting of contractions $\{S_1, \ldots, S_m\}$ on a closed subset D of $\mathbb{R}^n$ such that*

$$|S_i(x) - S_i(y)| \leqslant c_i|x - y|, \quad (x, y \in D)$$

with $0 < c_i < 1$ for each i. Then $\dim_{\mathrm{H}} F \leqslant s$ and $\overline{\dim}_{\mathrm{B}} F \leqslant s$, where $\sum_{i=1}^{m} c_i^s = 1$.

Next proposition gives a lower bound for dimension in the case, where the components $S_i(F)$ of F are disjoint. Note that this will certainly be the case if there is some non-empty compact set E with $S_i(E) \subset E$ for all i and with the $S_i(E)$ disjoint.

Proposition 1.12.6 ([5], Proposition 9.7)**.** .

Consider the IFS consisting of contractions $\{S_1, \ldots, S_m\}$ on a closed subset D of $\mathbb{R}^n$ such that

$$b_i|x - y| \leqslant |S_i(x) - S_i(y)| \quad (x, y \in D)$$

with $0 < b_i < 1$ for each i. Assume that the (non-empty compact) attractor F satisfies

$$F = \bigcup_{i=1}^{m} S_i(F),$$

with this union disjoint. Then F is totally disconnected and $\dim_{\mathrm{H}} F \geqslant s$, where

$$\sum_{i=1}^{m} b_i^s = 1.$$

1.13 Motivation for Current Work

In [26], Viswanathan et al. determined appropriate scaling factors such that function $f^\alpha \in \mathcal{C}(I)$ preserves the fundamental shape properties, including positivity, monotonicity, and convexity of the germ f^α. Chand et al. [64] presented results concerning a more general form of univariate constrained interpolation using fractal splines. Navascués and her coworkers [25] determined the box dimension of graph of f^α corresponding to uniformly Hölderian function in a compact subset of $\mathbb{R}$. In the same article, using a method of variation, the box dimension of graph of f^α is calculated. Many researchers have studied constrained fractal interpolants [5, 64, 65, 66, 67]. Our research work has been greatly inspired by the references we have quoted.

The exploration begins with an in-depth analysis of the FIF and its box dimension in relation to continuous functions defined on SG. Additionally, the thesis investigates the associated fractal operator, exploring its important aspects such as topological isomorphism, Fredholm, and more, within certain limitations. Furthermore, Fractal polynomials discussed through fractal perturbation of polynomials defined by Strichartz on SG. The thesis investigates constrained approximation and the best approximation properties of these fractal polynomials on SG. Interesting properties of the class of polynomials defined on SG are also discussed. It includes estimation of the fractal dimensions of the graph of a FIF using the method of oscillation of functions.

Furthermore, the thesis focuses on the approximation of functions by fractal functions in the context of $\mathcal{L}^p$-norm on SG. It defines the α-fractal function in $\mathcal{L}^p$ space and explores its properties, including topological isomorphism for the fractal operator. Set-valued mapping is also introduced and its useful properties in the context of fractal approximation are discussed.

In addition, the thesis establishes results related to bivariate constrained approximation in terms of dimension-preserving approximants, incorporating the concept of FIFs. The thesis also investigates multi-valued fractal operators associated with bivariate α-fractal functions.

During the preparation of this thesis, existing work on the upper bounds for the box dimension of bivariate α-fractal functions is acknowledged. However, the thesis adopts a different approach that leverages the variation of bivariate α-fractal functions over sub-rectangles to estimate bounds for their box dimensions.

At last, the continuous dependence of bivariate α-fractal functions on different parameters is studied. The thesis establishes results concerning the function's dimension and estimates box dimensions of α-fractal functions under certain conditions using the method of variation.

1.14 Organization of Thesis

- **Introduction and Preliminaries:** Chapter 1 serves as the introductory section of the thesis, providing an overview of the fundamental theory and relevant results that form the basis of our study in the following chapters. This chapter also incorporates a concise review of the existing literature pertaining to the topics of interest.

- **Theory of α-Fractal Function on SG:** In this chapter the work deals with the FIF and its box dimension corresponding to a continuous function, defined on SG. This work also explores the so-called fractal operator, which is associated with the α-fractal function. Under certain bounds, we shall demonstrate some significant properties of fractal operator such as topological isomorphism, Fredholm. we provide some results on constrained approximation by fractal polynomials and study best approximation properties of fractal polynomials defined on SG. Further, we discuss some interesting properties of the class of polynomials defined on SG. At the end, we try to estimate bounds for fractal dimensions of the graph of a FIF using the method of oscillation of functions.

- **$\mathcal{L}^p$-approximation using fractal functions on SG:** The chapter is concerned with the approximation of functions by fractal functions with respect to $\mathcal{L}^p$-norm on SG. We define α-fractal function in $\mathcal{L}^p$ space. The properties such as topological isomorphism and many others which are closely associated with the fractal operator will be discussed in more detail. Additionally, we define set-valued mapping and discuss some useful properties.

- **On bivariate fractal approximation:** In this chapter, the notion of dimension preserving approximation for real-valued bivariate continuous functions defined on a rectangular domain $\square$, has been discussed. We also define and study some multi-valued fractal operators associated with bivariate α-fractal functions.

- **A note on stability and fractal dimension of bivariate α-fractal functions:** In this chapter, we study the continuous dependence of so-called (bivariate) α-fractal function on the parameters such as scaling function α, net $\triangle$ of rectangular grid, and base function s involved in its construction. Further, we establish some results regarding its dimension.

- **Conclusions and Future Directions:** In the last chapter, we will provide the combined conclusions of the thesis. Furthermore, we will give some hints related to future research directions.

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