Chapter 4

Characterizations of the Bessel wavelet transform in Besov and Triebel-Lizorkin type spaces

4.1 Introduction

Besov space is an effective tool for measuring regular properties of functions and is a generalizations of Sobolev space. Exploiting the results of Fourier analysis, the characterizations of the spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ were introduced by Triebel [50]. Motivated from the result of multiplier theorem of Triebel [50], boundedness results of the wavelet transform in Besov space $B^s_{p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin space $F^s_{p,q}(\mathbb{R}^n)$ for $s \in \mathbb{R}$ and $0 < p,\ q \leq \infty$ are obtained by Pathak [46] by using Fourier transform technique. Later on, Betancor et al. [13, 14] introduced Besov- Hankel space $BH^{\alpha,\mu}_{p,q}$ and studied its various properties by using the Haimo Hankel transform theory [21].

From the aforesaid information, our main goal in this chapter is to introduce the Besov and Triebel-Lizorkin type spaces by using the Zemanian theory of Hankel transformation and to find the boundedness of the Bessel wavelet transform and its associated results.

The results of the present chapter are organized by the following way:
Section 4.1 is introductory, which provides the historical background of Besov space and Triebel-Lizorkin type space in Hankel setting. In Section 4.2, various definitions and properties of Besov space and Triebel-Lizorkin space associated with Hankel transform and Hankel convolution are given. Section 4.3 describes that the boundedness of the Bessel wavelet transform in Besov space and Triebel-Lizorkin space are proved by exploiting the Hankel transform theory.

4.2 Properties of the Besov space and Triebel-Lizorkin type space

In this section, some definitions and results of Triebel [50] in Hankel setting by using Zemanian space H_μ are constructed and some useful inequalities in the same setting are derived.

Definition 4.2.1. If N is a natural number then $A_\mu^N(\mathbb{R}^+)$ is the collection of all systems $\phi = \{\phi_j(x)\}_{j=0}^{\infty} \subset H_\mu(\mathbb{R}^+),\ \mu \geq -\frac{1}{2}$ of functions with compact supports such that

$$C_\mu(\phi) = \sup_{x\in\mathbb{R}^+} x^N \sum_{\alpha=0}^{N} \left|\left(x^{-1}D_x\right)^\alpha x^{-\mu-\frac{1}{2}}\phi_0(x)\right| + \sup_{\substack{x\in\mathbb{R}^+\\ j=1,2,3,\ldots}} \left(x^N + x^{-N}\right) \sum_{\alpha=0}^{N} \left|\left(x^{-1}D_x\right)^\alpha x^{-\mu-\frac{1}{2}}\phi_j(2^j x)\right| < \infty. \quad (4.2.1)$$

Definition 4.2.2. If N is a natural number, $\phi = \{\phi_j(x)\}_{j=0}^{\infty} \in A_\mu^N(\mathbb{R}^+)$, $f \in H'_\mu(\mathbb{R}^+)$ and $d > 0$ then

$$\left(\phi_j^* f\right)(x) = \sup_{y\in\mathbb{R}^+} \frac{\left|\left(\left(h_\mu^{-1}\phi_j\right)\#f\right)(x,y)\right|}{1+\left(2^j y\right)^{2d}}, \quad x \in \mathbb{R}^+, \quad j = 0,1,2,3,\dots \tag{4.2.2}$$

is called maximal function.

From (4.2.2), we get

$$\left|\left(\left(h_\mu^{-1}\phi_j\right)\#f\right)(x,y)\right| \le \left(1+\left(2^j y\right)^{2d}\right)\left(\phi_j^* f\right)(x). \tag{4.2.3}$$

Definition 4.2.3. Let $\Phi(\mathbb{R}^+)$ be the collection of all systems $\phi = \{\phi_j(x)\}_{j=0}^{\infty} \subset H_\mu(\mathbb{R}^+)$ such that

$$\begin{cases} \text{supp } x^{-\mu-\frac{1}{2}}\phi_0 \subset \{x : 0 \le x \le 2\}, \\ \text{supp } x^{-\mu-\frac{1}{2}}\phi_j \subset \{x : 2^{j-1} \le x \le 2^{j+1}\}, \quad j = 1,2,3,\dots \end{cases} \tag{4.2.4}$$

For every $\alpha \in \mathbb{N}_0$, there exists a positive number C_α such that

$$2^{j\alpha}\left|\left(x^{-1}D_x\right)^\alpha x^{-\mu-\frac{1}{2}}\phi_j(x)\right| \le C_\alpha \qquad \forall j = 0,1,2,3,\dots, \quad x \in \mathbb{R}^+ \tag{4.2.5}$$

and

$$\sum_{j=0}^{\infty} x^{-\mu-\frac{1}{2}}\phi_j(x) = 1, \qquad \forall\, x \in \mathbb{R}^+. \tag{4.2.6}$$

Definition 4.2.4. Let $-\infty < s < \infty$ and $1 \le q \le \infty$. Assume that $\{\phi_j(x)\}_{j=0}^{\infty} \in H_\mu(\mathbb{R}^+)$ and $\mu \ge -\frac{1}{2}$, then we define the following spaces:

$$\text{(i) } F_{p,q}^{\mu,s} = \left\{f : f \in H'_\mu(\mathbb{R}^+), \|f\|_{F_{p,q}^{\mu,s}} = \left\|2^{sj}x^{-\mu-\frac{1}{2}}\left(h_\mu^{-1}\phi_j\right)\#f\right\|_{L_p^\mu(l_q)} < \infty\right\}$$

for $1 \le p < \infty$.

$$\text{(ii) } B^{\mu,s}_{p,q} = \left\{ f : f \in H'_\mu(\mathbb{R}^+), \|f\|_{B^{\mu,s}_{p,q}} = \left\| 2^{sj} x^{-\mu-\frac{1}{2}} \left(h_\mu^{-1}\phi_j\right) \# f \right\|_{l_q(L^\mu_p)} < \infty \right\}$$

for $1 \leq p < \infty$.

Definition 4.2.5. Let $1 \leq p < \infty$, and $1 \leq q \leq \infty$. If $E = \{E_k\}_{k=0}^\infty$ is a sequence of compact subsets of $\mathbb{R}^+$, then

$$L^E_{p,\mu}(l_q) = \Big\{ f : f = \{f_k\}_{k=0}^\infty \subset H'_\mu(\mathbb{R}^+), \quad \text{supp } (h_\mu f_k) \subset E_k \quad \text{if } k = 0, 1, 2, \dots$$
$$\text{and } \left\| x^{-\mu-\frac{1}{2}} f_k \right\|_{L^\mu_p(l_q)} < \infty \Big\}.$$

Lemma 4.2.6. *Let $\phi, \psi \in L^1(\mathbb{R}^+)$, and $h_\mu^{-1}\phi, h_\mu^{-1}\psi \in L^1(\mathbb{R}^+)$. Then*

$$\left(h_\mu^{-1}\phi \# h_\mu^{-1}\psi\right)\left(2^{-k}y\right) = 2^{k-k(\mu+\frac{1}{2})}\left(h_\mu^{-1}\left(\phi(2^k\cdot)\right) \# h_\mu^{-1}\left(\psi(2^k\cdot)\right)\right)(y).$$

Proof. For $\phi \in L^1(\mathbb{R}^+)$, by the definition of inverse Hankel transformation

$$h_\mu^{-1}\left(\phi\left(2^k\cdot\right)\right)(y) = \int_0^\infty (xy)^{\frac{1}{2}} J_\mu(xy)\phi(2^k x)dx.$$

Putting $x = 2^{-k}t$ in the above expression, we get

$$\begin{aligned} h_\mu^{-1}\left(\phi\left(2^k\cdot\right)\right)(y) &= 2^{-k}\int_0^\infty (2^{-k}ty)^{\frac{1}{2}} J_\mu(2^{-k}ty)\phi(t)dt \\ &= 2^{-k}\left(h_\mu^{-1}\phi\right)\left(2^{-k}y\right). \end{aligned} \tag{4.2.7}$$

Using (1.4.9), we find

$$\begin{aligned} &h_\mu\left\{\left(h_\mu^{-1}\left(\phi(2^k\cdot)\right) \# h_\mu^{-1}\left(\psi(2^k\cdot)\right)\right)(y)\right\}(\eta) \\ &\quad = \eta^{-\mu-\frac{1}{2}} h_\mu\left\{h_\mu^{-1}\left(\phi(2^k\cdot)\right)\right\}(\eta)\ h_\mu\left\{h_\mu^{-1}\left(\psi(2^k\cdot)\right)(y)\right\}(\eta) = \eta^{-\mu-\frac{1}{2}}\phi(2^k\eta)\psi(2^k\eta). \end{aligned}$$

This implies that

$$\begin{aligned}\left(h_\mu^{-1}\left(\phi(2^k\cdot)\right)\#h_\mu^{-1}\left(\psi(2^k\cdot)\right)\right)(y) &= h_\mu^{-1}\left(\eta^{-\mu-\frac{1}{2}}\phi(2^k\eta)\psi(2^k\eta)\right)(y)\\ &= 2^{k\left(\mu+\frac{1}{2}\right)}h_\mu^{-1}\left(\left(2^k\eta\right)^{-\mu-\frac{1}{2}}\phi(2^k\eta)\psi(2^k\eta)\right)(y).\end{aligned}$$

Therefore, from (4.2.7) we get

$$\begin{aligned}\left(h_\mu^{-1}\left(\phi(2^k\cdot)\right)\#h_\mu^{-1}\left(\psi(2^k\cdot)\right)\right)(y) &= 2^{k\left(\mu+\frac{1}{2}\right)}2^{-k}h_\mu^{-1}\left(\eta^{-\mu-\frac{1}{2}}\phi(\eta)\psi(\eta)\right)(2^{-k}y)\\ &= 2^{-k+k\left(\mu+\frac{1}{2}\right)}\left\{\left(h_\mu^{-1}\phi\#h_\mu^{-1}\psi\right)\left(2^{-k}y\right)\right\}.\end{aligned}$$

Thus, we have

$$\left(h_\mu^{-1}\phi\#h_\mu^{-1}\psi\right)\left(2^{-k}y\right) = 2^{k-k(\mu+\frac{1}{2})}\left(h_\mu^{-1}\left(\phi(2^k\cdot)\right)\#h_\mu^{-1}\left(\psi(2^k\cdot)\right)\right)(y).$$

$\square$

Theorem 4.2.7. *Let $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $d > 0$. Let $E = \{E_k\}_{k=0}^{\infty}$ is a sequence of compact subsets of $\mathbb{R}^+$, then*

$$\left\|2^{ks}x^{-\mu-\frac{1}{2}}\sup_{y\in\mathbb{R}^+}\frac{|f_k(x,y)|}{1+(2^ky)^{2d}}\right\|_{L_p^\mu(l_q)} \leq c_\mu^{-1}\left\|2^{ks}x^{-\mu-\frac{1}{2}}f_k(x)\right\|_{L_p^\mu(l_q)}$$

holds for all $f \in L_{p,\mu}^E(l_q)$, where $f = \{f_k\}_{k=0}^{\infty}$, $\mu \geq -\frac{1}{2}$, $c_\mu = 2^\mu\Gamma(\mu+1)$, and

$$f_k(x,y) = \int_0^\infty f_k(z)D_\mu(x,y,z)dz$$

is the Hankel translation of $f_k(x)$.

Proof. From (1.4.4), we take

$$f_k(x,y) = \int_0^\infty f_k(z) D_\mu(x,y,z)dz. \tag{4.2.8}$$

Multiplying by 2^{ks} and taking l_q- norm on both sides, we get

$$\Big(\sum_{k=0}^{\infty} 2^{ksq}\, |f_k(x,y)|^q\Big)^{\frac{1}{q}} = \Big(\sum_{k=0}^{\infty} \Big| \int_0^\infty 2^{ks} f_k(z) D_\mu(x,y,z)dz\Big|^q\Big)^{\frac{1}{q}}.$$

Using Minkowski's integral inequality (1.7.1), we obtain

$$\begin{aligned}
&\Big(\sum_{k=0}^{\infty} 2^{ksq}\, |f_k(x,y)|^q\Big)^{\frac{1}{q}} \\
&\quad \le \int_0^\infty \Big(\sum_{k=0}^{\infty} 2^{ksq}\, |f_k(z) D_\mu(x,y,z)|^q\Big)^{\frac{1}{q}} dz \\
&\quad = \int_0^\infty \Big(\sum_{k=0}^{\infty} 2^{ksq} \Big|z^{-\mu-\frac{1}{2}} f_k(z)\Big|^q\Big)^{\frac{1}{q}} \Big(z^{\mu+\frac{1}{2}} D_\mu(x,y,z)\Big)^{\frac{1}{p}} \Big(z^{\mu+\frac{1}{2}} D_\mu(x,y,z)\Big)^{\frac{1}{p'}} dz,
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Now applying Holder's inequality, we get

$$\begin{aligned}
&\Big(\sum_{k=0}^{\infty} 2^{ksq}\, |f_k(x,y)|^q\Big)^{\frac{1}{q}} \\
&\quad \le \Big(\int_0^\infty \Big(\sum_{k=0}^{\infty} 2^{ksq} \Big|z^{-\mu-\frac{1}{2}} f_k(z)\Big|^q\Big)^{\frac{1}{q}\times p} z^{\mu+\frac{1}{2}} D_\mu(x,y,z)dz\Big)^{\frac{1}{p}} \\
&\qquad \times \Big(\int_0^\infty z^{\mu+\frac{1}{2}} D_\mu(x,y,z)dz\Big)^{\frac{1}{p'}} \\
&\quad = \Big(\int_0^\infty \Big(\sum_{k=0}^{\infty} 2^{ksq} \Big|z^{-\mu-\frac{1}{2}} f_k(z)\Big|^q\Big)^{\frac{1}{q}\times p} z^{\mu+\frac{1}{2}} D_\mu(x,y,z)dz\Big)^{\frac{1}{p}} \Big(c_\mu^{-1}(xy)^{\mu+\frac{1}{2}}\Big)^{\frac{1}{p'}},
\end{aligned}$$

since $\int_0^\infty z^{\mu+\frac{1}{2}} D_\mu(x,y,z)dz = c_\mu^{-1}(xy)^{\mu+\frac{1}{2}}, \ c_\mu = 2^\mu \Gamma(\mu+1)$ (1.4.8).

This implies that

$$(xy)^{\left(\mu+\frac{1}{2}\right)\left(-\frac{p}{p'}\right)}\Big(\sum_{k=0}^{\infty}2^{ksq}\left|f_k(x,y)\right|^q\Big)^{\frac{p}{q}}$$
$$\leq \left(c_\mu^{-1}\right)^{\frac{p}{p'}}\int_0^\infty\Big(\sum_{k=0}^{\infty}2^{ksq}\left|z^{-\mu-\frac{1}{2}}f_k(z)\right|^q\Big)^{\frac{p}{q}}z^{\mu+\frac{1}{2}}D_\mu(x,y,z)dz.$$

Now, multiplying by $x^{\mu+\frac{1}{2}}$ and taking integration with respect to x, we get

$$\int_0^\infty y^{\left(\mu+\frac{1}{2}\right)\left(-\frac{p}{p'}\right)}x^{\left(\mu+\frac{1}{2}\right)\left(1-\frac{p}{p'}\right)}\Big(\sum_{k=0}^{\infty}2^{ksq}\left|f_k(x,y)\right|^q\Big)^{\frac{p}{q}}dx$$
$$\leq \left(c_\mu^{-1}\right)^{\frac{p}{p'}}\int_0^\infty\int_0^\infty\Big(\sum_{k=0}^{\infty}2^{ksq}\left|z^{-\mu-\frac{1}{2}}f_k(z)\right|^q\Big)^{\frac{p}{q}}z^{\mu+\frac{1}{2}}x^{\mu+\frac{1}{2}}D_\mu(x,y,z)dzdx.$$

By Fubini's theorem, the above expression yields

$$\int_0^\infty y^{\left(\mu+\frac{1}{2}\right)(1-p)}x^{\left(\mu+\frac{1}{2}\right)(2-p)}\Big(\sum_{k=0}^{\infty}2^{ksq}\left|f_k(x,y)\right|^q\Big)^{\frac{p}{q}}dx$$
$$\leq \left(c_\mu^{-1}\right)^{\frac{p}{p'}}\int_0^\infty\Big(\sum_{k=0}^{\infty}2^{ksq}\left|z^{-\mu-\frac{1}{2}}f_k(z)\right|^q\Big)^{\frac{p}{q}}z^{\mu+\frac{1}{2}}\int_0^\infty x^{\mu+\frac{1}{2}}D_\mu(x,y,z)dxdz$$
$$\leq \left(c_\mu^{-1}\right)^{\frac{p}{p'}}\int_0^\infty\Big(\sum_{k=0}^{\infty}2^{ksq}\left|z^{-\mu-\frac{1}{2}}f_k(z)\right|^q\Big)^{\frac{p}{q}}z^{\mu+\frac{1}{2}}\left(c_\mu^{-1}(yz)^{\mu+\frac{1}{2}}\right)dz$$
$$\leq \left(c_\mu^{-1}\right)^{\left(1+\frac{p}{p'}\right)}\int_0^\infty\Big(\sum_{k=0}^{\infty}2^{ksq}\left|z^{-\mu-\frac{1}{2}}f_k(z)\right|^q\Big)^{\frac{p}{q}}z^{\mu+\frac{1}{2}}\left((yz)^{\mu+\frac{1}{2}}\right)dz.$$

Therefore

$$\int_0^\infty x^{2\mu+1}\Big(\sum_{k=0}^{\infty}2^{ksq}x^{-\mu-\frac{1}{2}}\left|\frac{f_k(x,y)}{y^{\mu+\frac{1}{2}}}\right|^q\Big)^{\frac{p}{q}}dx$$
$$\leq \left(c_\mu^{-1}\right)^p\int_0^\infty\Big(\sum_{k=0}^{\infty}2^{ksq}\left|z^{-\mu-\frac{1}{2}}f_k(z)\right|^q\Big)^{\frac{p}{q}}z^{2\mu+1}dz.$$

Now, for $y \in \mathbb{R}^+$

$$\Big(\int_0^\infty x^{2\mu+1}\Big(\sum_{k=0}^\infty 2^{ksq} x^{-\mu-\frac{1}{2}} \sup_{y\in\mathbb{R}^+}\Big|\frac{f_k(x,y)}{1+(2^k y)^{\mu+\frac{1}{2}}}\Big|^q\Big)^{\frac{p}{q}} dx\Big)^{\frac{1}{p}}$$
$$\leq c_\mu^{-1}\Big(\int_0^\infty z^{2\mu+1}\Big(\sum_{k=0}^\infty 2^{ksq}\Big|z^{-\mu-\frac{1}{2}} f_k(z)\Big|^q\Big)^{\frac{p}{q}} dz\Big)^{\frac{1}{p}}.$$

Thus, we get

$$\Big\|2^{ks}x^{-\mu-\frac{1}{2}} \sup_{y\in\mathbb{R}^+}\frac{|f_k(x,y)|}{1+(2^k y)^{\mu+\frac{1}{2}}}\Big\|_{L_p^\mu(l_q)} \leq c_\mu^{-1}\Big\|2^{ks}x^{-\mu-\frac{1}{2}} f_k(x)\Big\|_{L_p^\mu(l_q)}$$

for $1 \leq p < \infty$ and $1 \leq q < \infty$. Taking $\mu + \frac{1}{2} = 2d$, we get

$$\Big\|2^{ks}x^{-\mu-\frac{1}{2}} \sup_{y\in\mathbb{R}^+}\frac{|f_k(x,y)|}{1+(2^k y)^{2d}}\Big\|_{L_p^\mu(l_q)} \leq c_\mu^{-1}\Big\|2^{ks}x^{-\mu-\frac{1}{2}} f_k(x)\Big\|_{L_p^\mu(l_q)}.$$

For $q = \infty$, from (4.2.8), we have

$$\sup_k 2^{ks}|f_k(x,y)| = \sup_k\Big|\int_0^\infty 2^{ks} f_k(z) D_\mu(x,y,z)dz\Big|$$
$$\leq \int_0^\infty 2^{ks}\sup_k\Big|z^{-\mu-\frac{1}{2}} f_k(z)\Big| z^{\mu+\frac{1}{2}} D_\mu(x,y,z)dz.$$

Now, proceeding as in the case of $1 \leq q < \infty$ we get

$$\Big\|2^{ks}x^{-\mu-\frac{1}{2}} \sup_{y\in\mathbb{R}^+}\frac{|f_k(x,y)|}{1+(2^k y)^{2d}}\Big\|_{L_p^\mu(l_\infty)} \leq c_\mu^{-1}\Big\|2^{ks}x^{-\mu-\frac{1}{2}} f_k(x)\Big\|_{L_p^\mu(l_\infty)}.$$

□

Theorem 4.2.8. *Let $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $d > 0$. Let $E = \{E_k\}_{k=0}^\infty$ is a sequence of compact subsets of $\mathbb{R}^+$, then*

$$\left\| 2^{ks} x^{-\mu-\frac{1}{2}} \sup_{y\in\mathbb{R}^+} \frac{|f_k(x,y)|}{1+(2^k y)^{2d}} \right\|_{l_q(L_p^\mu)} \leq c_\mu^{-1} \left\| 2^{ks} x^{-\mu-\frac{1}{2}} f_k(x) \right\|_{l_q(L_p^\mu)}$$

holds for all $f_k \in L_{p,\mu}^{E_k}$*, where* $f = \{f_k\}_{k=0}^{\infty}$*,* $c_\mu = 2^\mu \Gamma(\mu+1)$ *and for* $\mu \geq -\frac{1}{2}$

$$f_k(x,y) = \int_0^\infty f_k(z) D_\mu(x,y,z) dz$$

is the Hankel translation of $f_k(x)$.

Proof. From (1.4.4), we have

$$f_k(x,y) = \int_0^\infty f_k(z) D_\mu(x,y,z) dz. \tag{4.2.9}$$

Therefore

$$2^{ks} |f_k(x,y)| \leq \int_0^\infty \left| 2^{ks} z^{-\mu-\frac{1}{2}} f_k(z) \left(z^{\mu+\frac{1}{2}} D_\mu(x,y,z) \right)^{\frac{1}{p}} \left(z^{\mu+\frac{1}{2}} D_\mu(x,y,z) \right)^{\frac{1}{p'}} \right| dz$$

for $\frac{1}{p} + \frac{1}{p'} = 1$. Applying Holder's inequality, we get

$$2^{ks} |f_k(x,y)| \leq \left(\int_0^\infty \left| 2^{ks} z^{-\mu-\frac{1}{2}} f_k(z) \right|^p z^{\mu+\frac{1}{2}} D_\mu(x,y,z) dz \right)^{\frac{1}{p}} \left(\int_0^\infty z^{\mu+\frac{1}{2}} D_\mu(x,y,z) dz \right)^{\frac{1}{p'}}.$$

From (1.4.8), we find

$$2^{ks} |f_k(x,y)| \leq \left(\int_0^\infty \left| 2^{ks} z^{-\mu-\frac{1}{2}} f_k(z) \right|^p z^{\mu+\frac{1}{2}} D_\mu(x,y,z) dz \right)^{\frac{1}{p}} \left(c_\mu^{-1} (xy)^{\mu+\frac{1}{2}} \right)^{\frac{1}{p'}}.$$

This implies that

$$2^{ks} (xy)^{\left(\mu+\frac{1}{2}\right)\left(-\frac{p}{p'}\right)} |f_k(x,y)| \leq \left(c_\mu^{-1}\right)^{\frac{p}{p'}} \int_0^\infty \left| 2^{ks} z^{-\mu-\frac{1}{2}} f_k(z) \right|^p z^{\mu+\frac{1}{2}} D_\mu(x,y,z) dz.$$

Therefore, we have

$$\int_0^\infty 2^{ksp} y^{\left(\mu+\frac{1}{2}\right)\left(-\frac{p}{p'}\right)} x^{\left(\mu+\frac{1}{2}\right)\left(1-\frac{p}{p'}\right)} \left|f_k(x,y)\right|^p dx \\ \leq \left(c_\mu^{-1}\right)^{\frac{p}{p'}} \int_0^\infty \int_0^\infty \left|2^{ks} z^{-\mu-\frac{1}{2}} f_k(z)\right|^p z^{\mu+\frac{1}{2}} x^{\mu+\frac{1}{2}} D_\mu(x,y,z) dz dx.$$

In view of the Fubini's theorem, we find

$$\begin{aligned} \int_0^\infty & 2^{ksp} y^{\left(\mu+\frac{1}{2}\right)(1-p)} x^{\left(\mu+\frac{1}{2}\right)(2-p)} \left|f_k(x,y)\right|^p dx \\ &\leq \left(c_\mu^{-1}\right)^{\frac{p}{p'}} \int_0^\infty \left|2^{ks} z^{-\mu-\frac{1}{2}} f_k(z)\right|^p z^{\mu+\frac{1}{2}} \int_0^\infty x^{\mu+\frac{1}{2}} D_\mu(x,y,z) dx dz \\ &\leq \left(c_\mu^{-1}\right)^{\frac{p}{p'}} \int_0^\infty \left|2^{ks} z^{-\mu-\frac{1}{2}} f_k(z)\right|^p z^{\mu+\frac{1}{2}} c_\mu^{-1} (yz)^{\mu+\frac{1}{2}} dz \\ &\leq \left(c_\mu^{-1}\right)^{\left(1+\frac{p}{p'}\right)} \int_0^\infty \left|2^{ks} z^{-\mu-\frac{1}{2}} f_k(z)\right|^p z^{2\mu+1} y^{\mu+\frac{1}{2}} dz. \end{aligned}$$

Therefore

$$\int_0^\infty x^{2\mu+1} 2^{ksp} \left(x^{-\mu-\frac{1}{2}} \left|\frac{f_k(x,y)}{y^{\mu+\frac{1}{2}}}\right|\right)^p dx \leq \left(c_\mu^{-1}\right)^p \int_0^\infty z^{2\mu+1} 2^{ksp} \left|z^{-\mu-\frac{1}{2}} f_k(z)\right|^p dz,$$

so that, for $y \in \mathbb{R}^+$

$$\left(\int_0^\infty x^{2\mu+1} 2^{ksp} \left(x^{-\mu-\frac{1}{2}} \sup_{y\in\mathbb{R}^+} \left|\frac{f_k(x,y)}{1+(2^k y)^{\mu+\frac{1}{2}}}\right|\right)^p dx\right)^{\frac{1}{p}} \\ \leq c_\mu^{-1} \left(\int_0^\infty z^{2\mu+1} 2^{ksp} \left|z^{-\mu-\frac{1}{2}} f_k(z)\right|^p dz\right)^{\frac{1}{p}}.$$

Applying l_q- norm (1.7.4) and (1.7.5), we obtain

$$\left\|2^{ks} x^{-\mu-\frac{1}{2}} \sup_{y\in\mathbb{R}^+} \frac{|f_k(x,y)|}{1+(2^k y)^{\mu+\frac{1}{2}}}\right\|_{l_q(L_p^\mu)} \leq c_\mu^{-1} \left\|2^{ks} x^{-\mu-\frac{1}{2}} f_k(x)\right\|_{l_q(L_p^\mu)}$$

for $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Taking $\mu + \frac{1}{2} = 2d$, we get

$$\left\| 2^{ks} x^{-\mu-\frac{1}{2}} \sup_{y \in \mathbb{R}^+} \frac{|f_k(x,y)|}{1+(2^k y)^{2d}} \right\|_{l_q(L_p^\mu)} \leq c_\mu^{-1} \left\| 2^{ks} x^{-\mu-\frac{1}{2}} f_k(x) \right\|_{l_q(L_p^\mu)} .$$

$\square$

4.3 Boundedness of the Bessel wavelet transform in Triebel-Lizorkin and Besov type spaces

In this section, a multiplier theorem and other results are discussed by exploiting the Hankel transform theory. Using the multiplier theorem and other results, boundedness of the continuous Bessel wavelet transform in the Besov space $B_{p,q}^{\mu,s}$ and the Triebel-Lizorkin space $F_{p,q}^{\mu,s}$ are investigated.

Theorem 4.3.1. *Let $a > 0$ be fixed. Let $-\infty < s < \infty$ and N be a natural number*
(i) If $1 \leq q \leq \infty$, and $1 \leq p < \infty$ then there exists a positive number D such that

$$\left\| 2^{ks} x^{-\mu-\frac{1}{2}} \left(h_\mu^{-1}\theta_k\right) \# f \right\|_{L_p^\mu(l_q)} \leq DC_\mu(\theta) \left\| 2^{ks} x^{-\mu-\frac{1}{2}} \left(\phi_k^* f\right) \right\|_{L_p^\mu(l_q)} \leq c_\mu^{-1} DC_\mu(\theta) \left\| f \right\|_{F_{p,q}^{\mu,s}}$$

holds for all $\phi = \{\phi_k(x)\}_{k=0}^\infty \in \Phi(\mathbb{R}^+)$, $\theta = \{\theta_k(x)\}_{k=0}^\infty \in A_\mu^N(\mathbb{R}^+)$ and $f \in H'_\mu(\mathbb{R}^+)$.
(ii) If $1 \leq q \leq \infty$ and $1 \leq p < \infty$, then there exists a positive number D such that

$$\left\| 2^{ks} x^{-\mu-\frac{1}{2}} \left(h_\mu^{-1}\theta_k\right) \# f \right\|_{l_q(L_p^\mu)} \leq DC_\mu(\theta) \left\| 2^{ks} x^{-\mu-\frac{1}{2}} \left(\phi_k^* f\right) \right\|_{l_q(L_p^\mu)} \leq c_\mu^{-1} DC_\mu(\theta) \left\| f \right\|_{B_{p,q}^{\mu,s}}$$

holds for all $\phi = \{\phi_k(x)\}_{k=0}^\infty \in \Phi(\mathbb{R}^+)$, $\theta = \{\theta_k(x)\}_{k=0}^\infty \in A_\mu^N(\mathbb{R}^+)$ and $f \in H'_\mu(\mathbb{R}^+)$.

Proof. Let $\{\psi_k(x)\}_{k=0}^{\infty} \in A_\mu^N(\mathbb{R}^+)$ with

$$\begin{cases} \text{supp } x^{-\mu-\frac{1}{2}}\psi_0(x) \subset \{x : 0 \leq x \leq 4\}, \\ \text{supp } x^{-\mu-\frac{1}{2}}\psi_k(x) \subset \{x : 2^{k-2} \leq x \leq 2^{k+2}\}, \quad k = 1, 2, 3, \ldots \end{cases} \tag{4.3.1}$$

and

$$x^{-\mu-\frac{1}{2}}\psi_k(x) = 1, \text{ if } x \in \text{supp}\left(x^{-\mu-\frac{1}{2}}\phi_k\right), \quad k = 0, 1, 2, 3, \ldots. \tag{4.3.2}$$

Now for $\phi = \{\phi_k(x)\}_{k=0}^{\infty} \in \Phi(\mathbb{R}^+), \theta = \{\theta_k(x)\}_{k=0}^{\infty} \in A_\mu^N(\mathbb{R}^+)$ and $f \in H'_\mu(\mathbb{R}^+)$, we have

$$h_\mu\left[\left(h_\mu^{-1}\theta_k\right)\#f\right] = \eta^{-\mu-\frac{1}{2}}\theta_k h_\mu f.$$

This implies that

$$\left(h_\mu^{-1}\theta_k\right)\#f = h_\mu^{-1}\left[\eta^{-\mu-\frac{1}{2}}\theta_k h_\mu f\right].$$

From (4.2.6) and (4.3.2), we find

$$\begin{aligned}
\left(h_\mu^{-1}\theta_k\right)\#f &= h_\mu^{-1}\Big[\eta^{-\mu-\frac{1}{2}}\theta_k\Big(\sum_{l=0}^{\infty}\eta^{-\mu-\frac{1}{2}}\psi_l\ \eta^{-\mu-\frac{1}{2}}\phi_l\Big)h_\mu f\Big] \\
&= \sum_{l=0}^{\infty} h_\mu^{-1}\left[\eta^{-\mu-\frac{1}{2}}\eta^{-\mu-\frac{1}{2}}h_\mu h_\mu^{-1}\theta_k h_\mu h_\mu^{-1}\psi_l\ \eta^{-\mu-\frac{1}{2}}h_\mu h_\mu^{-1}\phi_l h_\mu f\right] \\
&= \sum_{l=0}^{\infty} h_\mu^{-1}\left[\eta^{-\mu-\frac{1}{2}}h_\mu\left(h_\mu^{-1}\theta_k\#h_\mu^{-1}\psi_l\right)h_\mu\left(h_\mu^{-1}\phi_l\#f\right)\right] \\
&= \sum_{l=0}^{\infty} h_\mu^{-1}\left[h_\mu\left(\left(h_\mu^{-1}\theta_k\#h_\mu^{-1}\psi_l\right)\#\left(h_\mu^{-1}\phi_l\#f\right)\right)\right] \\
&= \sum_{l=0}^{\infty}\left[\left(h_\mu^{-1}\theta_k\#h_\mu^{-1}\psi_l\right)\#\left(h_\mu^{-1}\phi_l\#f\right)\right].
\end{aligned}$$

By multiplying 2^{ks} and taking absolute value, the above expression becomes

$$\begin{aligned}\left|2^{ks}\left(h_\mu^{-1}\theta_k\#f\right)(x)\right| &\le \sum_{l=0}^{\infty} 2^{ks}\left|\left(h_\mu^{-1}\theta_k\#h_\mu^{-1}\psi_l\right)\#\left(h_\mu^{-1}\phi_l\#f\right)(x)\right| \\ &\le \sum_{l=0}^{\infty} 2^{ks}\int_0^\infty \left|\left(h_\mu^{-1}\theta_k\#h_\mu^{-1}\psi_l\right)(y)\right|\left|\left(h_\mu^{-1}\phi_l\#f\right)(x,y)\right| dy.\end{aligned}$$

From (4.2.3), we have

$$\begin{aligned}\left|2^{ks}\left(h_\mu^{-1}\theta_k\#f\right)(x)\right| &\le \sum_{l=0}^{\infty} 2^{ks}\int_0^\infty \left|\left(h_\mu^{-1}\theta_k\#h_\mu^{-1}\psi_l\right)(y)\right|\left|\left(1+\left(2^l y\right)^{2d}\right)\left(\phi_l^* f\right)(x)\right| dy \\ &= \sum_{l=0}^{\infty} 2^{ks}\left(\phi_l^* f\right)(x)\int_0^\infty \left|\left(h_\mu^{-1}\theta_k\#h_\mu^{-1}\psi_l\right)(y)\right|\left(1+\left(2^l y\right)^{2d}\right) dy.\end{aligned}$$

This implies that

$$\left|2^{ks}\left(h_\mu^{-1}\theta_k\#f\right)(x)\right| \le \sum_{l=0}^{\infty} 2^{ks}\left(\phi_l^* f\right)(x)\int_0^\infty \left|\left(h_\mu^{-1}\theta_k\#h_\mu^{-1}\psi_l\right)(y)\right|\left(1+\left(2^l y\right)^{2d}\right) dy. \tag{4.3.3}$$

Now, we take

$$I_{k,l} = \int_0^\infty \left|\left(h_\mu^{-1}\theta_k\#h_\mu^{-1}\psi_l\right)(y)\right|\left(1+\left(2^l y\right)^{2d}\right) dy.$$

Putting $y = 2^{-k}z$ in the above expression, we get

$$\begin{aligned}I_{k,l} &= 2^{-k}\int_0^\infty \left|\left(h_\mu^{-1}\theta_k\#h_\mu^{-1}\psi_l\right)\left(2^{-k}z\right)\right|\left(1+\left(2^{l-k}z\right)^{2d}\right) dz \\ &\le 2^{-k+2d|l-k|}\int_0^\infty \left|\left(h_\mu^{-1}\theta_k\#h_\mu^{-1}\psi_l\right)\left(2^{-k}z\right)\right|\left(1+z^{2d}\right) dz \\ &\le 2^{-k+2d|l-k|}\int_0^\infty \left|\left(h_\mu^{-1}\theta_k\#h_\mu^{-1}\psi_l\right)\left(2^{-k}z\right)\right|\left(1+z^2\right)^d dz.\end{aligned}$$

Using Holder's inequality, we obtain

$$I_{k,l} \leq 2^{-k+2d|l-k|} \left\| \left(1+z^2\right)^{d+\frac{1}{2}} \left(h_\mu^{-1}\theta_k \# h_\mu^{-1}\psi_l\right) \left(2^{-k}z\right) \right\|_2 \left\| \left(1+z^2\right)^{-\frac{1}{2}} \right\|_2 .$$

In view of Lemma 4.2.6 and putting $\rho = E\left(d+\frac{1}{2}\right)+1$, we find

$$I_{k,l} \leq \sqrt{\frac{\pi}{2}}\, 2^{-k+2d|l-k|} 2^k 2^{-k(\mu+\frac{1}{2})} \left\| \left(1+z^2\right)^\rho \left(h_\mu^{-1}\left(\theta_k(2^k\cdot)\right) \# h_\mu^{-1}\left(\psi_l(2^k\cdot)\right)\right) \right\|_2 .$$

From (1.4.9) and (1.4.12), the above expression can be written as

$$I_{k,l} \leq \sqrt{\frac{\pi}{2}}\, 2^{2d|l-k|-k(\mu+\frac{1}{2})} \left\| h_\mu \left\{ \left(1+z^2\right)^\rho h_\mu^{-1} \left(\eta^{-\mu-\frac{1}{2}}\theta_k(2^k\eta)\psi_l(2^k\eta)\right) \right\} \right\|_2 .$$

From (1.7.8), we have

$$I_{k,l} \leq \sqrt{\frac{\pi}{2}}\, 2^{2d|l-k|-k(\mu+\frac{1}{2})} \left\| (1-S_{\mu,\eta})^\rho \left(\eta^{-\mu-\frac{1}{2}}\theta_k(2^k\eta)\psi_l(2^k\eta)\right) \right\|_2 . \tag{4.3.4}$$

The expression in the norm on right hand side of (4.3.4) will be

$$\begin{aligned}
&\left\| (1-S_{\mu,\eta})^\rho \left(\eta^{-\mu-\frac{1}{2}}\theta_k(2^k\eta)\psi_l(2^k\eta)\right) \right\|_2 \\
&\quad = \left\| \sum_{r=0}^{\rho} \binom{\rho}{r} (-1)^r S_{\mu,\eta}^r \left(\eta^{-\mu-\frac{1}{2}}\theta_k(2^k\eta)\psi_l(2^k\eta)\right) \right\|_2 \\
&\quad = \left\| \sum_{r=0}^{\rho} \binom{\rho}{r} \sum_{j=0}^{r} A_j \eta^{2j+\mu+\frac{1}{2}} \left(\eta^{-1}D_\eta\right)^{r+j} \left(\eta^{-\mu-\frac{1}{2}}\theta_k(2^k\eta)\eta^{-\mu-\frac{1}{2}}\psi_l(2^k\eta)\right) \right\|_2 \\
&\quad = \left\| \sum_{r=0}^{\rho} \binom{\rho}{r} \sum_{j=0}^{r} A_j \sum_{m=0}^{r+j} \binom{r+j}{m} \eta^{2j+\mu+\frac{1}{2}} \left(\eta^{-1}D_\eta\right)^m \left(\eta^{-\mu-\frac{1}{2}}\theta_k(2^k\eta)\right) \right. \\
&\qquad\qquad \left. \times \left(\eta^{-1}D_\eta\right)^{r+j-m} \left(\eta^{-\mu-\frac{1}{2}}\psi_l(2^k\eta)\right) \right\|_2
\end{aligned}$$

$$\leq \sum_{r=0}^{\rho}\binom{\rho}{r}\sum_{j=0}^{r}A_j\sum_{m=0}^{r+j}\binom{r+j}{m}\Big\|\left(\eta^{-1}D_\eta\right)^m\left(\eta^{-\mu-\frac{1}{2}}\theta_k(2^k\eta)\right)\eta^{2j+\mu+\frac{1}{2}} \times \left(\eta^{-1}D_\eta\right)^{r+j-m}\left(\eta^{-\mu-\frac{1}{2}}\psi_l(2^k\eta)\right)\Big\|_2$$

$$= \sum_{r=0}^{\rho}\binom{\rho}{r}\sum_{j=0}^{r}A_j\sum_{m=0}^{r+j}\binom{r+j}{m}\Big(\int_0^\infty\left|\left(\eta^{-1}D_\eta\right)^m\left(\eta^{-\mu-\frac{1}{2}}\theta_k(2^k\eta)\right)\right|^2 \times \left|\eta^{2j+\mu+\frac{1}{2}}\left(\eta^{-1}D_\eta\right)^{r+j-m}\left(\eta^{-\mu-\frac{1}{2}}\psi_l(2^k\eta)\right)\right|^2 d\eta\Big)^{\frac{1}{2}}.$$

From (4.3.1), $\psi_l(2^k\eta)$ vanishes outside the set

$$\left\{\eta : 2^{l-k-2} \leq \eta \leq 2^{l-k+2}\right\},$$

and $\psi_0(2^k\eta)$ vanishes outside the set

$$\left\{\eta : 0 \leq \eta \leq 2^{-k+2}\right\}.$$

So, if η belongs to that set then from (4.2.1), we have

$$\left|\left(\eta^{-1}D_\eta\right)^m\left(\eta^{-\mu-\frac{1}{2}}\theta_k(2^k\eta)\right)\right| \leq C_1 2^{-N|l-k|}C_\mu(\theta), \tag{4.3.5}$$

where C_1 is a constant and N is a natural number.

Using (4.3.5) and for $N \geq 2\rho$, we find that

$$\left\|(1-S_{\mu,\eta})^\rho\left(\eta^{-\mu-\frac{1}{2}}\theta_k(2^k\eta)\psi_l(2^k\eta)\right)\right\|_2$$
$$\leq C_1 2^{-N|l-k|}C_\mu(\theta)\sum_{r=0}^{\rho}\binom{\rho}{r}\sum_{j=0}^{r}A_j\sum_{m=0}^{r+j}\binom{r+j}{m} \times \left(\int_0^\infty\left|\eta^{2j+\mu+\frac{1}{2}}\left(\eta^{-1}D_\eta\right)^{r+j-m}\left(\eta^{-\mu-\frac{1}{2}}\psi_l(2^k\eta)\right)\right|^2 d\eta\right)^{\frac{1}{2}}.$$

Now, putting $\eta = 2^{l-k}t$ we get

$$
\begin{aligned}
&\left\|(1-S_{\mu,\eta})^{\rho}\left(\eta^{-\mu-\frac{1}{2}}\theta_k(2^k\eta)\psi_l(2^k\eta)\right)\right\|_2 \\
&\quad\leq C_1 2^{-N|l-k|}C_\mu(\theta)\sum_{r=0}^{\rho}\binom{\rho}{r}\sum_{j=0}^{r}A_j\sum_{m=0}^{r+j}\binom{r+j}{m}2^{(l-k)(2m-2r+\frac{1}{2})} \\
&\qquad\qquad\times\left(\int_0^\infty\left|t^{2j+\mu+\frac{1}{2}}\left(t^{-1}D_t\right)^{r+j-m}\left(t^{-\mu-\frac{1}{2}}\psi_l(2^l t)\right)\right|^2 dt\right)^{\frac{1}{2}} \\
&\leq C_1 2^{-N|l-k|}C_\mu(\theta)2^{(2\rho+\frac{1}{2})|l-k|} \\
&\qquad\times\sum_{r=0}^{\rho}\binom{\rho}{r}\sum_{j=0}^{r}A_j\sum_{m=0}^{r+j}\binom{r+j}{m}\left\|t^{2j+\mu+\frac{1}{2}}\left(t^{-1}D_t\right)^{r+j-m}\left(t^{-\mu-\frac{1}{2}}\psi_l(2^l t)\right)\right\|_2.
\end{aligned}
$$

Exploiting (4.3.1), we have

$$
\begin{aligned}
\left\|(1-S_{\mu,\eta})^{\rho}\left(\eta^{-\mu-\frac{1}{2}}\theta_k(2^k\eta)\psi_l(2^k\eta)\right)\right\|_2 &\leq C_1 2^{(-N+2\rho+\frac{1}{2})|l-k|}C_\mu(\theta)C_2C_3 \\
&\leq C_1C_2C_3 2^{(-N+2\rho+\frac{1}{2})|l-k|}C_\mu(\theta).
\end{aligned} \tag{4.3.6}
$$

Using (4.3.4) and (4.3.6), we find the estimate of $I_{k,l}$

$$
I_{k,l} \leq \sqrt{\frac{\pi}{2}}\,C_1C_2C_3C_\mu(\theta)2^{(2d+2\rho+\frac{1}{2}-N)|l-k|-k(\mu+\frac{1}{2})}. \tag{4.3.7}
$$

From (4.3.3) and (4.3.7), finally we get

$$
\begin{aligned}
&\left|2^{ks}\left(h_\mu^{-1}\theta_k\#f\right)(x)\right| \\
&\quad\leq\sum_{l=0}^{\infty}2^{s(k-l)}2^{ls}\left(\phi_l^*f\right)(x)\sqrt{\frac{\pi}{2}}\,C_1C_2C_3C_\mu(\theta)2^{(2d+2\rho+\frac{1}{2}-N)|l-k|-k(\mu+\frac{1}{2})} \\
&\quad\leq\sqrt{\frac{\pi}{2}}\,C_1C_2C_3C_\mu(\theta)\sum_{l=0}^{\infty}2^{(|s|+2d+2\rho+\frac{1}{2}-N)|l-k|-k(\mu+\frac{1}{2})}2^{ls}\left(\phi_l^*f\right)(x) \\
&\quad\leq CC_\mu(\theta)\sum_{l=0}^{\infty}2^{(|s|+2d+2\rho+\frac{1}{2}+\mu+\frac{1}{2}-N)|l-k|}2^{-l(\mu+\frac{1}{2})}2^{ls}\left(\phi_l^*f\right)(x),
\end{aligned}
$$

where $C = \sqrt{\frac{\pi}{2}}\ C_1C_2C_3$ is independent of k, l, x, f, θ and ϕ.

Taking $\sigma = |s| + 2d + 2\rho + \mu + 1$, we have

$$\left|2^{ks}\left(h_\mu^{-1}\theta_k\#f\right)(x)\right| \le CC_\mu(\theta)\sum_{l=0}^{\infty} 2^{(\sigma-N)|l-k|}2^{-l(\mu+\frac{1}{2})}2^{ls}\left(\phi_l^* f\right)(x). \tag{4.3.8}$$

For $N > \sigma$, from (4.3.8)

$$\left|2^{ks}\left(h_\mu^{-1}\theta_k\#f\right)(x)\right| \le CC_\mu(\theta)C'\Big(\sum_{l=0}^{\infty} 2^{(\sigma+\epsilon-N)|l-k|q}2^{-l(\mu+\frac{1}{2})q}2^{lsq}\left(\phi_l^* f\right)^q(x)\Big)^{\frac{1}{q}}$$

with $\epsilon > 0$ and $N > \sigma + \epsilon$. On taking l_q- norm on both sides, we obtain

$$\begin{aligned}
&\Big(\sum_{k=0}^{\infty}\left|2^{ks}\left(h_\mu^{-1}\theta_k\#f\right)(x)\right|^q\Big)^{\frac{1}{q}}\\
&\quad\le CC_\mu(\theta)C'\Big(\sum_{k=0}^{\infty}\Big|\sum_{l=0}^{\infty} 2^{(\sigma+\epsilon-N)|l-k|q}2^{-l(\mu+\frac{1}{2})q}2^{lsq}\left(\phi_l^* f\right)^q(x)\Big|^{\frac{1}{q}\times q}\Big)^{\frac{1}{q}}\\
&\quad\le CC_\mu(\theta)C'\Big(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} 2^{(\sigma+\epsilon-N)|l-k|q}2^{-l(\mu+\frac{1}{2})q}\left|2^{ls}\left(\phi_l^* f\right)(x)\right|^q\Big)^{\frac{1}{q}}\\
&\quad\le CC_\mu(\theta)C'C''\Big(\sum_{l=0}^{\infty}\left|2^{ls}\left(\phi_l^* f\right)(x)\right|^q\Big)^{\frac{1}{q}}.
\end{aligned}$$

Therefore

$$\left\|2^{ks}x^{-\mu-\frac{1}{2}}\left(h_\mu^{-1}\theta_k\#f\right)(x)\right\|_{L_p^\mu(l_q)} \le DC_\mu(\theta)\left\|2^{ks}x^{-\mu-\frac{1}{2}}\left(\phi_k^* f\right)(x)\right\|_{L_p^\mu(l_q)}, \tag{4.3.9}$$

with $N > |s| + \mu + 4d + 4,\ d > 0$ and $D = CC'C''$ for $1 \le q < \infty$. From Theorem 4.2.7 , taking $f_k = \left(h_\mu^{-1}\phi_k\right)\#f$, we have

$$\left\|2^{ks}x^{-\mu-\frac{1}{2}}\left(\phi_k^* f\right)(x)\right\|_{L_p^\mu(l_q)} \le c_\mu^{-1}\left\|2^{ks}x^{-\mu-\frac{1}{2}}\left(h_\mu^{-1}\phi_k\#f\right)(x)\right\|_{L_p^\mu(l_q)} = c_\mu^{-1}\left\|f\right\|_{F_{p,q}^{\mu,s}}. \tag{4.3.10}$$

From (4.3.9) and (4.3.10), we get the result **(i)** of Theorem 4.3.1:

$$\begin{aligned}\left\|2^{ks}x^{-\mu-\frac{1}{2}}\left(h_\mu^{-1}\theta_k\#f\right)(x)\right\|_{L_p^\mu(l_q)} &\le DC_\mu(\theta)\left\|2^{ks}x^{-\mu-\frac{1}{2}}\left(\phi_k^*f\right)(x)\right\|_{L_p^\mu(l_q)}\\ &\le c_\mu^{-1}DC_\mu(\theta)\left\|f\right\|_{F_{p,q}^{\mu,s}}\end{aligned}$$

with $N > |s| + \mu + 4d + 4,\ d > 0$.

Remark 4.3.2. In case of $q = \infty$; for $N > \sigma$, from (4.3.8) we have

$$\left|2^{ks}\left(h_\mu^{-1}\theta_k\#f\right)(x)\right| \le CC_\mu(\theta)\Big\{\sup_l 2^{-l(\mu+\frac{1}{2})}2^{ls}\left(\phi_l^*f\right)(x)\Big\}\sum_{l=0}^{\infty}2^{(\sigma-N)|l-k|}.$$

Now, taking l_∞-norm, we get

$$\begin{aligned}\sup_k\left|2^{ks}\left(h_\mu^{-1}\theta_k\#f\right)(x)\right| &\le CC_\mu(\theta)\Big\{\sup_l 2^{-l(\mu+\frac{1}{2})}2^{ls}\left(\phi_l^*f\right)(x)\Big\}\sup_k\sum_{l=0}^{\infty}2^{(\sigma-N)|l-k|}\\ &\le CC_\mu(\theta)\Big\{\sup_l 2^{ls}\left(\phi_l^*f\right)(x)\Big\}C_1.\end{aligned}$$

Therefore,

$$\left\|2^{ks}x^{-\mu-\frac{1}{2}}\left(h_\mu^{-1}\theta_k\#f\right)(x)\right\|_{L_p^\mu(l_\infty)} \le DC_\mu(\theta)\left\|2^{ks}x^{-\mu-\frac{1}{2}}\left(\phi_k^*f\right)(x)\right\|_{L_p^\mu(l_\infty)} \tag{4.3.11}$$

with $N > |s| + \mu + 4d + 4,\ d > 0$ and $D = CC_1$.

Now, to prove Theorem 4.3.1**(ii)**, we take $N > \sigma$. Then from (4.3.8), we have

$$\left|2^{ks}x^{-\mu-\frac{1}{2}}\left(h_\mu^{-1}\theta_k\#f\right)(x)\right| \le CC_\mu(\theta)\sum_{l=0}^{\infty}2^{(\sigma-N)|l-k|}2^{-l(\mu+\frac{1}{2})}2^{ls}x^{-\mu-\frac{1}{2}}\left(\phi_l^*f\right)(x).$$

Using the Minkowski's inequality, we obtain

$$\left\|2^{ks}x^{-\mu-\frac{1}{2}}\left(h_\mu^{-1}\theta_k\#f\right)(x)\right\|_{L_p^\mu} \leq CC_\mu(\theta)\sum_{l=0}^{\infty}2^{(\sigma-N)|l-k|}2^{-l(\mu+\frac{1}{2})}\left\|2^{ls}x^{-\mu-\frac{1}{2}}\left(\phi_l^*f\right)(x)\right\|_{L_p^\mu}. \tag{4.3.12}$$

From (4.3.12), we get

$$\begin{aligned}&\left\|2^{ks}x^{-\mu-\frac{1}{2}}\left(h_\mu^{-1}\theta_k\#f\right)(x)\right\|_{L_p^\mu}\\ &\qquad\leq CC_\mu(\theta)C'\Bigg(\sum_{l=0}^{\infty}2^{(\sigma+\epsilon-N)|l-k|q}2^{-l(\mu+\frac{1}{2})q}\left\|2^{ls}x^{-\mu-\frac{1}{2}}\left(\phi_l^*f\right)(x)\right\|_{L_p^\mu}^q\Bigg)^{\frac{1}{q}}\end{aligned}$$

with $\epsilon > 0$ and $N > \sigma + \epsilon$. Therefore

$$\begin{aligned}&\Bigg(\sum_{k=0}^{\infty}\left\|2^{ks}x^{-\mu-\frac{1}{2}}(h_\mu^{-1}\theta_k\#f)(x)\right\|_{L_p^\mu}^q\Bigg)^{\frac{1}{q}}\\ &\quad\leq CC_\mu(\theta)C'\Bigg(\sum_{k=0}^{\infty}\Bigg|\sum_{l=0}^{\infty}2^{(\sigma+\epsilon-N)|l-k|q}2^{-l(\mu+\frac{1}{2})q}\left\|2^{ls}x^{-\mu-\frac{1}{2}}\left(\phi_l^*f\right)(x)\right\|_{L_p^\mu}^q\Bigg|^{\frac{1}{q}\times q}\Bigg)^{\frac{1}{q}}\\ &\quad\leq CC_\mu(\theta)C'\Bigg(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}2^{(\sigma+\epsilon-N)|l-k|q}2^{-l(\mu+\frac{1}{2})q}\left\|2^{ls}x^{-\mu-\frac{1}{2}}\left(\phi_l^*f\right)(x)\right\|_{L_p^\mu}^q\Bigg)^{\frac{1}{q}}\\ &\quad\leq CC_\mu(\theta)C'C''\Bigg(\sum_{l=0}^{\infty}\left\|2^{ls}x^{-\mu-\frac{1}{2}}\left(\phi_l^*f\right)(x)\right\|_{L_p^\mu}^q\Bigg)^{\frac{1}{q}}.\end{aligned}$$

That is,

$$\left\|2^{ks}x^{-\mu-\frac{1}{2}}(h_\mu^{-1}\theta_k\#f)(x)\right\|_{l_q(L_p^\mu)} \leq DC_\mu(\theta)\left\|2^{ks}x^{-\mu-\frac{1}{2}}\left(\phi_k^*f\right)(x)\right\|_{l_q(L_p^\mu)} \tag{4.3.13}$$

with $N > |s| + \mu + 4d + 4,\ d > 0$ and $D = CC'C''$ for $1 \leq q < \infty$.

From Theorem 4.2.8, taking $f_k = (h_\mu^{-1}\phi_k)\#f$, we have

$$\left\|2^{ks}x^{-\mu-\frac{1}{2}}(\phi_k^* f)(x)\right\|_{l_q(L_p^\mu)} \leq c_\mu^{-1}\left\|2^{ks}x^{-\mu-\frac{1}{2}}(h_\mu^{-1}\phi_k\#f)(x)\right\|_{l_q(L_p^\mu)} = c_\mu^{-1}\left\|f\right\|_{B_{p,q}^{\mu,s}}. \tag{4.3.14}$$

From (4.3.13) and (4.3.14), we get the result **(ii)** of Theorem 4.3.1:

$$\begin{aligned}\left\|2^{ks}x^{-\mu-\frac{1}{2}}(h_\mu^{-1}\theta_k\#f)(x)\right\|_{l_q(L_p^\mu)} &\leq DC_\mu(\theta)\left\|2^{ks}x^{-\mu-\frac{1}{2}}(\phi_k^* f)(x)\right\|_{l_q(L_p^\mu)} \\ &\leq c_\mu^{-1}DC_\mu(\theta)\left\|f\right\|_{B_{p,q}^{\mu,s}}\end{aligned}$$

with $N > |s| + \mu + 4d + 4,\ d > 0.$ □

Remark 4.3.3. In case of $q = \infty$; for $N > \sigma$, from (4.3.12) we find

$$\begin{aligned}&\sup_k \left\|2^{ks}x^{-\mu-\frac{1}{2}}\left(h_\mu^{-1}\theta_k\#f\right)(x)\right\|_{L_p^\mu} \\ &\quad\leq CC_\mu(\theta)\left\{\sup_l 2^{-l(\mu+\frac{1}{2})}\left\|2^{ls}x^{-\mu-\frac{1}{2}}\left(\phi_l^* f\right)(x)|\right\|_{L_p^\mu}\right\}\sup_k\sum_{l=0}^{\infty}2^{(\sigma-N)|l-k|} \\ &\quad\leq CC_\mu(\theta)\left\{\sup_l\left\|2^{ls}x^{-\mu-\frac{1}{2}}\left(\phi_l^* f\right)(x)|\right\|_{L_p^\mu}\right\}C_1.\end{aligned}$$

Therefore

$$\left\|2^{ks}x^{-\mu-\frac{1}{2}}(h_\mu^{-1}\theta_k\#f)(x)\right\|_{l_\infty(L_p^\mu)} \leq DC_\mu(\theta)\left\|2^{ks}x^{-\mu-\frac{1}{2}}\left(\phi_k^* f\right)(x)\right\|_{l_\infty(L_p^\mu)} \tag{4.3.15}$$

with $N > |s| + \mu + 4d + 4,\ d > 0$ and $D = CC_1$.

Theorem 4.3.4. *Let $m(x)$ be a complex-valued function in $C^\infty(\mathbb{R}^+)$ such that*

$$\|m\|_N = \sup_{\alpha\leq N}\sup_{x\in\mathbb{R}^+}\left(1+x^2\right)^{\frac{\alpha}{2}}\left|\left(x^{-1}D_x\right)^\alpha x^{-\mu-\frac{1}{2}}m(x)\right| < \infty, \qquad N\in\mathbb{N}_0. \tag{4.3.16}$$

Let $-\infty < s < \infty$, $1 \leq q \leq \infty$ *and* $A(\mathbb{R}^+)$ *be either* $F^{\mu,s}_{p,q}(\mathbb{R}^+)$ *with* $1 \leq p < \infty$ *or* $B^{\mu,s}_{p,q}(\mathbb{R}^+)$ *with* $1 \leq p < \infty$*. Then there exists a positive number* C *such that*

$$\left\| \left(h_\mu^{-1} m\right) \# f \right\|_A \leq C \left\| m \right\|_N \left\| f \right\|_A$$

for all $m \in C^\infty(\mathbb{R}^+)$ *and* $f \in A$ *with* $N > |s| + \mu + 4d + 4,\ d > 0$.

Proof. We take $\phi = \{\phi_k(x)\}_{k=0}^\infty \in \Phi(\mathbb{R}^+), m \in C^\infty(\mathbb{R}^+)$ and $f \in A\big(= F^{\mu,s}_{p,q}(\mathbb{R}^+)$ or $B^{\mu,s}_{p,q}(\mathbb{R}^+)\big)$. From Definition 4.2.4 (i) and (1.4.9), we have

$$\begin{aligned} \left\| \left(h_\mu^{-1} m\right) \# f \right\|_{F^{\mu,s}_{p,q}} &= \left\| 2^{ks} x^{-\mu-\frac{1}{2}} h_\mu^{-1} \phi_k \# \left(h_\mu^{-1} m \# f\right) \right\|_{L^\mu_p(l_q)} \\ &= \left\| 2^{ks} x^{-\mu-\frac{1}{2}} \left(h_\mu^{-1} \phi_k \# h_\mu^{-1} m\right) \# f \right\|_{L^\mu_p(l_q)} \\ &= \left\| 2^{ks} x^{-\mu-\frac{1}{2}} \left(h_\mu^{-1} \left(\eta^{-\mu-\frac{1}{2}} m \phi_k\right)\right) \# f \right\|_{L^\mu_p(l_q)}. \end{aligned} \tag{4.3.17}$$

Similarily, from Definition 4.2.4 (ii) and (1.4.9)

$$\left\| \left(h_\mu^{-1} m\right) \# f \right\|_{B^{\mu,s}_{p,q}} = \left\| 2^{ks} x^{-\mu-\frac{1}{2}} \left(h_\mu^{-1} \left(\eta^{-\mu-\frac{1}{2}} m \phi_k\right)\right) \# f \right\|_{l_q(L^\mu_p)}. \tag{4.3.18}$$

From (4.3.17), (4.3.18) and Theorem 4.3.1 with $\theta = \{\theta_k(x)\}_{k=0}^\infty$ and $\theta_k = \eta^{-\mu-\frac{1}{2}} m \phi_k$, we get

$$\left\| \left(h_\mu^{-1} m\right) \# f \right\|_{F^{\mu,s}_{p,q}} \leq c_\mu^{-1} D C_\mu(\theta) \left\| f \right\|_{F^{\mu,s}_{p,q}} \text{ and } \left\| \left(h_\mu^{-1} m\right) \# f \right\|_{B^{\mu,s}_{p,q}} \leq c_\mu^{-1} D C_\mu(\theta) \left\| f \right\|_{B^{\mu,s}_{p,q}},$$

respectively for $N > |s| + \mu + 4d + 4,\ d > 0$. From (4.2.1) and (4.3.16), we have

$$C_\mu(\theta) \leq C_1 \left\| m \right\|_N,$$

where C_1 is a constant.

Thus, for $N > |s| + \mu + 4d + 4,\ d > 0$, we get

$$\left\| \left(h_\mu^{-1} m\right) \# f \right\|_A \leq C \, \|m\|_N \, \|f\|_A \, ,$$

where $C = c_\mu^{-1} D C_1$ and $A \left(= F_{p,q}^{\mu,s}(\mathbb{R}^+) \text{ or } B_{p,q}^{\mu,s}(\mathbb{R}^+)\right)$. □

Theorem 4.3.5. *Let $-\infty < s < \infty$, $\mu \geq -\frac{1}{2}$ and $1 \leq q \leq \infty$. Let $f \in A(\mathbb{R}^+)$, where A denotes $F_{p,q}^{\mu,s}(\mathbb{R}^+)$ with $1 \leq p < \infty$ or $B_{p,q}^{\mu,s}(\mathbb{R}^+)$ with $1 \leq p < \infty$. Let $N \in \mathbb{N}$ such that $N > |s| + \mu + 4d + 4$, $d > 0$. Assume that $h_\mu \psi \in C^\infty(\mathbb{R}^+)$ and (3.2.5) holds. Then there exists a positive number C such that*

$$\|B_\psi f\|_A \leq C \left(1 + a^2\right)^N a^{\mu + \frac{1}{2}} \sup_{\alpha \leq N} \|h_\mu \psi\|_{\mu,\alpha}^{m_1,\rho} \, \|f\|_A \tag{4.3.19}$$

for $m_1 + (1-\rho)N \leq 0$; $m_1 \in \mathbb{R}$, $\alpha \in \mathbb{N}_0$ and $0 \leq \rho \leq 1$.

Proof. Assume that

$$m(\omega) = (h_\mu \psi)(a\omega)$$

for $\omega \in \mathbb{R}^+$ and $a > 0$. Then

$$\|m\|_N = \sup_{\alpha \leq N} \sup_{\omega \in \mathbb{R}^+} \left(1 + \omega^2\right)^{\frac{\alpha}{2}} \left| \left(\omega^{-1} D_\omega\right)^\alpha \omega^{-\mu - \frac{1}{2}} \left(h_\mu \psi\right)(a\omega) \right| .$$

Putting $a\omega = u$, we get

$$\begin{aligned} \|m\|_N &= \sup_{\alpha \leq N} \sup_{u \in \mathbb{R}^+} \left(1 + \left(\frac{u}{a}\right)^2\right)^{\frac{\alpha}{2}} \left| a^{2\alpha + \mu + \frac{1}{2}} \left(u^{-1} D_u\right)^\alpha u^{-\mu - \frac{1}{2}} \left(h_\mu \psi\right)(u) \right| \\ &= \sup_{\alpha \leq N} \sup_{u \in \mathbb{R}^+} \left(a^2 + u^2\right)^{\frac{\alpha}{2}} a^{\alpha + \mu + \frac{1}{2}} \left| \left(u^{-1} D_u\right)^\alpha u^{-\mu - \frac{1}{2}} \left(h_\mu \psi\right)(u) \right| \end{aligned}$$

$$\leq \left(1+a^2\right)^{\frac{N}{2}}\left(1+a^2\right)^{\frac{N}{2}} a^{\mu+\frac{1}{2}} \sup_{\alpha\leq N}\sup_{u\in\mathbb{R}^+}\left(1+u^2\right)^{\frac{\alpha}{2}}\left|\left(u^{-1}D_u\right)^{\alpha} u^{-\mu-\frac{1}{2}}\left(h_\mu\psi\right)(u)\right|$$

$$\leq \left(1+a^2\right)^{N} a^{\mu+\frac{1}{2}}$$

$$\times \sup_{\alpha\leq N}\sup_{u\in\mathbb{R}^+}\frac{(1+u)^{\alpha}}{(1+u)^{-m_1+\rho\alpha}}\left[(1+u)^{-m_1+\rho\alpha}\left|\left(u^{-1}D_u\right)^{\alpha} u^{-\mu-\frac{1}{2}}\left(h_\mu\psi\right)(u)\right|\right]$$

$$\leq \left(1+a^2\right)^{N} a^{\mu+\frac{1}{2}} \sup_{\alpha\leq N}\|h_\mu\psi\|_{\mu,\alpha}^{m_1,\rho}$$

for $m_1+(1-\rho)N\leq 0$. From Theorem 4.3.4 and (1.5.4), we have

$$\begin{aligned}\|B_\psi f\|_A &= \left\|h_\mu^{-1}\{(h_\mu\psi)(a\cdot)\}\# f\right\|_A\\ &= \left\|h_\mu^{-1}m\# f\right\|_A \leq C\|m\|_N\|f\|_A \leq C\left(1+a^2\right)^N a^{\mu+\frac{1}{2}}\sup_{\alpha\leq N}\|h_\mu\psi\|_{\mu,\alpha}^{m_1,\rho}\|f\|_A\end{aligned}$$

for $m_1+(1-\rho)N\leq 0$ and $N>|s|+\mu+4d+4,\ d>0.$ □

4.4 Conclusions

Based on the results of Betancor et al. [13, 14], Triebel [50] and Pathak [46] and observations of the previous chapter, author extended the theory of Bessel wavelet transform in Besov and Triebel-Lizorkin type spaces and found a multiplier theorem and boundedness of the Bessel wavelet transform in Besov and Triebel-Lizorkin type spaces. These results are useful to find the applications in partial differential equations and other areas of mathematical sciences.

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