Chapter 2

Fractal Dimension of Univariate Vector-Valued Functions

Determining the fractal dimension of the graph of a function is not easy, even for real-valued functions. In this chapter, our focus is on examining the fractal dimension of univariate vector-valued functions. Specifically, we aim to approximate the fractal dimension of the Katugampola fractional integral of a vector-valued continuous function with bounded variation, defined on a closed and bounded interval in the real number system.

2.1 Introduction

Let $[a, b] \subset \mathbb{R}$ be an interval of real number, then the map $\boldsymbol{f} : [a, b] \to \mathbb{R}^n$ such that

$$\boldsymbol{f}(x) = (f_1(x), \dots, f_n(x)),$$

where $f_i : [a, b] \to \mathbb{R}$ is a real-valued map on interval $[a, b]$ for each $i \in \Sigma_n$, is known as a vector-valued map and f_i is known as coordinate map of $\boldsymbol{f}$. Katugampola fractional integral for an integrable vector-valued function $\boldsymbol{f} : [a, b] \to \mathbb{R}^n$ is given by

$${}^{\rho}_{a}\mathfrak{J}^{\mu}\boldsymbol{f}(x) = \Big({}^{\rho}_{a}\mathfrak{J}^{\mu} f_1(x), \dots, {}^{\rho}_{a}\mathfrak{J}^{\mu} f_n(x)\Big),$$

where

$$^{\rho}_{a}\mathfrak{J}^{\mu} f_i(x) = \frac{(\rho+1)^{1-\mu}}{\Gamma(\mu)} \int_a^x (x^{\rho+1} - t^{\rho+1})^{\mu-1} t^{\rho} f_i(t) \, \mathrm{d}t \text{ for } i \in \Sigma_n,$$

and $\mu > 0$, $\rho \neq -1$ are real numbers.

Definition 2.1. A function $\boldsymbol{f} : [a, b] \to \mathbb{R}^n$ is said to be Hölder continuous if there exists a positive constant K and $\sigma > 0$ such that

$$\|\boldsymbol{f}(x) - \boldsymbol{f}(y)\|_2 \leq K|x - y|^{\sigma} \text{ for all } x, y \in [a, b].$$

If $\sigma = 1$, the function $\boldsymbol{f}$ is called Lipschitz continuous.

Lemma 2.2. [36] *Let $f \in \mathcal{C}([a, b], \mathbb{R})$ be a real-valued continuous function such that $|f(x) - f(y)| \leq K|x - y|^{\sigma}$ for all $x, y \in [a, b]$, where $K > 0$ and $1 \leq \sigma \leq 2$. Then,* $\dim_H(\mathcal{G}(f)) \leq \underline{\dim}_B(\mathcal{G}(f)) \leq \overline{\dim}_B(\mathcal{G}(f)) \leq 2 - \sigma$.

2.1.1 Variation of a Vector-Valued Function

Definition 2.3. Let $\boldsymbol{f} : [a, b] \to \mathbb{R}^n$ be a function. For each partition $P : a = t_0 < t_1 < \cdots < t_n = b$ of the interval $[a, b]$, we define

$$V(\boldsymbol{f}, [a, b]) = \sup_P \sum_{i=1}^{n} \|\boldsymbol{f}(t_i) - \boldsymbol{f}(t_{i-1})\|_2,$$

where the supremum is taken over all partitions P of the interval $[a, b]$. If $V(\boldsymbol{f}, [a, b]) < \infty$, we say that $\boldsymbol{f}$ is of bounded variation. The set of all functions of bounded variation on $[a, b]$ will be denoted by $\mathcal{BV}([a, b], \mathbb{R}^n)$.

Remark 2.4. A vector-valued function $\boldsymbol{f} : [a, b] \to \mathbb{R}^n$ is of bounded variation if and only if each of its coordinate functions is of bounded variation.

Note 2.5. In view of the above remark, one can notice that the vector-valued function will be of unbounded variation if any of its coordinate functions is of unbounded variation.

Definition 2.6. Let $\boldsymbol{f} \in \mathcal{C}([a,b],\mathbb{R}^n)$. If $\boldsymbol{f}$ has a continuous derivative on $[a,b]$, then its arc length $L = \int_a^b \sqrt{1 + \|\boldsymbol{f}'(x)\|_2^2}\, \mathrm{d}x$.

Theorem 2.7. *The arc length of a curve* $\boldsymbol{y} = \boldsymbol{f}(x)$ *is finite if and only if* $\boldsymbol{f}$ *is of bounded variation on* $[a,b]$.

Theorem 2.8. *A function* $\boldsymbol{f}$ *is of bounded variation on an interval* $[a,b]$ *if and only if each coordinate function of* $\boldsymbol{f}$ *is decomposed as a difference of two increasing functions.*

Theorem 2.9. [52] *If* $f \in \mathcal{C}([a,b]) \cap \mathcal{BV}([a,b])$, *then* $\dim_H(\mathcal{G}(f)) = 1$.

2.1.2 Delineation

The structure of the current chapter is as follows. The subsequent section focuses on presenting fundamental findings regarding the dimension of the graph of a vector-valued function. In Section 2.3, we have examined the fractal dimension of the graph of the Katugampola fractional integral of a vector-valued function that is defined on a closed and finite interval. This chapter concludes in Section 2.4.

2.2 Dimension of the Graph of a Vector-Valued Function

In this section, we present a comprehensive analysis of the fundamental outcomes pertaining to the fractal dimension of the graph of a continuous vector-valued function. The proofs we provide are based on established dimensional findings documented in prior scholarly works such as [36].

Lemma 2.10. *Let* $\boldsymbol{f} : [a, b] \to \mathbb{R}^n$ *be a continuous function, and* $f_i : [a, b] \to \mathbb{R}$ *be the coordinate functions of* $\boldsymbol{f}$*. Then, we have*

(1). $\dim_H \big(\mathcal{G}(\boldsymbol{f})\big) \geq \max\{\dim_H(\mathcal{G}(f_i)) : i \in \Sigma_n\}$

(2). $\dim_H(\mathcal{G}(\boldsymbol{f})) = \dim_H(\mathcal{G}(f_i))$ *for some* $i \in \Sigma_n$*, provided that the coordinate maps* $f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_n$ *are all Lipschitz.*

Proof. (1). Let us begin the proof by defining a map $\Phi : \mathcal{G}(\boldsymbol{f}) \to \mathcal{G}(f_i)$ as

$$\Phi\big(x, \boldsymbol{f}(x)\big) = \Phi\big(x, f_1(x), \dots, f_n(x)\big) = \big(x, f_i(x)\big).$$

It is easy to check that the map Φ is surjective. By self-explanatory calculations, we have

$$\begin{aligned}
&\Big\|\Phi\big(x, f_1(x), \dots, f_n(x)\big) - \Phi\big(y, f_1(y), \dots, f_n(y)\big)\Big\|_2 \\
&= \big\|(x, f_i(x)) - (y, f_i(y))\big\|_2 \\
&= \sqrt{|x-y|^2 + \big(f_i(x) - f_i(y)\big)^2} \\
&\leq \sqrt{|x-y|^2 + \sum_{j=1}^{n} \big(f_j(x) - f_j(y)\big)^2}
\end{aligned}$$

$$= \Big\| \big(x, f_1(x), \dots, f_n(x)\big) - \big(y, f_1(y), \dots, f_n(y)\big) \Big\|_2.$$

That is, Φ is a Lipschitz map. Consequently, $\dim_H(\mathcal{G}(f_i)) = \dim_H\big(\Phi(\mathcal{G}(\boldsymbol{f}))\big) \le \dim_H(\mathcal{G}(\boldsymbol{f}))$ for each $i \in \Sigma_n$, proving the assertion.

(2). Here one may also start by defining a map $\Phi : \mathcal{G}(f_i) \to \mathcal{G}(\boldsymbol{f})$ by

$$\Phi(x, f_i(x)) = \big(x, f_1(x), \dots, f_n(x)\big).$$

It is easy to check that map Φ is onto. For $j = 1, \dots, i-1, i+1, \dots, n$, let L_{f_j} be the Lispchitz constant of f_j. Define $L = \max\{L_{f_j} : j = 1, \dots, i-1, i+1, \dots, n\}$ and $M_1 = \sqrt{1 + (n-1)L^2}$. It is simple to prove that

$$\begin{aligned} \|(x, f_i(x)) - (y, f_i(y))\|_2 \le & \|\Phi(x, f_i(x)) - \Phi(y, f_i(y))\|_2 \\ \le & M_1 \|(x, f_i(x)) - (y, f_i(y))\|_2. \end{aligned}$$

That is, Φ is a bi-Lipschitz map, and consequently, we have $\dim_H(\mathcal{G}(\boldsymbol{f})) = \dim_H(\mathcal{G}(f_i))$.

□

Corollary 2.11. *If* $\boldsymbol{f} : [0,1] \to \mathbb{R}^n$ *is a vector-valued bounded variation function on* $[0,1]$, *then* $\dim_H(\mathcal{G}(\boldsymbol{f})) = 1$.

Proof. From Theorem 2.9 and Lemma 2.10, it is straightforward to complete the proof. □

Remark 2.12. We first note that a space-filling curve $f : [0,1] \to [0,1] \times [0,1]$ is a continuous and surjective function. In [49], the author proved that both the coordinate functions of the Peano space-filling curve $f : [0,1] \to [0,1] \times [0,1]$ are generated by self-affine iterated function systems and further that they have positive finite $\frac{3}{2}$-dimensional Hausdorff measure. Consequently, $\dim_H\big(\mathcal{G}(f_1)\big) = \dim_H\big(\mathcal{G}(f_2)\big) = 1.5$. However, the Peano space-filling curve $f : [0,1] \to [0,1] \times [0,1]$ is a continuous and onto function satisfying $\dim_H\big(\mathcal{G}(f)\big) \geq 2$. This exemplifies that the inequality in 1st result of Lemma 2.10 may be strict. Further, we remark that the coordinate functions of the Hilbert space-filling curve also have graphs of Hausdorff dimension 1.5. For more details, the reader can consult [61, 82].

Proposition 2.13. *If $\boldsymbol{f} : [a,b] \to \mathbb{R}^n$ is an Hölder continuous function with exponent σ, then each coordinate function is also Hölder continuous function with exponent σ. Conversely, if all the coordinate functions of $\boldsymbol{f} : [a,b] \to \mathbb{R}^n$ are Hölder continuous with the same exponent σ, then so is $\boldsymbol{f}$.*

Proof. Since $\|\boldsymbol{f}(x) - \boldsymbol{f}(y)\|_2 \leq K_{\boldsymbol{f}}\ |x-y|^{\sigma}$ for all $x, y \in [a,b]$ and for some $K_{\boldsymbol{f}} > 0$, we have

$$\big|f_i(x) - f_i(y)\big| \leq \|\boldsymbol{f}(x) - \boldsymbol{f}(y)\|_2 \leq K_{\boldsymbol{f}}\ |x-y|^{\sigma},$$

where f_i denotes the i-th coordinate function of $\boldsymbol{f}$.

Similarly,

$$\|\boldsymbol{f}(x) - \boldsymbol{f}(y)\|_2 \leq \sqrt{n} \max_{1 \leq i \leq n} |f_i(x) - f_i(y)| \leq \sqrt{n} K_{\boldsymbol{f}} |x-y|^{\sigma},$$

where $K_{\boldsymbol{f}} = \max\{K_{f_1}, K_{f_2}, \ldots, K_{f_n}\}$ and K_{f_i} is the Lipschitz constant of f_i. □

Recall that the Hausdorff dimension of the graph of a real-valued Hölder continuous function with the Hölder exponent $\sigma \in (0,1)$, i.e., defined on $[0,1]$ is less than or equal to $2 - \sigma$ [36]. The previous proposition, taken in conjunction with the result mentioned above, provides the following:

Corollary 2.14. *If* $\boldsymbol{f} : [0,1] \to \mathbb{R}^n$ *is a Hölder continuous function with the exponent* σ*, then* $\dim_H \big(\mathcal{G}(f_i)\big) \leq 2 - \sigma$ *for each* $i \in \Sigma_n$*.*

Remark 2.15. Note that the Peano space filling curve $\boldsymbol{f} : [0,1] \to [0,1] \times [0,1]$ is $\frac{1}{2}$-Hölder continuous. As mentioned in Remark 2.12, the component functions $f_1,\ f_2$ of the function $\boldsymbol{f}$ satisfy $\dim_H \big(\mathcal{G}(f_1)\big) = \dim_H \big(\mathcal{G}(f_2)\big) = 1.5$. On the other hand, we have $\dim_H(\mathcal{G}(\boldsymbol{f})) \geq 2$. This illustrative example serves to persuade the reader that, unlike the case of a real-valued function, it is generally not feasible to establish an upper bound for the Hausdorff dimension of the graph of a vector-valued function in terms of its Hölder exponent.

2.3 Fractal Dimension and Katugampola Fractional Integral

In this section, we will present the main results of our study. We demonstrate these findings using straightforward and commonly used reasoning. We will utilize a constant $K > 0$, which may vary across different equations or lines of our discussion. For our convenience, let us define $K_{\mu,\rho} = \frac{(\rho+1)^{1-\mu}}{\Gamma(\mu)}$. It is important to note that throughout this chapter, we will make the following assumptions: $0 \leq a < b < \infty$, $-1 < \rho$, and $\mu > 0$.

Theorem 2.16. *If* $\boldsymbol{f} : [a,b] \to \mathbb{R}^n$ *is a bounded function, then* ${}^{\rho}_{a}\mathfrak{J}^{\mu}\boldsymbol{f}$ *is bounded.*

Proof. Since $\boldsymbol{f}$ is bounded, all coordinate functions $f_1, \dots f_n$ are bounded. Then, we can choose $K > 0$ such that $|f_i(x)| \leq K$ for all $x \in [a,b]$ and $i \in \Sigma_n$. Now

$$\| {}^{\rho}_{a}\mathfrak{J}^{\mu}\boldsymbol{f}(x)\|_2 \leq \sqrt{n} \max_{1 \leq i \leq n} | {}^{\rho}_{a}\mathfrak{J}^{\mu} f_i(x)|$$

$$
\begin{aligned}
&= \sqrt{n} \max_{1\le i\le n} \left| K_{\mu,\rho} \int_a^x (x^{\rho+1} - t^{\rho+1})^{\mu-1} t^\rho f_i(t)\ \mathrm{d}t \right| \\
&\le \sqrt{n} \max_{1\le i\le n} K_{\mu,\rho} \int_a^x |(x^{\rho+1} - t^{\rho+1})^{\mu-1} t^\rho|\ \ |f_i(t)|\ \mathrm{d}t \\
&\le \sqrt{n} \max_{1\le i\le n} K_{\mu,\rho} K \int_a^x |(x^{\rho+1} - t^{\rho+1})^{\mu-1} t^\rho|\ \mathrm{d}t.
\end{aligned}
$$

For $0 < a < b < \infty$ and $-1 < \rho$, the expression inside the modulus on the right side of the above inequality is nonnegative. Applying the change of variable $t^{\rho+1} = u$, we obtain

$$
\begin{aligned}
\| {}^{\rho}_{a}\mathfrak{J}^{\mu} \boldsymbol{f}(x)\|_2 \le & \frac{K\sqrt{n}(\rho+1)^{1-\mu}}{(\rho+1)\Gamma(\mu)} \int_{a^{\rho+1}}^{x^{\rho+1}} (x^{\rho+1} - u)^{\mu-1} \mathrm{d}u \\
= & \frac{K\sqrt{n}(\rho+1)^{-\mu}}{\mu\Gamma(\mu)} (x^{\rho+1} - a^{\rho+1})^{\mu} \\
\le & \frac{K\sqrt{n}(\rho+1)^{-\mu}}{\Gamma(\mu+1)} (b^{\rho+1} - a^{\rho+1})^{\mu},
\end{aligned}
$$

for all $x \in [a, b]$. This completes the proof. □

Remark 2.17. Theorem 2.16 deduces that for all $\boldsymbol{f} \in L^\infty([a, b], \mathbb{R}^n)$, $\| {}^{\rho}_{a}\mathfrak{J}^{\mu} \boldsymbol{f}\|_\infty \le K\|\boldsymbol{f}\|_\infty$ for some constant K and hence it shows that ${}^{\rho}_{a}\mathfrak{J}^{\mu} \boldsymbol{f}$ is a bounded linear operator on $L^\infty([a, b], \mathbb{R}^n)$.

Example 2.18. Consider

$$
f(x) = \begin{cases} 0, & \text{whenever } 1 \le x \le 2 \\ 1, & \text{whenever } 2 \le x \le 3, \end{cases}
$$

then f is a bounded function. We calculate its Katugampola fractional integral in the following way: for $1 \le x \le 2$, we get ${}_1^{\rho}\mathfrak{J}^{\mu} f(x) = 0$. For $x > 2$,

$$\begin{aligned} {}_1^{\rho}\mathfrak{J}^{\mu} f(x) =&\ K_{\mu,\rho} \int_1^x (x^{\rho+1} - t^{\rho+1})^{\mu-1} t^{\rho} f(t)\ \mathrm{d}t \\ =&\ K_{\mu,\rho} \int_2^x (x^{\rho+1} - t^{\rho+1})^{\mu-1} t^{\rho}\ \mathrm{d}t \\ =&\ \frac{(\rho+1)^{-\mu}}{\Gamma(\mu+1)} (x^{\rho+1} - 2^{\rho+1})^{\mu}. \end{aligned} \tag{2.1}$$

The second equality comes from the definition of f, with a change of variable (put $t^{\rho+1} = u$), and the calculation of integral produces the last equality in the above equation. Finally, one can see that the fractional integral ${}_1^{\rho}\mathfrak{J}^{\mu} f$ is a bounded function. More precisely, $\sup_{x\in[0,3]} |{}_1^{\rho}\mathfrak{J}^{\mu} f(x)| \le \frac{(\rho+1)^{-\mu}}{\Gamma(\mu+1)} (3^{\rho+1} - 2^{\rho+1})^{\mu}$.

Theorem 2.19. *If* $\boldsymbol{f} \in \mathcal{C}([0,1], \mathbb{R}^n)$*, then* ${}_a^{\rho}\mathfrak{J}^{\mu} \boldsymbol{f} \in \mathcal{C}([0,1], \mathbb{R}^n)$.

Proof. Since $\boldsymbol{f}$ is continuous, then all coordinate functions $f_1, \dots, f_n$ of $\boldsymbol{f}$ are continuous on $[a, b]$. Now for $0 < a < x < x + h \le b$, we have

$$\begin{aligned} \| {}_a^{\rho}\mathfrak{J}^{\mu} \boldsymbol{f}(x+h) - {}_a^{\rho}\mathfrak{J}^{\mu} \boldsymbol{f}(x) \|_2 \le&\ \sqrt{n} \max_{1\le i\le n} \left| {}_a^{\rho}\mathfrak{J}^{\mu} f_i(x+h) - {}_a^{\rho}\mathfrak{J}^{\mu} f_i(x) \right| \\ =& \sqrt{n} \max_{1\le i\le n} \left| K_{\mu,\rho} \int_a^{x+h} \left((x+h)^{\rho+1} - t^{\rho+1}\right)^{\mu-1} t^{\rho} f_i(t)\ \mathrm{d}t \right. \\ &\left. - K_{\mu,\rho} \int_a^x (x^{\rho+1} - t^{\rho+1})^{\mu-1} t^{\rho} f_i(t)\ \mathrm{d}t \right| \\ =& \sqrt{n} \max_{1\le i\le n} \left| K_{\mu,\rho} \int_a^{a+h} ((x+h)^{\rho+1} - t^{\rho+1})^{\mu-1} t^{\rho} f_i(t)\ \mathrm{d}t \right. \\ &+ K_{\mu,\rho} \int_{a+h}^{x+h} ((x+h)^{\rho+1} - t^{\rho+1})^{\mu-1} t^{\rho} f_i(t)\ \mathrm{d}t \\ &\left. - K_{\mu,\rho} \int_a^x (x^{\rho+1} - t^{\rho+1})^{\mu-1} t^{\rho} f_i(t)\ \mathrm{d}t \right|. \end{aligned}$$

The above equation can be written as

$$\| {}_{a}^{\rho}\mathfrak{I}^{\mu}\boldsymbol{f}(x+h) - {}_{a}^{\rho}\mathfrak{I}^{\mu}\boldsymbol{f}(x)\|_2 \leq \sqrt{n}\max_{1\leq i\leq n}|J_{i1}+J_{i2}+J_{i3}|, \tag{2.2}$$

where

$$\begin{aligned}
J_{i1} &= K_{\mu,\rho}\int_a^{a+h}((x+h)^{\rho+1}-t^{\rho+1})^{\mu-1}t^{\rho}f_i(t)\ \mathrm{d}t,\\
J_{i2} &= K_{\mu,\rho}\int_{\sigma}^{a}(x^{\rho+1}-t^{\rho+1})^{\mu-1}t^{\rho}f_i\big(\sqrt[\rho+1]{t^{\rho+1}+(x+h)^{\rho+1}-x^{\rho+1}}\big)\ \mathrm{d}t,\\
J_{i3} &= K_{\mu,\rho}\int_{a}^{x}(x^{\rho+1}-t^{\rho+1})^{\mu-1}t^{\rho}\Big[f_i\big(\sqrt[\rho+1]{t^{\rho+1}+(x+h)^{\rho+1}-x^{\rho+1}}\big)-f_i(t)\Big]\ \mathrm{d}t,\\
\sigma &= \sqrt[\rho+1]{x^{\rho+1}-(x+h)^{\rho+1}+(a+h)^{\rho+1}}.
\end{aligned}$$

Using the fact that $\boldsymbol{f}\in\mathcal{C}([a,b],\mathbb{R}^n)$, it follows that $f_i\in\mathcal{C}([a,b],\mathbb{R})$, This now implies that there exists $K>0$ such that $\sup_{t\in[a,b]}|f(t)|\leq K$. Further, for given $\epsilon>0$, we may choose $\delta>0$ such that

$$|f_i(z)-f_i(w)|<\epsilon \text{ whenever } |z-w|<\delta$$

holds for all $1\leq i\leq n$. Hence, we have the following:

$$\begin{aligned}
|J_{i1}| &\leq K_{\mu,\rho}Kh,\\
|J_{i2}| &\leq K_{\mu,\rho}K\ \Big|a-\sqrt[\rho+1]{x^{\rho+1}-(x+h)^{\rho+1}+(a+h)^{\rho+1}}\Big|,\\
|J_{i3}| &\leq \frac{\epsilon(\rho+1)^{1-\mu}}{\Gamma\mu}\int_a^x|(x^{\rho+1}-t^{\rho+1})^{\mu-1}t^{\rho}|\ \mathrm{d}t,
\end{aligned}$$

for all $1\leq i\leq n$. Applying the change of variable $t^{\rho+1}=u$, we bound $|J_{i3}|$ as below

$$|J_{i3}|\leq\frac{\epsilon}{(\rho+1)^{\mu}\Gamma(\mu+1)}\ (b^{\rho+1}-a^{\rho+1})^{\mu}.$$

From these estimates and (2.2), we have

$$\begin{aligned}\| {}_{a}^{\rho}\mathfrak{I}^{\mu}\boldsymbol{f}(x+h) - {}_{a}^{\rho}\mathfrak{I}^{\mu}\boldsymbol{f}(x)\|_2 \leq & \sqrt{n}\Bigg(K_{\mu,\rho}K\ h + \\ & + K_{\mu,\rho}K\ \left| a - \sqrt[\rho+1]{x^{\rho+1} - (x+h)^{\rho+1} + (a+h)^{\rho+1}} \right| \\ & + \frac{\epsilon}{(\rho+1)^{\mu}\Gamma(\mu+1)}\ (b^{\rho+1} - a^{\rho+1})^{\mu} \Bigg),\end{aligned}$$

proving the assertion. ☐

Remark 2.20. Recall that a vector-valued function $\boldsymbol{g} = (g_1, \ldots, g_n)$ is continuous if and only if all coordinate functions g_i are continuous. One can prove the above theorem by demonstrating that all coordinate functions ${}_{a}^{\rho}\mathfrak{I}^{\mu} f_i$ of ${}_{a}^{\rho}\mathfrak{I}^{\mu}\boldsymbol{f}$ are continuous.

The following result has appeared in [89] regarding real-valued functions of bounded variation. However, we include the proof for completeness.

Lemma 2.21. *If $\boldsymbol{f} : [a, b] \to \mathbb{R}^n$ is of bounded variation on $[a, b]$, then the following holds:*

1. *If $f_i(a) \geq 0$, then there exist increasing functions g_i and h_i such that $f_i = g_i - h_i$, $g_i(a) \geq 0$ and $h_i(a) = 0$.*
2. *If $f_i(a) < 0$, then there exist increasing functions g_i and h_i such that $f_i = g_i - h_i$, $g_i(a) = 0$ and $h_i(a) > 0$.*

Proof. Since $\boldsymbol{f} = (f_1, \ldots, f_n)$ is of bounded variation on $[a, b]$, all coordinate functions of $\boldsymbol{f}$ are of bounded variation on $[a, b]$. f_i can be written as $f_i = \phi_i - \xi_i$, where ϕ_i and ξ_i are increasing functions for each $1 \leq i \leq n$. Define $g_i(x) = \phi_i(x) + f_i(a) - \phi_i(a)$ and $h_i(x) = \xi_i(x) + f_i(a) - \phi_i(a)$. Then, both g_i and h_i are increasing functions and satisfy the required conditions. Similarly, for the second part of the lemma, we take $g_i(x) = \phi_i(x) - f_i(a) - \xi_i(a)$ and $h_i(x) = \phi_i(x) - f_i(a) - \xi_i(a)$. This completes the proof of the assertion. $\square$

Theorem 2.22. *If* $\boldsymbol{f} \in \mathcal{BV}([a,b],\mathbb{R}^n)$, *then* ${}_a^{\rho}\mathfrak{J}^{\mu}\boldsymbol{f} \in \mathcal{BV}([a,b],\mathbb{R}^n)$.

Proof. In view of Theorem 2.21, it is enough to show that each coordinate function ${}_a^{\rho}\mathfrak{J}^{\mu}f_i$ of ${}_a^{\rho}\mathfrak{J}^{\mu}\boldsymbol{f}$ is a difference of two monotone increasing functions. Since $\boldsymbol{f} = (f_1,\ldots,f_n)$ is of bounded variation, each f_i can be written as a difference of two monotone increasing functions. That is, $f_i(x) = g_i(x) - h_i(x)$ for $x \in [a,b]$, where g_i and h_i are monotone increasing functions. Without loss of generality, we assume that $f_i(a) \geq 0$. By Lemma 2.21, we can choose $g_i(a) \geq 0$ and $h_i(a) = 0$. Define functions $\boldsymbol{G}$ and $\boldsymbol{H}$ as

$$\boldsymbol{G}(x) := {}_a^{\rho}\mathfrak{J}^{\mu}\boldsymbol{g}(x) \text{ and } \boldsymbol{H}(x) := {}_a^{\rho}\mathfrak{J}^{\mu}\boldsymbol{h}(x),$$

where $\boldsymbol{g} = (g_1,\ldots,g_n)$, $\boldsymbol{h} = (h_1,\ldots,h_n)$. Linearity of Katugampola fractional integral yields, ${}_a^{\rho}\mathfrak{J}^{\mu}\boldsymbol{f}(x) = \boldsymbol{G}(x) - \boldsymbol{H}(x)$. It is enough to show that G_i and H_i are monotone increasing functions, where $G_i(x) = {}_a^{\rho}\mathfrak{J}^{\mu}g_i(x)$ and $H_i(x) = {}_a^{\rho}\mathfrak{J}^{\mu}h_i(x)$. To this end, let $a \leq x \leq y \leq b$,

$$\begin{aligned} G_i(y) - G_i(x) &= {}_a^{\rho}\mathfrak{J}^{\mu}g_i(y) - {}_a^{\rho}\mathfrak{J}^{\mu}g_i(x) \\ &= K_{\mu,\rho}\int_a^y (y^{\rho+1} - t^{\rho+1})^{\mu-1}t^{\rho}g_i(t)dt - K_{\mu,\rho}\int_a^x (x^{\rho+1} - t^{\rho+1})^{\mu-1}t^{\rho}g_i(t)dt. \end{aligned}$$

By a change of variable in the second integral using $x^{\rho+1} - t^{\rho+1} = y^{\rho+1} - u^{\rho+1}$, we may deduce that

$$G_i(y) - G_i(x) = I_4 + I_5,$$

where

$$\begin{aligned}
I_4 &= K_{\mu,\rho} \int_a^{\sqrt[\rho+1]{y^{\rho+1}-x^{\rho+1}+a^{\rho+1}}} (y^{\rho+1} - t^{\rho+1})^{\mu-1} t^{\rho} g_i(t) \ \mathrm{d}t, \\
I_5 &= K_{\mu,\rho} \int_{\tau}^{y} (y^{\rho+1} - t^{\rho+1})^{\mu-1} t^{\rho} \Big[g_i(t) - g_i\big(\sqrt[\rho+1]{t^{\rho+1} + x^{\rho+1} - y^{\rho+1}} \big) \Big] \ \mathrm{d}t, \\
\tau &= \sqrt[\rho+1]{y^{\rho+1} - x^{\rho+1} + a^{\rho+1}}.
\end{aligned}$$

Since $\sqrt[\rho+1]{t^{\rho+1} + x^{\rho+1} - y^{\rho+1}} \leq t$, $g_i(a) \geq 0$ and g_i is monotone increasing, we conclude that all terms under the integration signs are nonnegative. Therefore, $G_i(y) - G_i(x) \geq 0$, that is, G_i is monotone increasing function. Similarly, one can deduce that H_i is also a monotone-increasing function. If $f_i(a) < 0$, using Lemma 2.21, one can select monotone increasing functions g_i, h_i satisfying $g_i(a) = 0$ and $h_i(a) > 0$, and the rest of the proof follow on similar lines. Thus, we complete the proof of the theorem. □

The following theorem is widely known; its statement can be found in [36]. However, we have provided proof here for the reader's convenience.

Theorem 2.23. *If* $\boldsymbol{f}$ *is of bounded variation on* $[a, b]$*, then* $\dim_H(\mathcal{G}(\boldsymbol{f})) = 1$.

Proof. We know that a function of bounded variation can have, at most, a countable number of discontinuous points. Let $\{a \leq x_1 < x_2 < \cdots \leq b\}$ be the set of points of discontinuity of $\boldsymbol{f}$. For notational consistency, let $a = x_0$ and G_i be the graph of function $\boldsymbol{f}$ restricted to the interval $[x_i, x_{i+1}]$ for $i \in \mathbb{N} \cup \{0\}$. Clearly $\mathcal{G}(\boldsymbol{f}) = \cup_{i=0}^{\infty} G_i$. Using the countable stability of the Hausdorff dimension we obtain $\dim_H(\mathcal{G}(\boldsymbol{f})) = \dim_H(\cup_{i=0}^{\infty} G_i) = \sup_{0 \leq i < \infty} \dim_H(G_i)$. It is obvious that on each intervals $[x_i, x_{i+1}]$, f is a continuous function of bounded variation. Therefore, in view of Theorem 2.9, we have $\dim_H(G_i) = 1$ for each $i \in \mathbb{N} \cup \{0\}$, and whence $\dim_H(\mathcal{G}(\boldsymbol{f})) = 1$. □

Remark 2.24. One can try to construct a 1-dimensional vector-valued function of unbounded variation. A hint for such construction can be found in [89]. In the mentioned paper, the authors have provided a clue on constructing a real-valued function in 1- dimension with unbounded variation. If we use this function as one of the coordinate functions in our vector-valued function and ensure that all the other coordinate functions have bounded variation, then using Note 2.5 and Lemma 2.10, we get the desired outcome.

In view of Theorems 2.23 and 2.22, we immediately obtain

Theorem 2.25. *If* $\boldsymbol{f} \in \mathcal{BV}([a,b],\mathbb{R}^n)$, *then* $\dim_H(\mathcal{G}({}_{a}^{\rho}\mathfrak{J}^{\mu}\boldsymbol{f})) = 1$, *whenever* $0 \leq a < b < \infty$, $-1 < \rho$ *and* $\mu > 0$.

Combining Theorems 2.9, 2.19 and 2.22, we have

Theorem 2.26. *If* $\boldsymbol{f} \in \mathcal{C}([a,b],\mathbb{R}^n) \cap \mathcal{BV}([a,b],\mathbb{R}^n)$, *then* $\dim_B(\mathcal{G}({}_{a}^{\rho}\mathfrak{J}^{\mu}\boldsymbol{f})) = 1$.

Remark 2.27. One can observe that the above theorem is a generalized version of [51, Theorem 1.5] and [94, Lemma 2.2]. To be precise, if $n = 1$ and $\rho = 0$, the above theorem will reduce to [51, Theorem 1.5] and if $n = 1$ and $\rho \to -1^+$ it will reduce to [94, Lemma 2.2].

In the upcoming theorem, we shall give an upper bound for the upper box dimensions of Katugampola fractional integrals of coordinate functions of a vector-valued continuous function.

Theorem 2.28. *If* $\boldsymbol{f} \in \mathcal{C}([a,b],\mathbb{R}^n)$, *then* ${}_{a}^{\rho}\mathfrak{J}^{\mu}\boldsymbol{f}$ *will be Hölder continuous with Hölder exponent* μ *provided that* $-1 < \rho \leq 0$ *and* $0 < \mu < 1$. *Moreover, we have* $\overline{\dim}_B(\mathcal{G}({}_{a}^{\rho}\mathfrak{J}^{\mu}f_i)) \leq 2 - \mu$ *for all* $i \in \Sigma_n$.

Proof. For $0 < a \leq x < x + h \leq b$, we have

$$\begin{aligned}
\| {}^{\rho}_{a}\mathfrak{J}^{\mu}\boldsymbol{f}(x+h) - {}^{\rho}_{a}\mathfrak{J}^{\mu}\boldsymbol{f}(x)\|_2 \leq{}& \sqrt{n} \max_{1\leq i \leq n} \left| {}^{\rho}_{a}\mathfrak{J}^{\mu} f_i(x+h) - {}^{\rho}_{a}\mathfrak{J}^{\mu} f_i(x) \right| \\
={}& \sqrt{n} \max_{1\leq i \leq n} \Bigg| K_{\mu,\rho} \int_a^{x+h} ((x+h)^{\rho+1} - t^{\rho+1})^{\mu-1} t^{\rho} f_i(t) dt \\
& - K_{\mu,\rho} \int_a^{x} (x^{\rho+1} - t^{\rho+1})^{\mu-1} t^{\rho} f_i(t) dt \Bigg| \\
={}& \sqrt{n} \max_{1\leq i \leq n} |J_{i6} + J_{i7}| \\
\leq{}& \sqrt{n} \max_{1\leq i \leq n} \{|J_{i6}| + |J_{i7}|\},
\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}
J_{i6} &= K_{\mu,\rho} \int_a^x \Big[((x+h)^{\rho+1} - t^{\rho+1})^{\mu-1} - (x^{\rho+1} - t^{\rho+1})^{\mu-1} \Big] t^{\rho} f_i(t) \ \mathrm{d}t \\
J_{i7} &= K_{\mu,\rho} \int_x^{x+h} \Big((x+h)^{\rho+1} - t^{\rho+1} \Big)^{\mu-1} t^{\rho} f_i(t) \ \mathrm{d}t.
\end{aligned}$$

Using the fact that $\boldsymbol{f} \in \mathcal{C}([a,b],\mathbb{R}^n)$, it follows that $f_i \in \mathcal{C}([a,b],\mathbb{R})$, this now implies that there exists $K > 0$ such that $\max_{1\leq i\leq n} \sup_{t\in[a,b]} |f_i(t)| \leq K$. Further, the term J_{i6} can be estimated as follows.

$$|J_{i6}| \leq K_{\mu,\rho} K \int_a^x \Big[(x^{\rho+1} - t^{\rho+1})^{\mu-1} - (x+h)^{\rho+1} - t^{\rho+1})^{\mu-1} \Big] t^{\rho} \ \mathrm{d}t.$$

To reduce the integral appearing on the right side of the above inequality, we let $t^{\rho+1} = u$. Simple calculations yield

$$\begin{aligned}
|J_{i6}| \leq{}& K_{\mu,\rho} K \int_{a^{\rho+1}}^{x^{\rho+1}} \Big[(x^{\rho+1} - u)^{\mu-1} - ((x+h)^{\rho+1} - u)^{\mu-1} \Big] \ \mathrm{d}u \\
={}& K_{\mu,\rho} K \ \Big[\big((x+h)^{\rho+1} - x^{\rho+1} \big)^{\mu} - \big((x+h)^{\rho+1} - a^{\rho+1} \big)^{\mu} + (x^{\rho+1} - a^{\rho+1})^{\mu} \Big].
\end{aligned}$$

With the help of the Bernoulli's inequality $(1+x)^r \leq 1 + rx$ for $0 \leq r \leq 1$ and $x \geq -1$, we find that

$$|J_{i6}| \leq K_{\mu,\rho} K h^{\mu} \text{ for all } 1 \leq i \leq n.$$

Turning to J_7, we observe that

$$|J_{i7}| \leq K_{\mu,\rho} K h^{\mu} \text{ for all } 1 \leq i \leq n.$$

Plugging the previous estimates for J_{i6} and J_{i7} into Equation 2.3, we have

$$\| {}^{\rho}_{a}\mathfrak{I}^{\mu}\boldsymbol{f}(x+h) - {}^{\rho}_{a}\mathfrak{I}^{\mu}\boldsymbol{f}(x)\|_2 \leq \sqrt{n} \max_{1 \leq i \leq n} \{|J_{i6}| + |J_{i7}|\} \leq 2\sqrt{n} K_{\mu,\rho} K h^{\mu}.$$

That is, ${}^{\rho}_{a}\mathfrak{I}^{\mu}\boldsymbol{f}$ is Hölder continuous with exponent μ. From Lemma 2.2 and Proposition 2.13, we obtain $\overline{dim}_B(\mathcal{G}({}^{\rho}_{a}\mathfrak{I}^{\mu} f_i)) \leq 2 - \mu$. This completes the proof. □

Remark 2.29. One may also prove the above theorem by showing that the coordinate functions ${}^{\rho}_{a}\mathfrak{I}^{\mu} f_i$ of ${}^{\rho}_{a}\mathfrak{I}^{\mu}\boldsymbol{f}$ are Hölder continuous. Then, Proposition 2.13 in turn yields that ${}^{\rho}_{a}\mathfrak{I}^{\mu}\boldsymbol{f}$ is Hölder continuous with exponent μ.

Remark 2.30. In the previous theorem, we have shown that ${}^{\rho}_{a}\mathfrak{I}^{\mu}\boldsymbol{f}$ is Hölder continuous on $[a,b]$, whenever $\boldsymbol{f}$ is a continuous function on $[a,b]$ with $0 \leq a < b < \infty$, $-1 < \rho \leq 0$ and $\mu > 0$. One may compare this with Theorem 2.19, wherein we proved that for $-1 < \rho$, the Katugampola integral ${}^{\rho}_{a}\mathfrak{I}^{\mu}\boldsymbol{f}$ is continuous, whenever $\boldsymbol{f}$ is continuous. A natural question arises whether the Hölder continuity of $\boldsymbol{f}$ will be preserved by the Katugampola integral beyond the values $-1 < \rho \leq 0$ or not.

As a prelude to the following theorem, we note the semigroup property of Katugampola integral for a “sufficiently good” function:

$$ {}^{\rho}_{a}\mathfrak{J}^{\mu} \; {}^{\rho}_{a}\mathfrak{J}^{\mu_1} \boldsymbol{f} = \; {}^{\rho}_{a}\mathfrak{J}^{\mu+\mu_1} \boldsymbol{f}. $$

Now, the previous theorem yields the following:

Theorem 2.31. *Let* $\boldsymbol{f}$ *be continuous on* $[a, b]$, $0 < a < b < \infty$ *and* $\rho > -1$.

1. *If* $0 < \mu < 1$, *then*

$$ 1 \leq \dim_H(\mathcal{G}(\, {}^{\rho}_{a}\mathfrak{J}^{\mu} f_i)) \leq \overline{\dim}_B(\mathcal{G}(\, {}^{\rho}_{a}\mathfrak{J}^{\mu} f_i)) \leq 2 - \mu. $$

2. *If* $\mu \geq 1$, *then*

$$ \dim_H(\mathcal{G}(\, {}^{\rho}_{a}\mathfrak{J}^{\mu} f_i)) = \dim_B(\mathcal{G}(\, {}^{\rho}_{a}\mathfrak{J}^{\mu} f_i)) = 1. $$

2.4 Conclusion

In this chapter, we have examined the idea of the fractal dimension of vector-valued functions (Lemma 2.10, Corollary 2.11, Corollary 2.14). Later on, in Section 2.3, we have presented our main findings regarding the fractal dimension of the Katugampola fractional integral of vector-valued functions, as well as some properties of the integral (Theorem 2.16, Theorem 2.19, Theorem 2.22, Theorem 2.23, Theorem 2.25, Theorem 2.26, Theorem 2.28).