Chapter 3

A Newton-type Globally Convergent Interior-Point Method to Solve Multiobjective Optimization Problems

3.1 Introduction

El-Bakry et al. [90] extended the interior-point formulation from linear programming to general nonlinear programming. In addition, they considered a linesearch globalization strategy for their local algorithm that uses the ℓ_2-norm of the residual function of the KKT conditions as merit function. While this choice for the merit function has apparent advantages, it also has disadvantages. This algorithm is convergent whenever the Jacobian of the KKT conditions is nonsingular (see [90]). The proposed algorithm in [90] was tested on the Hock and Schittkowski test problems by Shanno and Simantiraki [92]. They found that the algorithm fails on some problems. One of the objectives of this chapter is to construct an appropriate merit function that couples the objective function with the constraints in such a way that progress in the merit function effectively means real progress in solving the single objective optimization subproblems.

3.2 Motivation

As described in previous chapter (see Section 2.6.1), if the search direction is descent with respect to the merit function then there are values of step length for which the merit function value decreases. The merit function needs to ensure the decrease is not negligible and the step taken is not too small. The merit function that is discussed in the previous chapter may produce very small step length along the Newton direction [86]. To overcome this deficiency in this chapter, a new differentiable merit function is proposed.

3.3 Contributions

In this chapter, a Newton-type IPM is proposed for detecting the nondominated points of multiobjective optimization problems using the direction-based cone method. Cone method decomposes the multiobjective optimization problems into a set of the single-objective optimization subproblems. Under some mild conditions, the proposed algorithm is shown to be globally convergent. Numerical results of unconstrained and constrained multiobjective optimization test problems are presented.

The novel contributions of this chapter is as follows:

(i) We introduce a new merit function which has the following two properties: the KKT point is also the stationary point of the merit function and Newton direction is descent for the merit function.

(ii) The performance of the proposed algorithm is tested on SCH, FON, ZTD test suit and CEC09 test suite.

(iii) Three performance measures (GD+, HV, IGD) has taken to test the efficiency of the proposed algorithm on ZDT and CEC09 test suite. The performance of the proposed algorithm in comparison with some existing popular algorithms found efficient.

In this chapter, the following MOP is considered:

$$\left.\begin{array}{ll} \text{minimize} & F(x) \\ \text{subject to} & x \in \mathcal{X}, \end{array}\right\} \tag{3.1}$$

where $F(x) = (f_1(x), f_2(x), \ldots, f_{\mathcal{P}}(x))^\top$, $\mathcal{P} \geq 2$ is a vector-valued function (multiobjective function), and $\mathcal{X} = \{x \in \mathbb{R}^n : h_j(x) \geq 0,\ x \geq 0,\ j = 1, 2, \ldots, \mathcal{J}\}$.

3.3.1 Description of the cone method associated to MOP (3.1)

As described in [47], firstly, it determines the ideal point $F^* = (f_1^*, f_2^*, \ldots, f_{\mathcal{P}}^*)^\top$, where $f_i^* = \min\{f_i(x) : x \in \mathcal{X}\}$. Then, by solving the following minimization problem corresponding to a particular $\hat{\beta} \in \mathbb{S}^{\mathcal{P}-1}_{\geqq} = \mathbb{S}^{\mathcal{P}-1} \cap \mathbb{R}^{\mathcal{P}}_{\geqq}$ (where $\mathbb{S}^{\mathcal{P}-1}$ represents the unit sphere in $\mathbb{R}^{\mathcal{P}}$), the solution of the MOP (3.1) can be obtained:

$$\left.\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & t\hat{\beta} \geqq F(x) - F^*, \\ & h_j(x) \geq 0,\ j = 1, 2, \ldots, \mathcal{J}, \\ & x \geqq 0,\ t \geq 0, \end{array}\right\} \tag{3.2}$$

where the expression of $\hat{\beta}$ is calculated from (2.3).

The next section formulates the Newton scheme for an IPM to solve the parametric problem (3.2).

3.4 Interior-point method

In this section, an IPM is discussed to solve (3.2) for each given $\hat{\beta} \in \mathbb{S}^{\mathcal{P}-1}$. In the sequel, (3.2) is formulated into the barrier problem and then KKT conditions are derived. Thereafter, IPM takes the advantage of Newton method to solve the system of KKT.

Introducing the vectors $\mathrm{x} = (x_1, x_2, \ldots, x_n, t)^\top$, $c = (0, 0, \ldots, 0, 1)^\top$, $\hat{\beta} = (\beta_1, \beta_2, \ldots, \beta_{\mathcal{P}})^\top$, and denoting $f_i(\mathrm{x}) = f_i(x)$, $i = 1, 2, \ldots, \mathcal{P}$ and $f(\mathrm{x}) = F(\mathrm{x}) - F^*$ reduces the parametric problem (3.2) into the following problem:

$$\left.\begin{array}{ll} \text{minimize} & c^\top \mathrm{x} \\ \text{subject to} & \hat{\beta} c^\top \mathrm{x} - f(\mathrm{x}) - v = 0, \\ & h(\mathrm{x}) - w = 0, \\ & \mathrm{x} \geqq 0, \;\; v \geqq 0, \;\; w \geqq 0, \end{array}\right\} \tag{3.3}$$

where $v = (v_1, v_2, \ldots, v_{\mathcal{P}})^\top$, $w = (w_1, w_2, \ldots, w_{\mathcal{J}})^\top$ and $h(\mathrm{x}) = (h_1(x), h_2(x), \ldots, h_{\mathcal{J}}(x))^\top$.

In the problem (3.3), the exclusion of the non-negative vectors x, v and w are achieved by setting them within a barrier function as follows:

$$\left.\begin{array}{ll} \text{minimize} & b\left(\mathrm{x}, v, w, \mu\right) \\ \text{subject to} & \hat{\beta} c^\top \mathrm{x} - f(\mathrm{x}) - v = 0, \\ & h(\mathrm{x}) - w = 0, \end{array}\right\} \tag{3.4}$$

where $b\left(\mathrm{x}, v, w, \mu\right) = c^\top \mathrm{x} - \mu \left(\sum_{l=1}^{n+1} \log(\mathrm{x}_l) + \sum_{i=1}^{\mathcal{P}} \log(v_i) + \sum_{j=1}^{\mathcal{J}} \log(w_j) \right)$ and $\mu > 0$ is the barrier parameter.

In this chapter, our main focus will be on solving the following first order perturbed KKT conditions. For $\mu > 0$ and $(\mathrm{x}, v, w, s, y, z) > 0$:

$$\begin{bmatrix} c - s - \nabla_{\mathrm{x}} \left(\hat{\beta} c^{\top} \mathrm{x} - f(\mathrm{x}) \right)^{\top} y - (\nabla_{\mathrm{x}} h(\mathrm{x}))^{\top} z \\ -\mu e + VYe \\ -\mu e + WZe \\ -\mu e + S\mathrm{X}e \\ -f(\mathrm{x}) + \hat{\beta} c^{\top} \mathrm{x} - v \\ h(\mathrm{x}) - w \end{bmatrix} = 0, \tag{3.5}$$

where $s = \mu \mathrm{X}^{-1} e$, $\mathrm{X} = \mathrm{diag}(\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n, t)$, $V = \mathrm{diag}(v_1, v_2, \ldots, v_{\mathcal{P}})$, $W = \mathrm{diag}(w_1, w_2, \ldots, w_{\mathcal{J}})$, $Y = \mathrm{diag}(y_1, y_2, \ldots, y_{\mathcal{P}})$, $Z = \mathrm{diag}(z_1, z_2, \ldots, z_{\mathcal{J}})$ and $S = \mathrm{diag}(s_1, s_2, \ldots, s_n, s_{n+1})$. For the rest of this chapter, we denote the matrix in the left-hand side of (3.5) by $\mathcal{D}_{\mu}(\mathrm{x}, v, w, s, y, z)$.

The KKT system (3.5) need not an ill-conditioned system of equations [90], and also the vector $(\mathrm{x}, v, w, s, y, z)$ keeps away from the zero at every iteration. For any $\mu > 0$, we call $\Lambda_{\mu} = (\mathrm{x}_{\mu}, v_{\mu}, w_{\mu}, s_{\mu}, y_{\mu}, z_{\mu})$ a perturbed KKT point if it satisfies the perturbed KKT conditions (3.5). Clearly, for $\mu = 0$, the perturbed KKT system is the KKT system corresponding to the problem (3.3).

Definition 3.1 *(i) A point $\Lambda = (\mathrm{x}, v, w, s, y, z)$ is said to be an interior point for the barrier problem (3.4) if $(\mathrm{x}, v, w, s, y, z) > 0$.*

(ii) A point $\Lambda = (\mathrm{x}, v, w, s, y, z)$ is said to be a quasi-central point for the problem (3.4) if it satisfies the following conditions for any $\mu > 0$:

$$\left.\begin{aligned} -\mu e + VYe &= 0 \\ -\mu e + WZe &= 0 \\ -\mu e + S\mathrm{X}e &= 0 \\ -f(\mathrm{x}) + \hat{\beta} c^{\top} \mathrm{x} - v &= 0 \\ h(\mathrm{x}) - w &= 0. \end{aligned}\right\} \tag{3.6}$$

The set of all points that satisfy (3.6) is called quasi-central path. We simplified the expressions below by using the following notations:

$$A_{\hat{\beta}}(\mathrm{x}) = \nabla_{\mathrm{x}}\left(\hat{\beta}c^{\top}\mathrm{x} - f(\mathrm{x})\right), \ B(\mathrm{x}) = \nabla_{\mathrm{x}}h(\mathrm{x})$$

and

$$H(\mathrm{x}, y, z) = \sum_{j=1}^{s} y_j \nabla^2 f_j(\mathrm{x}) - \sum_{i=1}^{m+1} z_i \nabla^2 h_i(\mathrm{x}), \quad y \geqq 0, \quad z \geqq 0. \tag{3.7}$$

For a fixed $\mu > 0$, the Newton step $\Delta\Lambda = (\Delta\mathrm{x}, \Delta v, \Delta w, \Delta s, \Delta y, \Delta z)$ at the interior-point $\Lambda = (\mathrm{x}, v, w, s, y, z)$ is obtained by solving the following system

$$\bar{\mathcal{D}}_{\mu}(\Lambda)\Delta\Lambda = -q_{\hat{\beta}}(\Lambda, \mu), \tag{3.8}$$

where

$$\bar{\mathcal{D}}_{\mu}(\Lambda) = \begin{bmatrix} H(\mathrm{x}, y, z) & 0 & 0 & -I & -(A_{\hat{\beta}}(\mathrm{x}))^{\top} & -(B(\mathrm{x}))^{\top} \\ 0 & Y & 0 & 0 & V & 0 \\ 0 & 0 & Z & 0 & 0 & W \\ S & 0 & 0 & X & 0 & 0 \\ A_{\hat{\beta}}(\mathrm{x}) & -I & 0 & 0 & 0 & 0 \\ B(\mathrm{x}) & 0 & -I & 0 & 0 & 0 \end{bmatrix}$$

and

$$q_{\hat{\beta}}(\Lambda, \mu) = \begin{bmatrix} c - s - (A_{\hat{\beta}}(\mathrm{x}))^{\top} y - (B(\mathrm{x}))^{\top} z \\ -\mu e + VYe \\ -\mu e + WZe \\ -\mu e + S\mathrm{X}e \\ \hat{\beta}c^{\top}\mathrm{x} - f(\mathrm{x}) - v \\ h(\mathrm{x}) - w \end{bmatrix}.$$

The matrix $\bar{\mathcal{D}}_{\mu}(\Lambda)$ is not symmetric. However, it can be made symmetric by multiplying the first row by -1, the second row by $-V^{-1}$, the third row by $-W^{-1}$ and the forth row by S^{-1}. Accordingly, we get

$$\begin{bmatrix} -H(\mathrm{x},y,z) & 0 & 0 & I & (A_{\hat{\beta}}(\mathrm{x}))^\top & (B(\mathrm{x}))^\top \\ 0 & -V^{-1}Y & 0 & 0 & -I & 0 \\ 0 & 0 & -W^{-1}Z & 0 & 0 & -I \\ I & 0 & 0 & S^{-1}X & 0 & 0 \\ A_{\hat{\beta}}(\mathrm{x}) & -I & 0 & 0 & 0 & 0 \\ B(\mathrm{x}) & 0 & -I & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta \mathrm{x} \\ \Delta v \\ \Delta w \\ \Delta s \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} \vartheta_{\hat{\beta}} \\ -\gamma_1 \\ -\gamma_2 \\ \gamma_3 \\ \varrho_{\hat{\beta}} \\ \rho \end{bmatrix}, \tag{3.9}$$

where the expressions of $\vartheta_{\hat{\beta}},\ \gamma_1,\ \gamma_2,\ \gamma_3,\ \varrho_{\hat{\beta}}$ and ρ are as follows:

$$\left.\begin{aligned} \vartheta_{\hat{\beta}}(\mathrm{x},s,y,z) &= c - s - (A_{\hat{\beta}}(\mathrm{x}))^\top y - (B(\mathrm{x}))^\top z, \\ \gamma_1(v,y) &= \mu V^{-1}e - y, \\ \gamma_2(w,z) &= \mu W^{-1}e - z, \\ \gamma_3(x,s) &= \mu S^{-1}e - \mathrm{x}, \\ \varrho_{\hat{\beta}}(\mathrm{x},v) &= f(\mathrm{x}) + v - \hat{\beta}c^\top \mathrm{x}, \\ \text{and} \ \ \rho(\mathrm{x},w) &= w - h(\mathrm{x}). \end{aligned}\right\} \tag{3.10}$$

Note that $\varrho_{\hat{\beta}}$ and ρ together find *primal infeasibility* and $\vartheta_{\hat{\beta}}$ gives *dual infeasibility*. If $\rho_{\hat{\beta}}$ and ρ vanish at a point, then the point is primal feasible. Moreover, let

$$\nu(\Lambda;\mu) = \max\{\|\rho\|, \|\rho_{\hat{\beta}}\|, \|\vartheta_{\hat{\beta}}\|, \|\gamma_1\|, \|\gamma_2\|, \|\gamma_3\|\}. \tag{3.11}$$

For a given $\epsilon > 0$, we define the approximated perturbed KKT point as an interior point Λ that satisfies $\nu(\Lambda;\mu) \leq \epsilon$. Note that $\nu(\Lambda;\mu) = 0$ and Λ is interior point if and only if Λ is a perturbed KKT point. Also, $\nu(\Lambda;0) = 0$ and $\Lambda \geq 0$ if and only if Λ is a KKT point of the problem (3.3).

We note that second and third equations of (3.9) can be used to eliminate Δv and Δw without producing any off-diagonal fill-in in the remaining system with the help of the following equations:

$$\left.\begin{aligned} \Delta v &= VY^{-1}(\gamma_1 - \Delta y) \\ \Delta w &= WZ^{-1}(\gamma_2 - \Delta z) \\ \Delta s &= S\mathrm{X}^{-1}\left(\gamma_3 - \Delta \mathrm{x}\right). \end{aligned}\right\} \tag{3.12}$$

Accordingly, from (3.9), the resulting *reduced KKT system* is given by

$$\begin{bmatrix} -\left(H(\mathrm{x},y,z) + S\mathrm{X}^{-1}\right) & (A_{\hat{\beta}}(\mathrm{x}))^\top & (B(\mathrm{x}))^\top \\ A_{\hat{\beta}}(\mathrm{x}) & VY^{-1} & 0 \\ B(\mathrm{x}) & 0 & WZ^{-1} \end{bmatrix} \begin{bmatrix} \Delta \mathrm{x} \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} \vartheta_{\hat{\beta}} - S\mathrm{X}^{-1}\gamma_3 \\ \varrho_{\hat{\beta}} + VY^{-1}\gamma_1 \\ \rho + WZ^{-1}\gamma_2 \end{bmatrix}. \tag{3.13}$$

The system (3.13) has a unique solution, provided the matrix $H(\mathrm{x},y,z) + S\mathrm{X}^{-1}$ is nonsingular (see [87]). The solution of the system (3.13) provides $\Delta \mathrm{x}$, Δy and Δz. Then, by using (3.12), one can obtain Δv, Δw and Δs. The following theorem gives explicit formulas of the solution to system (3.13).

Theorem 3.1 *We denote*

$$N_{\hat{\beta}}(\Lambda) = H(\mathrm{x},y,z) + S\mathrm{X}^{-1} + (A_{\hat{\beta}}(\mathrm{x}))^\top V^{-1} Y A_{\hat{\beta}}(\mathrm{x}) + (B(\mathrm{x}))^\top W^{-1} Z B(\mathrm{x}).$$

If at a point $\Lambda = (\mathrm{x},v,w,s,y,z)$, $N_{\hat{\beta}}(\Lambda)$ is nonsingular, then the system (3.9) has a unique solution. In particular,

$$\left.\begin{aligned} \Delta \mathrm{x} &= N_{\hat{\beta}}^{-1}\left(-\vartheta_{\hat{\beta}} + S\mathrm{X}^{-1}\gamma_3 + (A_{\hat{\beta}}(\mathrm{x}))^\top(\gamma_1 + V^{-1}Y\varrho_{\hat{\beta}}) + (B(\mathrm{x}))^\top(\gamma_2 + W^{-1}Z\rho)\right) \\ \Delta v &= A_{\hat{\beta}}(\mathrm{x})\Delta \mathrm{x} - \varrho_{\hat{\beta}} \\ \Delta w &= B(\mathrm{x})\Delta \mathrm{x} - \rho. \end{aligned}\right\} \tag{3.14}$$

Proof: By solving the second and the third equations of (3.13) for Δy and Δz, we obtain $\Delta y = V^{-1}Y\varrho_{\hat{\beta}} + \gamma_1 - V^{-1}YA_{\hat{\beta}}(\mathrm{x})\Delta \mathrm{x}$ and $\Delta z = W^{-1}Z\rho + \gamma_2 - W^{-1}ZB(\mathrm{x})\Delta \mathrm{x}$.

Eliminating Δy and Δz from the first block of the system (3.13), we get

$$\Delta \mathrm{x} \;=\; N_{\hat{\beta}}^{-1}\left(-\vartheta_{\hat{\beta}} + S\mathrm{X}^{-1}\gamma_3 + (A_{\hat{\beta}}(\mathrm{x}))^\top(\gamma_1 + V^{-1}Y\varrho_{\hat{\beta}}) + (B(\mathrm{x}))^\top(\gamma_2 + W^{-1}Z\rho)\right).$$

We notice that the square matrix of order $(n+\mathcal{J}+\mathcal{P}+1)\times(n+\mathcal{J}+\mathcal{P}+1)$ in the left side of the system (3.13) is quasi-definite, and therefore is nonsingular in nature (see [87])). Hence, we can compute Δv and Δw uniquely as $\Delta v = A_{\hat{\beta}}(\mathrm{x})\Delta\mathrm{x} - \varrho_{\hat{\beta}}$, $\Delta w = B(\mathrm{x})\Delta\mathrm{x} - \rho$. $\square$

Note 3.1 *Let $\Lambda = (\mathrm{x}, v, w, s, y, z)$ be the current point of iteration and the matrix $N_{\hat{\beta}}(\Lambda)$ is nonsingular. Then, Theorem 3.1 provides the solution of the system (3.13). If at any iteration the matrix $N_{\hat{\beta}}(\Lambda)$ is singular, we replace the matrix $H(\mathrm{x}, y, z)$ by $\hat{H}(\mathrm{x}, y, z)$ to make the matrix $N_{\hat{\beta}}(\Lambda)$ nonsingular, where $\hat{H}(\mathrm{x}, y, z) = H(\mathrm{x}, y, z) + \lambda I$ and $\lambda > 0$ is chosen such that the matrix $\hat{H}(\mathrm{x}, y, z)$ is positive definite, I being the identity matrix of the order of $\hat{H}$.*

To find a solution of (3.5), the algorithm that we propose below proceeds from an initial point $\left(\mathrm{x}^{(0)}, v^{(0)}, w^{(0)}, s^{(0)}, y^{(0)}, z^{(0)}\right)$; then, at the k-th iteration, it determines a search direction $\left(\Delta\mathrm{x}^{(k)}, \Delta v^{(k)}, \Delta w^{(k)}, \Delta s^{(k)}, \Delta y^{(k)}, \Delta z^{(k)}\right)$ with the help of Theorem 3.1 at $\left(\mathrm{x}^{(k)}, v^{(k)}, w^{(k)}, s^{(k)}, y^{(k)}, z^{(k)}\right)$; lastly, it chooses a step length $\alpha^{(k)}$, and then finds the next iterate by

$$\left.\begin{aligned}
\mathrm{x}^{(k+1)} &= \mathrm{x}^{(k)} + \alpha^{(k)}\Delta\mathrm{x}^{(k)},\\
v^{(k+1)} &= v^{(k)} + \alpha^{(k)}\Delta v^{(k)},\\
w^{(k+1)} &= w^{(k)} + \alpha^{(k)}\Delta w^{(k)},\\
s^{(k+1)} &= s^{(k)} + \alpha^{(k)}\Delta s^{(k)},\\
y^{(k+1)} &= y^{(k)} + \alpha^{(k)}\Delta y^{(k)},\\
z^{(k+1)} &= z^{(k)} + \alpha^{(k)}\Delta z^{(k)},
\end{aligned}\right\} \tag{3.15}$$

where the choice of $\alpha^{(k)}$ is detailed in the next section.

As described in (3.15), to move from current to the next iterate, the proposed algorithm first determines the search direction and then the step length. To make the barrier function in (3.4) well defined across the iterates, the step length is suitably chosen. Also, the identification of every iteration towards the solution of the KKT system (3.5) is discussed in the upcoming section.

3.5 Computation of step size

In this section, we discuss the computation of step length to be taken along the search directions in Theorem 3.1. The proposed algorithm updates the iteration point by (3.15). To guarantee that the successive points $\mathrm{x}_+ = \mathrm{x} + \alpha\Delta\mathrm{x},\ v_+ = v + \alpha\Delta v,\ w_+ = w + \alpha\Delta w,\ s_+ = s + \alpha\Delta s,\ y_+ = y + \alpha\Delta y$ are interior points, we choose the step length α by the following standard ratio formula (see [89]):

$$\alpha \ = \ \min\left\{\delta\left(\max_{i,j,l}\left\{-\frac{\Delta\mathrm{x}_l}{\mathrm{x}_l}, -\frac{\Delta v_j}{v_j}, -\frac{\Delta y_j}{y_j}, -\frac{\Delta w_i}{w_i}, -\frac{\Delta s_l}{s_l}, -\frac{\Delta z_i}{z_i}\right\}\right)^{-1}, 1\right\}, \qquad (3.16)$$

for some $0 < \delta \leq 1$.

Although the step length calculated by (3.16) ensures that the vectors $\mathrm{x},\ v,\ w$ and s remain in the interior, there is no guarantee of the reduction in the objective function and convergence of the generated sequence to a minimum point. This can be seen by taking the unconstrained optimization problem $f(x) = (1 + x^2)^{1/2}$ with an initial $|x_0| > 1$. Merit functions shorten the interval $(0, \alpha]$ such that an appropriate reduction towards optimality can be made along the search direction.

3.6 Merit function

To solve nonlinear constrained optimization problems, IPMs simultaneously minimizes both of objective function and an infeasibility measure. Therefore, the progress in the proposed IPM towards optimality is measured by a merit function that incorporates both the objective function and the infeasibility terms. In this article, we introduce a merit function that is corresponding to the barrier problem (3.4).

For any $\mu > 0$, consider the merit function $\mathcal{M}_{\eta,\mu} : \mathbb{R}^{(n+1)+\mathcal{P}+\mathcal{J}+(n+1)+\mathcal{P}+\mathcal{J}} \to \mathbb{R}$, defined by

$$\mathcal{M}_{\eta,\mu}(\Lambda) = c^\top \mathrm{x} + \varrho_{\hat{\beta}}^\top y + \rho^\top z - s^\top \mathrm{x} + \eta \Psi_\mu(\Lambda), \tag{3.17}$$

where η is the nonnegative penalty parameter, and

$$\Psi_\mu(\Lambda) = \frac{1}{2}\left(\varrho_{\hat{\beta}}^\top \rho_{\hat{\beta}} + \rho^\top \rho\right) + \mathrm{x}^\top s + v^\top y + w^\top z - \mu\left(\sum_{i=1}^{n+1} \log(\mathrm{x}_i s_i) + \sum_{j=1}^{\mathcal{P}} \log(v_j y_j) + \sum_{l=1}^{m} \log(w_l z_l)\right)$$

is the penalty term of the merit function $\mathcal{M}_{\eta,\mu}$.

Clearly, the penalty term is well-defined for $\Lambda > 0$. If the penalty parameter $\eta > 0$ is big and we attempt to minimize the merit function $\mathcal{M}_{\eta,\mu}$, then a lot of computaional effort will be concentrated on making the penalty term Ψ_μ towards zero.

The next two theorems address the global property of the merit function.

Theorem 3.2 *Consider the barrier problem (3.4). Let the interior point $\Lambda_\mu^* = (\mathrm{x}_\mu^*, v_\mu^*, w_\mu^*, s_\mu^*, y_\mu^*, z_\mu^*)$ be such that $\mathcal{D}_\mu(\Lambda_\mu^*) = 0$ for $\mu > 0$. Then, for any $\eta > 0$, the point x_μ^* is the stationary point of $\mathcal{M}_{\eta,\mu}(\mathrm{x}, v_\mu^*, w_\mu^*, s_\mu^*, y_\mu^*, z_\mu^*)$.*

Proof: The gradient of the merit function $\mathcal{M}_{\eta,\mu}$ at point $(\mathrm{x}, v_\mu^*, w_\mu^*, s_\mu^*, y_\mu^*, z_\mu^*)$ is

$$\begin{aligned}\nabla_{\mathrm{x}} \mathcal{M}_{\eta,\mu}(\mathrm{x}, v_\mu^*, w_\mu^*, s_\mu^*, y_\mu^*, z_\mu^*) = c + \left(\nabla_{\mathrm{x}} \varrho_{\hat{\beta}}(\mathrm{x}, v_\mu^*)\right)^\top y_\mu^* + \left(\nabla_{\mathrm{x}} \rho(\mathrm{x}, w_\mu^*)\right)^\top z_\mu^* - s_\mu^* \\ + \eta\left[\left(\varrho_{\hat{\beta}}(\mathrm{x}, v_\mu^*)\right)^\top \nabla_{\mathrm{x}} \varrho_{\hat{\beta}}(\mathrm{x}, v_\mu^*) + \left(\rho(\mathrm{x}, w_\mu^*)\right)^\top \nabla_{\mathrm{x}} \rho(\mathrm{x}, w_\mu^*) + s_\mu^* - \mu \mathrm{X}^{-1} e\right].\end{aligned}$$

For any $\mu > 0$, at the interior point Λ_μ^*, the primal infeasibilities $\varrho_{\hat{\beta}}(\mathrm{x}_\mu^*, v_\mu^*)$ and $\rho(\mathrm{x}_\mu^*, w_\mu^*)$ vanish. Therefore, after replacing x by x_μ^* in the last expression, we get

$$\begin{aligned}\nabla_{\mathrm{x}} \mathcal{M}_{\eta,\mu}(\Lambda_\mu^*) &= c - s_\mu^* - \left(A_{\hat{\beta}}(\mathrm{x}_\mu^*)\right)^\top y_\mu^* - \left(B(\mathrm{x}_\mu^*)\right)^\top z_\mu^* + \eta\left[s_\mu^* - \mu(\mathrm{X}_\mu^*)^{-1} e\right] \\ &= \vartheta_{\hat{\beta}}(\mathrm{x}_\mu^*, s_\mu^*, y_\mu^*, z_\mu^*) + \eta\left[s_\mu^* - \mu(\mathrm{X}_\mu^*)^{-1} e\right] \\ &= 0,\end{aligned}$$

because the dual infeasibility $\vartheta_{\hat{\beta}}$ and the complementarity terms vanish at Λ^*_μ (see (3.5)). $\square$

Theorem 3.3 *Consider the barrier problem (3.4). Let the interior point $\Lambda^*_\mu = (\mathrm{x}^*_\mu, v^*_\mu, w^*_\mu, y^*_\mu, z^*_\mu)$ be such that $\mathcal{D}_\mu(\Lambda^*_\mu) = 0$ for $\mu > 0$. Then, there exists $\bar{\eta} > 0$ such that for $\eta \geq \bar{\eta}$, the Hessian of the merit function $\nabla^2_x \mathcal{M}_{\eta,\mu}(\Lambda^*_\mu)$ is positive definite.*

Proof: The Hessian of the merit function $\mathcal{M}_{\eta,\mu}$ is

$$\begin{aligned}\nabla^2_{\mathrm{X}} \mathcal{M}_{\eta,\mu}(\Lambda^*_\mu) &= H(\mathrm{x}^*_\mu, y^*_\mu, z^*_\mu) + \eta\Big[\left(\nabla_x \varrho_{\hat{\beta}}(\mathrm{x}^*_\mu, v^*_\mu)\right)^\top \nabla_x \varrho_{\hat{\beta}}(\mathrm{x}^*_\mu, v^*_\mu)\left(\nabla_{\mathrm{X}} \rho(\mathrm{x}^*_\mu, w^*_\mu)\right)^\top \nabla_{\mathrm{X}} \rho(\mathrm{x}^*_\mu, w^*_\mu) + \mu(\mathrm{X}^*_\mu)^{-2}\Big] \\ &= H(\mathrm{x}^*_\mu, y^*_\mu, z^*_\mu) + \eta S^*_\mu \left(\mathrm{X}^*_\mu\right)^{-1} + \eta\Big[\left(\nabla_x \varrho_{\hat{\beta}}(\mathrm{x}^*_\mu, v^*_\mu)\right)^\top \nabla_x \varrho_{\hat{\beta}}(\mathrm{x}^*_\mu, v^*_\mu) + \left(\nabla_{\mathrm{X}} \rho(\mathrm{x}^*_\mu, w^*_\mu)\right)^\top \nabla_{\mathrm{X}} \rho(\mathrm{x}^*_\mu, w^*_\mu)\Big].\end{aligned}$$

Choosing $\tilde{\eta} = \max\{|\lambda_H| : \lambda_H$ is an eigenvalue of $H(\mathrm{x}^*_\mu, y^*_\mu, z^*_\mu)\}$, we notice that $\tilde{\eta} > 0$ and the matrix $H(\mathrm{x}^*_\mu, y^*_\mu, z^*_\mu) + \tilde{\eta} S^*_\mu \left(\mathrm{X}^*_\mu\right)^{-1}$ is positive definite. Hence, for all $\eta \geq \tilde{\eta}$, $\nabla^2_x \mathcal{M}_{\eta,\mu}(\Lambda^*)$ is positive definite. $\square$

Note 3.2 *By Theorem 3.2 and Theorem 3.3, for any $\mu > 0$, if Λ^*_μ satisfies $\mathcal{D}_\mu(\Lambda^*_\mu) = 0$, then there exists a $\bar{\eta} > 0$ such that $\mathrm{x}^*_\mu = \arg\min \mathcal{M}_{\eta,\mu}(\mathrm{x}, v^*_\mu, w^*_\mu, y^*_\mu, z^*_\mu), \quad$ for all $\eta \geq \bar{\eta}$.*

Theorem 3.4 *For any $\mu > 0$, the penalty term Ψ_μ has a unique minimum value $3\mu\left(1 - \log \mu\right)$.*

Proof: The result can be obtained by computing the minimum of the following function

$$\varphi(\xi, \varsigma, \omega) = \xi + \varsigma + \omega - \mu\left(\log(\xi) + \log(\varsigma) + \log(\omega)\right).$$

The function φ has the minimizer $\xi = \mu,\ \varsigma = \mu,$ and $\omega = \mu$. Therefore, the minimum value of the function φ is $3\mu\left(1 - \log \mu\right)$. Also, the function φ is convex. Hence, $\xi = \mu,\ \varsigma = \mu,$ and $\omega = \mu$ is the unique minimizer of the function φ. $\square$

3.7 Descent direction

This section ensures that the Newton direction (3.14) is descent for the penalty function Ψ_μ and the merit function $\mathcal{M}_{\eta,\mu}$.

Theorem 3.5 *Consider the barrier problem (3.4). Suppose that the point* $\Lambda = (\mathrm{x}, v, w, s, y, z)$ *is an interior point. Then, for any* $\mu > 0$*, the following results hold:*

(i) The Newton direction $\Delta\Lambda = (\Delta\mathrm{x}, \Delta v, \Delta w, \Delta s, \Delta v, \Delta w)$ *is descent at* Λ *for the penalty term* Ψ_μ *if and only if* Λ *is not quasi-central point.*

(ii) The Newton direction $\Delta\Lambda = (\Delta\mathrm{x}, \Delta v, \Delta w, \Delta s, \Delta y, \Delta z)$ *is descent at* Λ *for the merit function* $\mathcal{M}_{\eta,\mu}$ *if and only if* Λ *is not quasi-central point.*

Proof: (i) We can easily compute the following gradients: $\nabla_{\mathrm{x}}\Psi_\mu = -\left(A_{\hat\beta}(\mathrm{x})\right)^\top \varrho_{\hat\beta} - (B(\mathrm{x}))^\top \rho + s - \mu x^{-1}$, $\nabla_v\Psi_\mu = \varrho_{\hat\beta} + y - \mu v^{-1}$, $\nabla_w\Psi_\mu = \rho + z - \mu w^{-1}$, $\nabla_s\Psi_\mu = \mathrm{x} - \mu s^{-1}$, $\nabla_y\Psi_\mu = v - \mu y^{-1}$ and $\nabla_z\Psi_\mu = w - \mu z^{-1}$.

The directional derivative of Ψ_μ in the direction $\Delta\Lambda$ is

$$
\begin{aligned}
(\nabla\Psi_\mu)^\top \Delta\Lambda &= (\nabla_{\mathrm{x}}\Psi_\mu)^\top \Delta\mathrm{x} + (\nabla_v\Psi_\mu)^\top \Delta v + (\nabla_w\Psi_\mu)^\top \Delta w + (\nabla_s\Psi_\mu)^\top \Delta s + (\nabla_y\Psi_\mu)^\top \Delta y + (\nabla_z\Psi_\mu)^\top \Delta z \\
&= \left(-\left(A_{\hat\beta}(\mathrm{x})\right)^\top \varrho_{\hat\beta} - (B(\mathrm{x}))^\top \rho\right)^\top \Delta\mathrm{x} + s^\top\Delta\mathrm{x} - \mu(\mathrm{x}^{-1})^\top\Delta\mathrm{x} + \varrho_{\hat\beta}^\top\Delta v + \rho^\top\Delta w + \mathrm{x}^\top\Delta s \\
&\quad - \mu(s^{-1})^\top\Delta s - \mu\left(v^{-1}\right)^\top \Delta v - \mu\left(w^{-1}\right)^\top\Delta w - \mu\left(y^{-1}\right)^\top\Delta y - \mu\left(z^{-1}\right)^\top\Delta z. \\
&= -\varrho_{\hat\beta}^\top\left(A_{\hat\beta}(\mathrm{x})\Delta\mathrm{x} - \Delta v\right) - \rho^\top(B(\mathrm{x}) - \Delta w) + \mathrm{x}^\top\Delta s + s^\top\Delta\mathrm{x} - \mu\left((\mathrm{x}^{-1})^\top\Delta\mathrm{x} + (s^{-1})^\top\Delta s\right) \\
&\quad - \mu\left(\left(v^{-1}\right)^\top\Delta v + \left(y^{-1}\right)^\top\Delta y\right) - \mu\left(\left(w^{-1}\right)^\top\Delta w + \left(z^{-1}\right)^\top\Delta z\right) \\
&= -\varrho_{\hat\beta}^\top\varrho_{\hat\beta} - \rho^\top\rho + 2(n+1)\mu - \mu\left((\mathrm{x}^{-1})^\top\Delta\mathrm{x} + (s^{-1})^\top\Delta s\right) - \mu\left(\left(v^{-1}\right)^\top\Delta v + \left(y^{-1}\right)^\top\Delta y\right) \\
&\quad - \mu\left(\left(w^{-1}\right)^\top\Delta w + \left(z^{-1}\right)^\top\Delta z\right),
\end{aligned}
$$

where the last equality is obtained by the last two equations of the system (3.9) and the complementarity conditions.

Now, if we set $\Theta = (\mathrm{X}S)^{1/2}\,e$, $\Upsilon = (VY)^{1/2}e$ and $\Phi = (WZ)^{1/2}e$ then we obtain

$$
(\nabla\Psi_\mu)^\top \Delta\Lambda = -\left(\|\varrho_{\hat\beta}\|^2 + \|\rho\|^2 + \|\Theta - \mu\Theta^{-1}\|^2 + \|\Upsilon - \mu\Upsilon^{-1}\|^2 + \|\Phi - \mu\Phi^{-1}\|^2\right) < 0. \tag{3.18}
$$

(ii) We have

$$
\begin{aligned}
(\nabla \mathcal{M}_{\eta,\mu})^\top \Delta\Lambda = {} & (\nabla_{\mathrm{x}} \mathcal{M}_{\eta,\mu})^\top \Delta \mathrm{x} + (\nabla_v \mathcal{M}_{\eta,\mu})^\top \Delta v + (\nabla_w \mathcal{M}_{\eta,\mu})^\top \Delta w + (\nabla_s \mathcal{M}_{\eta,\mu})^\top \Delta s \\
& + (\nabla_y \mathcal{M}_{\eta,\mu})^\top \Delta y + (\nabla_z \mathcal{M}_{\eta,\mu})^\top \Delta z \\
= {} & \vartheta_{\hat{\beta}}^\top \Delta \mathrm{x} - \mathrm{x}^\top \Delta s + y^\top \Delta v + z^\top \Delta w + \varrho_{\hat{\beta}}^\top \Delta y + \rho^\top \Delta z + \eta \, (\nabla \Psi_\mu)^\top \Delta\Lambda.
\end{aligned}
$$

Since $(\nabla \Psi_\mu)^\top \Delta\Im < 0$, the least value of penalty parameter η to make the Newton direction as descent for the merit function $\mathcal{M}_{\eta,\mu}$ is

$$
\tilde{\eta} = \frac{\vartheta_{\hat{\beta}}^\top \Delta \mathrm{x} - \mathrm{x}^\top \Delta s + y^\top \Delta v + z^\top \Delta w + \varrho_{\hat{\beta}}^\top \Delta y + \rho^\top \Delta z}{|\,(\nabla \Psi_\mu)^\top \Delta\Lambda|}. \tag{3.19}
$$

Hence, for all $\eta > \tilde{\eta}$, the Newton direction $\Delta\Lambda$ is descent for the merit function if and only if Λ is not quasi-central point.

□

3.8 Sufficient decrease

If the penalty parameter η is such that $\eta > \tilde{\eta}$, then, we can write $\eta = \tilde{\eta} + \delta,$ where $\delta > 0$. Since

$$
(\nabla \mathcal{M}_{\eta,\mu}(\Lambda))^\top \Delta\Lambda = \delta \, (\nabla \Psi_\mu)^\top \Delta\Lambda \tag{3.20}
$$

and $(\nabla \Psi_\mu)^\top \Delta\Lambda < 0$, then for any $\kappa \in (0,1)$, there exists a positive number $\bar{\alpha}$ such that for any $\alpha \in (0, \bar{\alpha})$, the following condition is satisfied:

$$
\Psi_\mu(\Lambda + \alpha\Delta\Lambda) \leq \Psi_\mu(\Lambda) + \alpha\kappa \, (\nabla \Psi_\mu)^\top \Delta\Lambda. \tag{3.21}
$$

As $(\nabla \mathcal{M}_{\eta,\mu})^\top \Delta\Lambda < 0,$ for $\eta > \tilde{\eta}$, for some $\alpha' \in (0, \bar{\alpha})$, we get

$$\mathcal{M}_{\eta,\mu}(\Lambda + \alpha'\Delta\Lambda) \leq \mathcal{M}_{\eta,\mu}(\Lambda) + \delta\alpha'\kappa\left(\nabla\mathcal{M}_{\eta,\mu}\right)^{\top}\Delta\Lambda. \tag{3.22}$$

Definition 3.2 *The following set of points represents the γ-neighbourhood of the quasi-central path corresponding to μ :*

$$\begin{aligned}\mathcal{N}_{\gamma}(\Lambda;\mu) = \{\Lambda : \mathrm{x} > 0, s > 0, v > 0, w > 0, y > 0, z > 0,\ \|\varrho_{\hat{\beta}}\|^2 + \|\rho\|^2 \\ + \|\Theta - \mu\Theta^{-1}\|^2 + \|\Upsilon - \mu\Upsilon^{-1}\|^2 + \|\Phi - \mu\Phi^{-1}\|^2 \leq \gamma\mu\},\end{aligned} \tag{3.23}$$

where $(\gamma,\mu) > 0,\ \Theta = (\mathrm{X}S)^{1/2}e,\ \Upsilon = (VY)^{1/2}e,$ and $\Phi = (WZ)^{1/2}e.$

By this definition, we are able to calculate the distance of an interior point from the KKT point corresponding to a $\mu > 0$.

3.9 Update of the penalty parameter

The penalty parameter η is chosen such that the Newton direction is a descent for the merit function. Corresponding to a μ and a given $\tilde{\delta}$, the current penalty parameter η_{new} is updated as follows:

$$\eta_{\text{new}} = \begin{cases} \tilde{\eta} + \tilde{\delta}, & \text{if } \tilde{\eta} + \tilde{\delta} > \eta, \\ \tilde{\eta} + \delta, & \text{otherwise,} \end{cases} \tag{3.24}$$

where $\tilde{\eta}$ is calculated by (3.19).

3.10 Description of the initial point and the barrier parameter

In Algorithm 4, the initial point is chosen such that all the components of the vectors $\mathrm{x}^{(0)}, s^{(0)}, v^{(0)}, w^{(0)}, y^{(0)}$ and $z^{(0)}$ are positive. If an MOP has bound constraints with negative values, then we can easily formulate the problem in the form of (3.1). Consequently, we can always initialize x such that $x_i > 0$ for all $i \in \{1, 2, \ldots, n\}$. Algorithm 4 is based on the strategy of the quasi-central path (3.6). It solves the KKT system (3.5) for decreasing values of $\mu > 0$ until $\nu(\Lambda;\mu)$ is not less than a given precision parameter ϵ

(see (3.11)). In the proposed algorithm, we initialize the value of the barrier parameter μ_0 as (see [93])

$$\mu_0 = \frac{\mathrm{x}^\top s + v^\top y + w^\top z}{\mathcal{P} + \mathcal{J}}. \tag{3.25}$$

Then, we solve the KKT system (3.5) for μ_0 until it satisfies the (μ, γ)-neighbourhood condition (3.2). Suppose an iterate $\Lambda = (\mathrm{x}, v, w, s, y, z)$ lies in (μ, γ)-neighbourhood (see (3.2)) but fails the stopping criteria condition $\nu(\Lambda; \mu) \leq \epsilon$ for the outer loop, then we decrease the barrier parameter at the $(k+1)$-th iteration by the value

$$\mu^{(k+1)} = \wp \left(\|\varrho_{\hat{\beta}}^{(k)}\|^2 + \|\rho^{(k)}\|^2 + \|\Theta^{(k)} - \mu^{(k)}(\Theta^{(k)})^{-1}\|^2 + \|\Upsilon^{(k)} - \mu(\Upsilon^{(k)})^{-1}\|^2 + \|\Phi^{(k)} - \mu(\Phi^{(k)})^{-1}\|^2 \right), \tag{3.26}$$

where $\wp \in (0,1)$, $\Theta^{(k)} = (\mathrm{X}^{(k)} S^{(k)})^{1/2} e$, $\Upsilon^{(k)} = (V^{(k)} Y^{(k)})^{1/2} e$ and $\Phi^{(k)} = (W^{(k)} Z^{(k)})^{1/2} e$.

In Algorithm 4, we describe a step-by-step procedure for finding Pareto optimal points for a given optimization problem with the help of the above process.

3.11 Global convergence results

The following section discusses the global convergence theory for the Algorithm 4. The process of global convergence analysis begins with an additional set of assumptions and lemmas. Assumptions for developing the convergence theory of Algorithm 4 are as follows:

(A1) for $\mathrm{x} \geq 0$, the set $\{\nabla h_1(\mathrm{x}), \nabla h_2(\mathrm{x}), \ldots, \nabla h_{\mathcal{J}}(\mathrm{x})\}$ is linearly independent.

(A2) The iteration sequences $\mathrm{x}^{(k)}$, $v^{(k)}$, $w^{(k)}$, $y^{(k)}$ and $z^{(k)}$ generated by the Algorithm 4 is bounded above.

(A3) The matrix $H(\mathrm{x}, y, z) + S\mathrm{X}^{-1}$ is positive definite.

Lemma 3.1 *Assume $\Lambda = (\mathrm{x}, v, w, s, y, z)$ is an interior point. Then, for any $\mu > 0$ and under the assumption (A3), the matrix $D_{\hat{\beta}}(\Lambda)$ is nonsingular.*

Proof: See [90]. □

Algorithm 4 Ideal Cone-IPM (IC-IPM) for MOPs

Inputs:
(a) Given MOP:

$$\begin{cases} \text{minimize} & F(x) \\ \text{subject to} & x \in \mathcal{X} \end{cases}$$

(b) Provide the number of subproblems to be solved, N
Finding the Ideal Point: Find $f_i^* = \min\{f_i(x) : x \in \mathcal{X}\}$ for each $i \in \{1, 2, \dots, \mathcal{P}\}$ using IPM [89], and then set ideal point $F^* = (f_1^*, f_2^*, \dots, f_{\mathcal{P}}^*)^\top$
Initialization:
Set Pareto set $\leftarrow \emptyset$
Give an initial point such that $\Lambda^{(k)} = (\mathrm{x}^{(k)}, s^{(k)}, v^{(k)}, w^{(k)}, y^{(k)}, z^{(k)}) > 0$
Choose the values of the parameters $\tilde{\delta} = 2$ and $\kappa = 0.85$
Give a value of the precision parameter $\epsilon > 0$ for the optimum solutions to (3.4)
Set $k \leftarrow 0$
for $i = 1 : 1 : N$ **do**
  Choose randomly a direction $\hat{\beta}$ from (2.3)
  while $\nu(\Lambda^{(k)}) \geq \epsilon$ **do**
    Choose $\mu^{(k)}$ by (3.25)
    while $\Lambda^{(k)} \notin \mathcal{N}_{\mu^{(k)}}(\gamma)$ **do** (inner loop)
      Calculate the direction $(\Delta\mathrm{x}^{(k)}, \Delta s, \Delta v^{(k)}, \Delta w^{(k)})$ by using (3.14)
      Choose step length α by the formula (3.16)
      Calculate $\tilde{\eta}$ by the expression (3.19)
      Calculate $\eta^{(k)}$ and $\delta^{(k)}$ from to ensure that the Newton direction is descent for $\mathcal{M}_{\eta,\mu}$
      Find $\alpha^{(k)} \in (0, \alpha)$ such that the following Armijo condition is satisfied

$$\mathcal{M}_{\eta,\mu^{(k)}}(\Lambda^{(k)} + \alpha^{(k)}\Delta\Lambda^{(k)}) \leq \mathcal{M}_{\eta,\mu^{(k)}}(\Lambda^{(k)}) + \delta^{(k)}\alpha^{(k)}\kappa \left(\nabla\mathcal{M}_{\eta,\mu^{(k)}}\right)^\top \Delta\Lambda^{(k)}$$

      Set $\Lambda^{(k+1)} = \Lambda^{(k)} + \alpha\Delta\Lambda^{(k)}$
    end while
  end while
  Calculate $F(\mathrm{x}^{(k)}) = \hat{\beta}c^T\mathrm{x}^{(k)} - v^{(k)}$
  Update Pareto set $\leftarrow$ Pareto set $\bigcup\{F(\mathrm{x}^{(k)})\}$
end for
return Pareto set

Lemma 3.2 *For any $\mu > 0$, the sequence of interior points $\{\Lambda^{(k)} : \Lambda^{(k)} > 0\}$, produced by the inner loop of Algorithm 4 is bounded whenever the conditions (A2) and (A3) follows. Moreover, the components $\mathrm{x}_i^{(k)}$, $s_i^{(k)}$, $v_j^{(k)}$, $y_j^{(k)}$, $w_l^{(k)}$ and $z_l^{(k)}$ are bounded away from zero.*

Proof: By assumption (A2), the sequences $\{\mathrm{x}^{(k)}\}$, $\{v^{(k)}\}$ and $\{w^{(k)}\}$ are bounded above. Let $\Psi_\mu^{(0)}$ be the value of penalty term at the initial point. Note that the inequality

$$3\mu\,(1 - \log\mu) \leq \Psi_\mu(\Lambda^{(k)} + \alpha^{(k)}\Delta\Lambda^{(k)}) \leq \Psi_\mu^{(0)} \tag{3.27}$$

holds by Theorem 3.4 and Theorem 3.5. Further, observe that

$$\begin{aligned}
\mathrm{x}_j^{(k)} s_j^{(k)} - \mu \log\left(\mathrm{x}_j^{(k)} s_j^{(k)}\right) &\to \infty \quad \text{if either } \mathrm{x}_j^{(k)} s_j^{(k)} \to 0 \text{ or } \mathrm{x}_j^{(k)} s_j^{(k)} \to \infty \\
v_i^{(k)} y_i^{(k)} - \mu \log\left(v_i^{(k)} y_i^{(k)}\right) &\to \infty \quad \text{if either } v_i^{(k)} y_i^{(k)} \to 0 \text{ or } v_i^{(k)} y_i^{(k)} \to \infty \\
w_l^{(k)} z_l^{(k)} - \mu \log\left(w_l^{(k)} z_l^{(k)}\right) &\to \infty \quad \text{if either } w_l^{(k)} z_l^{(k)} \to 0 \text{ or } w_l^{(k)} z_l^{(k)} \to \infty.
\end{aligned}$$

However, by (3.27), we see that the products $\mathrm{x}_j^{(k)} s_j^{(k)}$, $v_i^{(k)} y_i^{(k)}$ and $w_l^{(k)} z_l^{(k)}$ are bounded above and bounded away from zero. Therefore, the sequences $s^{(k)}$, $y^{(k)}$ and $z^{(k)}$ are bounded above and the components $\mathrm{x}_j^{(k)}$, $s_j^{(k)}$, $v_i^{(k)}$, $y_i^{(k)}$, $w_l^{(k)}$ and $z_l^{(k)}$ are bounded away from zero. □

Lemma 3.3 *Under the assumption (A1), (A2) and (A3), the merit function $\mathcal{M}_{\eta,\mu}$ is bounded below.*

Proof: By Theorem 3.4, the penalty term is bounded below and the term $c^\top \mathrm{x} + \varrho_{\hat\beta}^\top y + \rho^\top z - s^\top \mathrm{x}$ is bounded below due to Assumptions (A1) and (A2), and Lemma 3.2. Hence, the merit function $\mathcal{M}_{\eta,\mu}$ is bounded below. □

Theorem 3.6 *(i) Consider $\mu > 0$. Assume that the sequence $\{\Lambda^{(k)} = (\mathrm{x}^{(k)}, v^{(k)}, w^{(k)}, s^{(k)}$ $y^{(k)}, z^{(k)})\}$ is created by the inner loop of Algorithm 4 and its limit point is $\Lambda_\mu^* = (\mathrm{x}_\mu^*, v_\mu^*, w_\mu^*, s_\mu^*, y_\mu^*, z_\mu^*)$. If $\bar{\mathcal{D}}_{\hat\beta}$ is continuous at Λ^* and $\bar{\mathcal{D}}_{\hat\beta}(\Lambda^*)$ is nonsingular then, Λ_μ^* lies on the quasi-central path (3.6).*

(ii) Suppose the problem (3.4) has an optimal point in $\mathcal{N}_\gamma(\Lambda;\mu)$ for a fixed μ and γ. Let the starting point $\Lambda^{(0)}$ is an interior point. Then Algorithm 4 will terminate in at most $\frac{\log\left(\frac{\gamma\mu}{\Xi(\Lambda^{(0)})}\right)}{\log(1-\tau)}$ iterations, where

$$\Xi(\Lambda^{(0)}) = \left(\|\varrho_{\hat\beta}^{(0)}\|^2 + \|\rho^{(0)}\|^2 + \|\Theta^{(0)} - \mu\left(\Theta^{(0)}\right)^{-1}\|^2 + \|\Upsilon^{(0)} - \mu\left(\Upsilon^{(0)}\right)^{-1}\|^2 + \|\Phi^{(0)} - \mu\left(\Phi^{(0)}\right)^{-1}\|^2\right).$$

Proof: (i) Since the limit point of the sequence $\{\Lambda^{(k)}\}$ is Λ_μ^* then, there exists a convergent subsequence $\Lambda^{(k_l)}$ such that $\Lambda^{(k_l)} \to \Lambda_\mu^*$.

We need to prove three properties regarding the subsequence $\Lambda^{(k_l)}$:

(a) The limit point Λ_μ^* is an interior point.

(b) The sequence of search direction $\{\Delta\Lambda^{(k_l)}\}$ is bounded.

(c) The sequence of steplengths $\{\alpha^{(k_l)}\}$ is bounded away from zero.

The first property can be obtained by Lemma 3.2, stating that the components $\mathrm{x}_j^{(k_l)}$ and $s_j^{(k_l)}$ are bounded away from zero. Therefore, $\mathrm{x}^{(k_l)} \to \mathrm{x}_\mu^* > 0$ and $s^{(k_l)} \to s_\mu^* > 0$. The property (b) holds because the set $E = \{\Lambda^{(k_l)}, \Lambda_\mu^*\}$ is compact and $\mathcal{D}$, $(\bar{\mathcal{D}})^{-1}$ are continuous functions on E.

To prove the property (c), if possible let the steplength sequence $\{\alpha^{k_l}\}$ generated by the Algorithm 4 is not bounded away from zero. Then,

$$\lim_{l\to\infty} \frac{\mathrm{x}_j^{(k_l)}}{|\Delta \mathrm{x}_j|^{(k_l)}} = 0 \quad \text{holds for at least one } j \in \{1, 2, \ldots\}.$$

Since $\mathrm{x}_j^{(k_l)}$ is bounded away from zero, $|\Delta \mathrm{x}_j|^{(k_l)}$ tends to infinity. However, this goes against the fact that $\{\Delta\Lambda^{(k)}\}$ is bounded. Hence, (c) holds.

Now, we are able to proof for the key part of the theorem. Consider the iterative sequence

$$\Lambda^{(k+1)} = \Lambda^{(k)} + \alpha^{(k)}\Delta\Lambda^{(k)},$$

where $\alpha^{(k)}$ is computed by (3.16) and $\Delta\Lambda^{(k)}$ is determined by solving (3.13). At k-th iteration, let $\eta^{(k)} > 0$ and $\mu^{(k)} > 0$ stand for the value of the penalty parameter and barrier parameter, respectively. Also, $\Lambda^{(k)}$ satisfies (see Theorem 3.5) $\left(\nabla \mathcal{M}_{\eta^{(k)},\mu^{(k)}}(\Lambda^{(k)})\right)^\top \Delta\Lambda^{(k)} < 0$. Since the step length $\alpha^{(k)}$ is bounded away from zero, the sequence $\{\Lambda^{(k)}\}$ satisfies (see [101]) $\left(\nabla \mathcal{M}_{\eta^{(k)},\mu^{(k)}}(\Lambda^{(k)})\right)^\top \frac{\Delta\Lambda^{(k)}}{\|\Delta\Lambda^{(k)}\|} \to$

0. However, the sequence $\{\Delta\Lambda^{(k)}\}$ is bounded and by (3.20), we obtain

$$\left(\nabla\mathcal{M}_{\eta^{(k)},\mu^{(k)}}(\Lambda^{(k)})\right)^\top \Delta\Lambda^{(k)} = \delta^{(k)}\left(\nabla\Psi_\mu(\Lambda^{(k)})\right)^\top \Delta\Lambda^{(k)} \to 0.$$

Since $\delta^{(k)} > 0$ and by (3.2), we obtain

$$\begin{aligned}\left(\nabla\Psi_\mu(\Lambda^{(k)})\right)^\top \Delta\Lambda^{(k)} = -\Bigg(&\|\varrho_{\hat\beta}^{(k)}\|^2 + \|\rho^{(k)}\|^2 + \|\Theta^{(k)} - \mu\left(\Theta^{(k)}\right)^{-1}\|^2 \\ &+ \|\Upsilon^{(k)} - \mu\left(\Upsilon^{(k)}\right)^{-1}\|^2 + \|\Phi^{(k)} - \mu\left(\Phi^{(k)}\right)^{-1}\|^2\Bigg) \to 0. \end{aligned} \tag{3.28}$$

This directly implies that $\lim_{k\to\infty}\varrho_{\hat\beta}^{(k)} = 0,\ \lim_{k\to\infty}\rho^{(k)} = 0,\ \lim_{k\to\infty}\mathrm{X}^{(k)}S^{(k)}e = \mu e,\ \lim_{k\to\infty}V^{(k)}Y^{(k)}e = \mu e,\ \lim_{k\to\infty}W^{(k)}Z^{(k)}e = \mu e.$

(ii) From (3.28), we can conclude that there exists $\tau_1,\ \tau_2,\ \tau_3,\ \tau_4,$ and τ_5 such that

$\|\varrho_{\hat\beta}^{(k+1)}\|^2 \le (1-\tau_1)\|\varrho_{\hat\beta}^{(k)}\|^2,\ \ \|\rho^{(k+1)}\|_2^2 \le (1-\tau_2)\|\rho^{(k)}\|^2,\ \ \|\Theta^{(k+1)} - \mu\left(\Theta^{(k+1)}\right)^{-1}\|^2 \le (1-\tau_3)\|\Theta^{(k)} - \mu\left(\Theta^{(k)}\right)^{-1}\|^2$

$\|\Upsilon^{(k+1)} - \mu\left(\Upsilon^{(k+1)}\right)^{-1}\|^2 \le (1-\tau_4)\|\Upsilon^{(k)} - \mu\left(\Upsilon^{(k)}\right)^{-1}\|^2$ and $\|\Phi^{(k+1)} - \mu\left(\Phi^{(k+1)}\right)^{-1}\|^2 \le (1-\tau_5)\|\Phi^{(k)} - \mu\left(\Phi^{(k)}\right)^{-1}\|^2.$

Denoting

$$\Xi(\Lambda^{(k)}) = \left(\|\varrho_{\hat\beta}^{(k)}\|^2 + \|\rho^{(k)}\|^2 + \|\Theta^{(k)} - \mu\left(\Theta^{(k)}\right)^{-1}\|^2 + \|\Upsilon^{(k)} - \mu\left(\Upsilon^{(k)}\right)^{-1}\|_2^2 + \|\Phi^{(k)} - \mu\left(\Phi^{(k)}\right)^{-1}\|^2\right)$$

and $\tau = \min\{\tau_1,\tau_2,\tau_3,\tau_4,\tau_5\}$, we obtain

$$\Xi(\Lambda^{(k)}) \le (1-\tau)\Xi(\Lambda^{(k-1)}) \le (1-\tau)^2\Xi(\Lambda^{(k-1)}) \le \cdots \le (1-\tau)^k\Xi(\Lambda^{(0)}). \tag{3.29}$$

Suppose that at the k-th iteration, $\Lambda^{(k)}$ lies in the γ-neighbourhood of the quasi-central path (3.2) corresponding to μ. Therefore, $\Xi(\Lambda^{(k)}) \le (1-\tau)^k\Xi(\Lambda^{(0)}) \le \gamma\mu$. Hence, the number of iterations to reach γ-neighbourhood of the quasi-central path is $\frac{\log\left(\frac{\gamma\mu}{\Xi(\Lambda^{(0)})}\right)}{\log(1-\tau)}$.

$\square$

We now show the global convergence of the Algorithm 4.

Theorem 3.7 *Assume that the assumptions (A1), (A2) and (A3) hold true. Then, the sequence* $\{\Lambda^{(k)} = \left(\mathrm{x}^{(k)}, v^{(k)}, w^{(k)}, s^{(k)}, y^{(k)}, z^{(k)}\right)\}$ *generated by Algorithm 4 has a limit point* $\Lambda^* = (\mathrm{x}^*, v^*, w^*, s^*, y^*, z^*)$*. Moreover, the limit point* Λ^* *satisfies* $\mathcal{D}(\Lambda^*) = 0$.

Proof: Since Λ^* is the limit point of the sequence $\{\Lambda^{(k)}\}$, there exists a convergent subsequence $\{\Lambda^{(k_l)}\}$ such that $\Lambda^{(k_l)} \to \Lambda^*$, where $\Lambda^{(k_l)} \in \mathcal{N}_{\mu_k}(\gamma)$ and $\mu_k \to 0$. As the step length is bounded away from zero (by Lemma 3.2), $\left(\Delta \mathrm{x}^{(k_l)}, \Delta v^{(k_l)}, \Delta w^{(k_l)}, \Delta s^{(k_l)}, \Delta y^{(k_l)}, \Delta z^{(k_l)}\right) \to 0$. From the first equation of (3.8), we have

$$c - s^{(k_l)} - \Delta s^{(k_l)} - (A_{\hat{\beta}}(\mathrm{x}^{(k_l)}))^\top \left(y^{(k_l)} + \Delta y^{(k_l)}\right) - (B(\mathrm{x}^{(k_l)}))^\top \left(z^{(k_l)} + \Delta z^{(k_l)}\right) = -H(\mathrm{x}^{(k_l)}, y^{(k_l)}, z^{(k_l)}) \Delta \mathrm{x}^{(k_l)}.$$

Taking $k_l \to \infty$, and since $\{H(\mathrm{x}^{(k_l)}, y^{(k_l)}, z^{(k_l)})\}$ is bounded, we obtain

$$c - s^* - (A_{\hat{\beta}}(\mathrm{x}^*))^\top y^* - (B(\mathrm{x}^*))^\top z^* = 0. \tag{3.30}$$

Since $\Lambda^{(k_l)} \in \mathcal{N}_{\mu_{k_l}}(\gamma)$,

$$\begin{aligned} &\|\varrho_{\hat{\beta}}(\mathrm{x}^{(k_l)}, v^{(k_l)})\|^2 + \|\rho(\mathrm{x}^{(k_l)}, w^{(k_l)})\|^2 + \|\Theta^{(k_l)} - \mu^{(k_l)} \left(\Theta^{(k_l)}\right)^{-1}\|^2 \\ &\quad + \|\Upsilon^{(k_l)} - \mu^{(k_l)} \left(\Upsilon^{(k_l)}\right)^{-1}\|^2 + \|\Phi^{(k_l)} - \mu^{(k_l)} \left(\Phi^{(k_l)}\right)^{-1}\|^2 \leq \gamma \mu^{k_l}. \end{aligned}$$

Again taking $k_l \to \infty$ then $\mu^{k_l} \to 0$ and

$$\left.\begin{aligned} \varrho_{\hat{\beta}}(\mathrm{x}^*, v^*) &= 0 \\ \rho(\mathrm{x}^*, w^*) &= 0 \\ \mathrm{X}^* S^* e &= 0 \\ V^* Y^* e &= 0 \\ \text{and} \quad W^* Z^* e &= 0. \end{aligned}\right\} \tag{3.31}$$

Hence, from (3.31), $\mathcal{D}(\Lambda^*) = 0$. Therefore, x^* is a KKT point. $\square$

3.12 Numerical experiments

This subsection reports the outcomes of several types of test problems found in the literature. The performance of Algorithm 4 is tested on constrained multiobjective as well as box constrained multiobjective problems. The test have been carried out on a PC with Intel Core i7-4770U 3.40 GHz CPU and 4GB RAM in MATLAB 2020a. We take some widely used multiobjective test problems (BNH, SRN, TNK, CONSTR, Kita, SWG) to test the performance of Algorithm 4. The test problem Kita is a maximization problem and remaining are the minimization problems. The details of these five constrained test problems are shown in Table 3.1. The Pareto points of these problems are shown in blue (see Figure 3.1, Figure 3.2 and Figure 3.3).

Table 3.1: Constrained test problems used in this study

Problem	Source	n	$\mathcal{P}$	Number of subproblems	Accuracy
BNH	[94]	2	2	300	$1.0E-6$
SRN	[96]	2	2	75	$1.0E-6$
TNK	[95]	2	2	100	$1.0E-6$
CONSTR	[97]	2	3	400	$1.0E-6$
Kita	[97]	2	3	150	$1.0E-6$
SGW	[102]	2	2	300	$1.0E-6$

3.13 Performance metrics

To measure the performance of Algorithm 4, it is necessary to ascertain how close obtained solution is to the actual solution. For this propose, the literature contains various evaluation criteria (see [103]). In this chapter, we used three main performance measures such as GD+ (modified generational distance) [104], HV (hyper volume) [105] and IGD (inverted generational distance) [106].

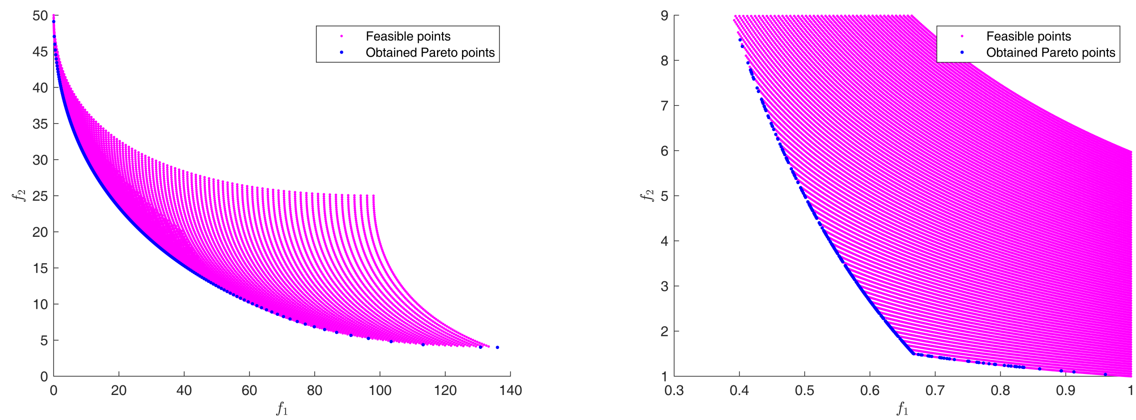

Figure 3.1: Obtained Pareto points of BNH and CONSTR problems by Algorithm 4

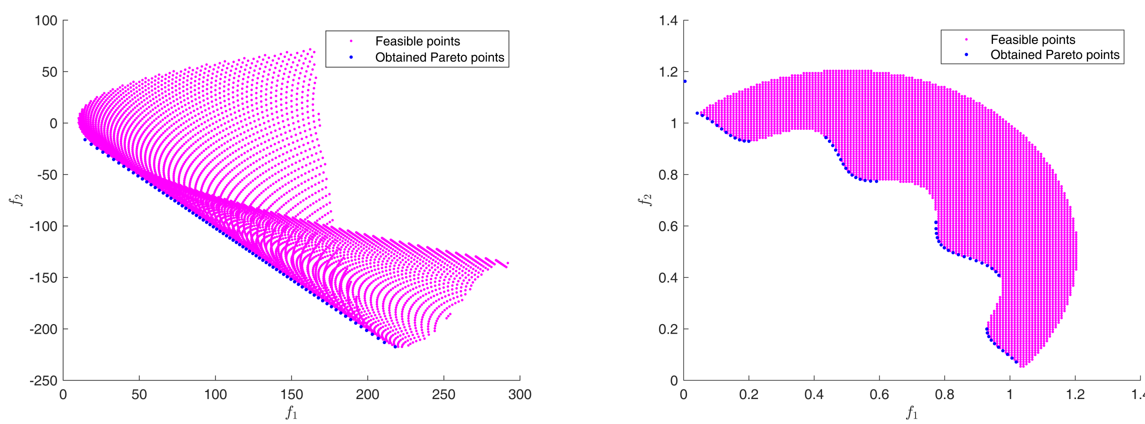

Figure 3.2: Obtained Pareto points of SRN and TNK problems by Algorithm 4

3.14 Performance of the Algorithm 4 on some test problems

We take the following optimization problem (FON [107]) to test the performance measure of Algorithm 4:

$$\left.\begin{array}{ll}\text{minimize} & \left(1-\exp\left(-\sum_{i=1}^{n}\left(x_i-\frac{1}{\sqrt{n}}\right)^2\right), 1-\exp\left(-\sum_{i=1}^{n}\left(x_i+\frac{1}{\sqrt{n}}\right)^2\right)\right)^\top \\ \text{subject to} & x\in[-4,4].\end{array}\right\} \tag{3.32}$$

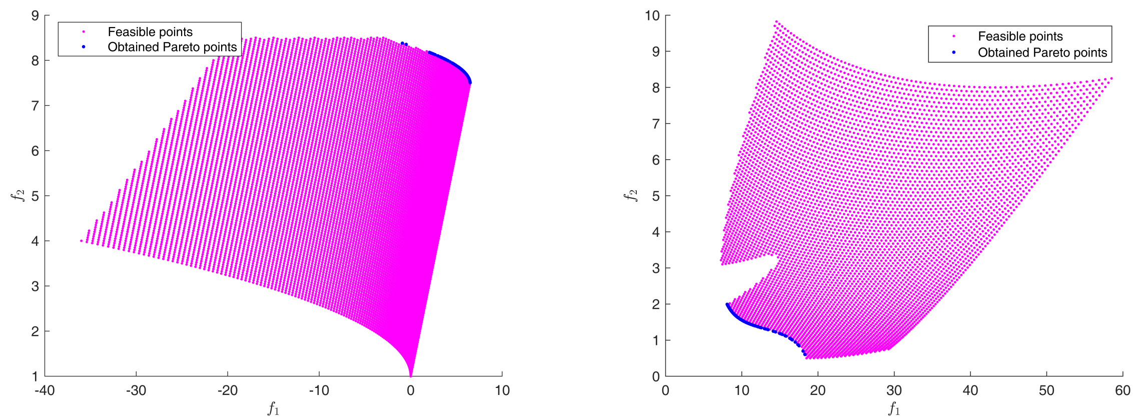

Figure 3.3: Obtained Pareto points of Kita and SGW problem by Algorithm 4

The efficient set is

$$\left\{x \in \mathbb{R}^n : \ x_i \in \left[-\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right], \ i = 1, 2, \ldots, n\right\}.$$

The performance measures (GD+, HV, IGD) of Algorithm 4 for the problem (3.32) are calculated by taking different values of n (see Table 3.2). Also, the obtained Pareto points comparison with true Pareto front are shown in Figures 3.5 and 3.6.

CEC09 and Zitzler, Deb and Thiele (ZDT) test suit

So far, we have seen the performance of IC-IPM on constrained MOPs and an unconstrained MOP with bound constraints (FON). The efficiency of IC-IPM is demonstrated by the good approximation of the Pareto fronts in all the above problems. Below, we use CEC09 test problems and ZDT test suite to further test the algorithm's capabilities. We evaluate the problems from CEC09 test suite [28] that are smooth. In comparison to the previous test problems, these are more complex problems. Table 3.3 shows the results of CEC09, obtained by IC-IPM and other well-known techniques, such as MOEA [108], ENS-MOEA [108], etc. The comparison Table 3.3 shows that the performance (based on IGD values) of Algorithm 4 (IC-IPM) is good compared to other algorithms on test suite CEC09. The obtained Pareto points, along with the true

Pareto front, are depicted below in Figures 3.5 and 3.6.

For the ZDT test suite, the outcomes achieved by IC-IPM and the reputed methods such as MOEA/D PBI Approach, MOEA/D Tchebycheff (TE) Approach, MOEA/D Weighted Sum (WS) Approach, Pareto-adaptive weight vectors ($pa\lambda$) based MoEA/D approach, NSGA-II, Cultural MOQPSO and MOWOATS are presented in Table 3.4. The obtained Pareto points, along with the true Pareto front, are depicted below in Figure 3.7.

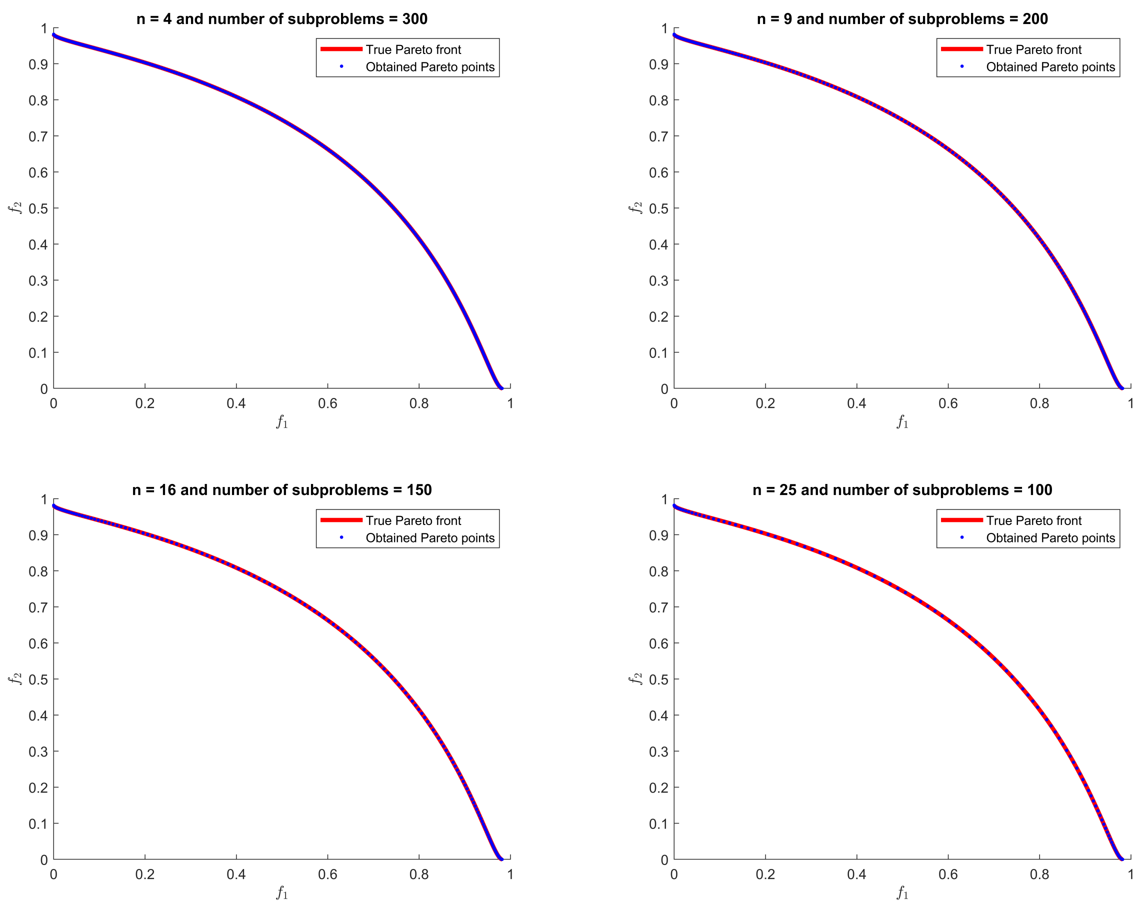

Figure 3.4: Obtained Pareto points of test problem FON by Algorithm 4

Table 3.2: Performance measures of test problem FON

Number of decision variables (n)	Number of subproblems	GD+	HV	IGD
4	300	$0.1760E-4$	0.34047	$6.2512E-4$
9	200	$1.4097E-4$	0.33941	$8.8491E-4$
16	150	$0.1171E-4$	0.33880	$8.8334E-4$
25	100	$8.1465E-4$	0.33716	$8.1986E-4$

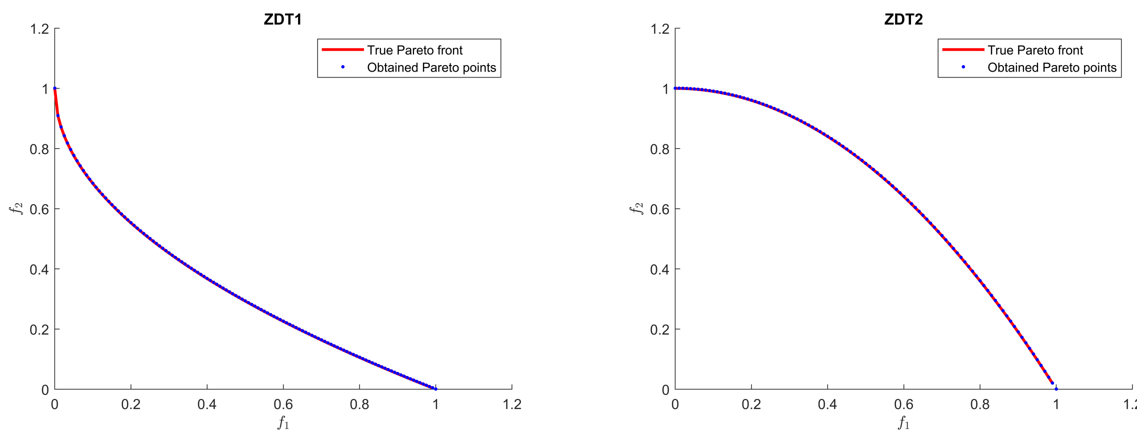

Figure 3.5: Obtained Pareto points of test problem ZDT1 and ZDT2 by Algorithm 4

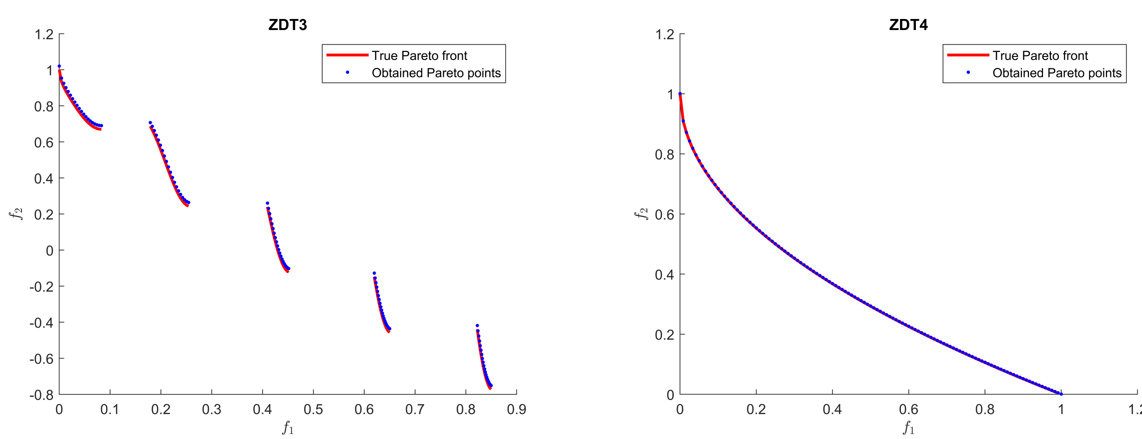

Figure 3.6: Obtained Pareto points of test problem ZDT3 and ZDT4 by Algorithm 4

Table 3.3: Comparison of IGD scores for CEC09

Problem	FRD [28]	FD [28]	RD [28]	OD [28]	MOEA/D [108]	ENS-MOEA/D [108]	Cultural MOQPSO [109]	MOWOATS [109]	IC-IPM
UF1	9.61E-3	6.40E-3	2.52E-3	2.78E-3	2.01E-3	1.64E-3	1.11E-2	2.32E-3	1.04E-4
UF2	8.41E-3	7.20E-3	9.43E-3	9.80E-3	4.82E-3	4.03E-3	2.15E-2	2.21E-3	1.12E-4
UF3	4.72E-3	3.11E-3	9.30E-3	1.05E-2	1.06E-2	2.66E-3	3.75E-2	9.77E-3	1.43E-4
UF4	5.92E-2	7.88E-2	8.81E-2	8.58E-2	6.24E-2	4.21E-2	5.98E-2	1.83E-3	1.33E-4
UF7	5.62E-3	6.30E-3	5.41E-3	3.24E-3	1.80E-3	1.72E-3	1.13E-2	2.12E-3	2.35E-4
UF8	6.60E-2	6.11E-2	5.69E-2	5.62E-2	4.28E-2	3.10E-2	1.18E-2	3.61E-3	5.78E-3

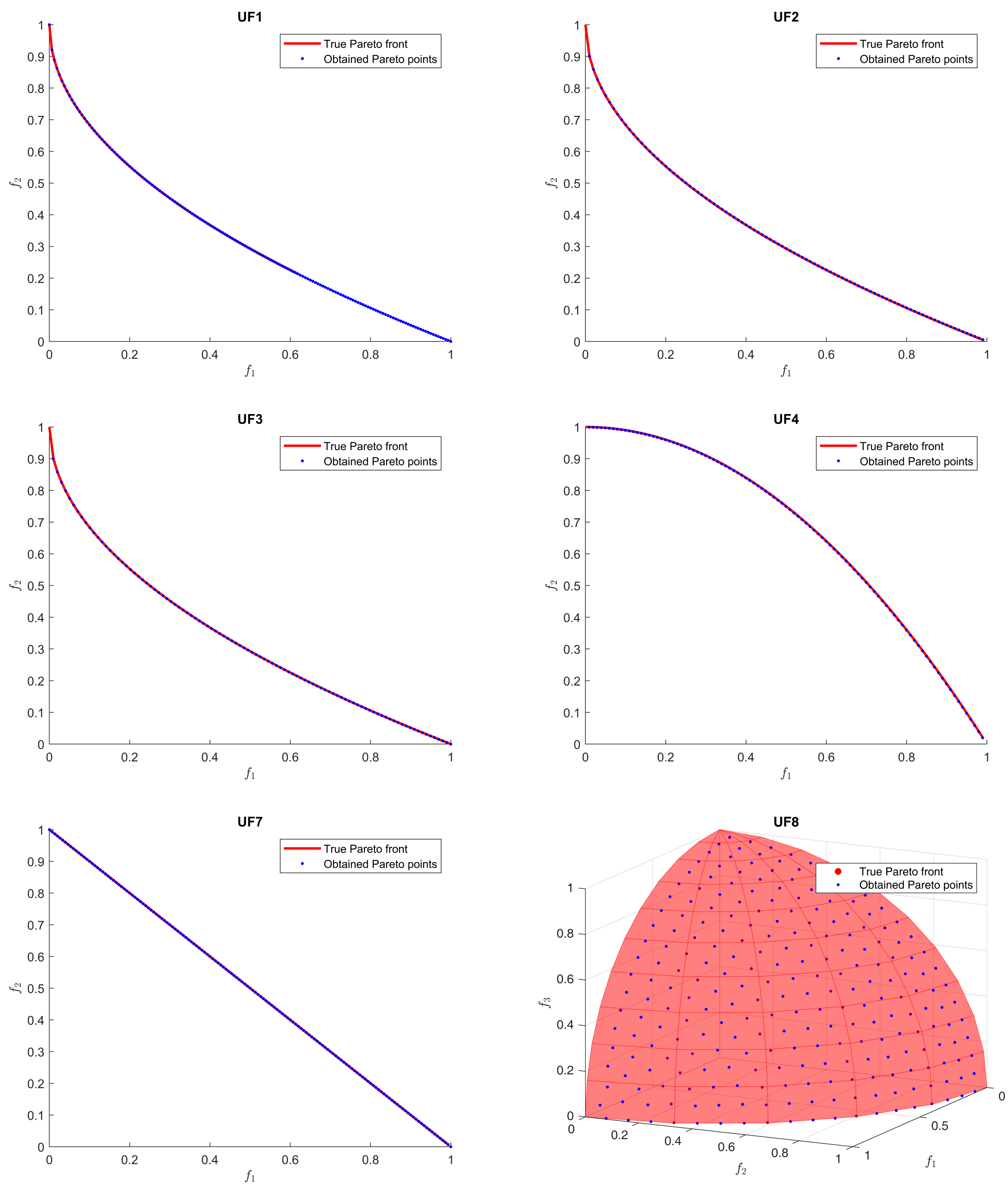

Figure 3.7: Obtained Pareto points of UF1, UF2, UF3, UF4, UF7 and UF8 by Algorithm 4

Table 3.4: Comparison of IGD values for the ZDT benchmark suite

Problem	WS [100]	TE [100]	PBI [100]	NSGA-II [100]	$pa\lambda$-MOEA/D [100]	Cultural MOQPSO [109]	MOWOATS [109]	IC-IPM
ZDT1	5.42E-4	6.48E-4	1.14E-4	7.94E-4	5.79E-4	6.13E-3	1.30E-3	1.05E-4
ZDT2	1.30E-2	5.84E-4	7.02E-4	8.15E-4	6.02E-4	4.87E-3	7.32E-4	1.27E-4
ZDT3	4.93E-3	2.01E-3	2.06E-3	1.19E-3	1.97E-3	1.89E-1	2.87E-3	1.01E-4
ZDT4	7.01E-3	6.65E-4	7.85E-4	8.14E-4	5.93E-4	5.26E-3	2.85E-4	1.45E-4

3.15 Conclusion

This chapter has introduced an interior-point approach, with the help of the cone method (IC-IPM), to find a subset of nondominated points of an MOPs. In proposed method, IC has been used to transform an MOP into a collection of single objective optimization problems. Each single objective optimization problem of the collection has been solved by IPM. To find the solution of each subproblem by IPM, a barrier problem and its KKT conditions have been derived. In order to the solve KKT conditions, the iteration started with the initial point and then calculated the direction by Newton method. Thereafter, step length has been chosen so that the nonnegative variables remain nonnegative. A new merit function has been also proposed with global properties (Theorem 3.2, Theorem 3.3 and Theorem 3.4). Merit function has helped to take a suitable step length along the search direction.

Theorem 3.5 shows that the search direction calculated by Theorem 3.1 is descent for the merit function (3.17). Furthermore, we have proved the global convergence of the proposed algorithm under standard assumptions. We demonstrate through numerical experiments that Algorithm 4 can solve constrained MOPs efficiently. We have used three performance measures (GD+, HV, IGD) to test the efficiency of the Algorithm 4 on some standard test suite. The performance of Algorithm 4 has compared with some existing popular algorithms. Table 3.3 and 3.4 has shown that Algorithm 4 is comparatively efficient.

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