Chapter 4

Approximation Aspects of Multivariate FIF

In this chapter, we discuss the approximation aspects of multivariate α-fractal functions. We prove the existence of a one-sided approximation for multivariate functions using fractal functions. Furthermore, we establish the existence of a Schauder basis composed of multivariate fractal functions for the set of all continuous real-valued functions defined on the interval $[0,1]^N$.

4.1 Introduction

Let $f : I^N \to \mathbb{R}$ be a continuous multivariate real-valued function on I^N. For each $k_i \in \mathbb{N}$ with $i \in \Sigma_N$, let $D^{(k_1,\ldots,k_N)}f = \frac{\partial^{k_1+\cdots+k_N} f}{\partial x_1^{k_1}\cdots\partial x_N^{k_N}}$ be defined as $(k_1,\ldots,k_N)^{th}$ order derivative of f.

Definition 4.1. For $n_1,\ldots,n_N \in \mathbb{N}$, define

$$\mathcal{C}^{n_1,\ldots,n_N}(I^N) = \{f : I^N \to \mathbb{R} : \ D^{(k_1,\ldots,k_N)}f \in \mathcal{C}\big(I^N\big) \text{ for all } 0 \le k_i \le n_i\}.$$

If $D^{(k_1,\ldots,k_N)}f(x_1,\ldots,x_N) \ge 0$ for all $(x_1,\ldots,x_N) \in I^N$ and $0 \le k_i \le n_i$ with $i \in \Sigma_N$, then the function f is known as $(n_1,\ldots,n_N)$-convex.

Consider $g \in \mathcal{C}\big(I^N\big)$ be a multivariate continuous function such that $\dim(\mathcal{G}(g)) > q$ (for the existence of such functions one can refer [86]). Then, the function $f : I^N \to$

$\mathbb{R}$ defined by $f(x_1,\ldots,x_N)=\int_0^{x_1}\cdots\int_0^{x_N} g(t_1,\ldots,t_N)dt_1\cdots dt_N$ satisfies the following,

$$\dim(\mathcal{G}(f))=N \quad \text{and} \quad \dim(\mathcal{G}(D^{(1,\ldots,1)}f))=\dim(\mathcal{G}(g))>N. \tag{4.1}$$

Now recall the Bernstein polynomial defined on I^N:

$$B_{m_1,\ldots,m_N}(f)(x_1,\ldots,x_N)=\sum_{i_1=0}^{m_1}\cdots\sum_{i_N=0}^{m_N} f\left(\frac{i_1}{m_1},\ldots,\frac{i_N}{m_N}\right)\binom{m_1}{i_1}\cdots\binom{m_N}{i_N}x_1^{i_1}(1-x_1)^{m_1-i_1}\cdots x_N^{i_N}(1-x_N)^{m_N-i_N}.$$

We know that if a multivariate function $f\in\mathcal{C}^{k_1,\ldots,k_N}(I^N)$ with $B_{m_1,\ldots,m_N}(f)$ is approximated by Bernstein polynomial, then we obtain the following :

- $B_{m_1,\ldots,m_N}(f)\to f$ uniformly on I^N
- $\left(D^{(k_1,\ldots,k_N)}(B_{m_1,\ldots,m_N}(f))\right)\to D^{(k_1,\ldots,k_N)}f$ uniformly on I^N
- since $B_{m_1,\ldots,m_N}(f)$ and $D^{(k_1,\ldots,k_N)}(B_{m_1,\ldots,m_N}(f))$ are polynomials, then

$$\dim\left(Gr\big(D^{(k_1\ldots,k_N)}(B_{m_1,\ldots,m_N}(f))\big)\right)=\dim(\mathcal{G}(B_{m_1,\ldots,m_N}(f)))=\dim(\mathcal{G}(f))=N. \tag{4.2}$$

Now, it can be observed from (4.1) and (4.2) that the smoothness of a function is maintained when it is approximated using Bernstein polynomials. However, the dimensions of its partial derivatives are not guaranteed. This motivated us to focus on investigating the fractal dimension of a function and its partial derivatives in relation to approximation in this chapter.

Note 4.2. We have the following.

$$B_{m_1,\ldots,m_N}(f)(x_1,\ldots,x_N)=\sum_{i_1=0}^{m_1}\cdots\sum_{i_N=0}^{m_N} f\left(\frac{i_1}{m_1},\ldots,\frac{i_N}{m_N}\right)\binom{m_1}{i_1}\cdots\binom{m_N}{i_N}x_1^{i_1}$$
$$(1-x_1)^{m_1-i_1}\cdots x_N^{i_N}(1-x_N)^{m_N-i_N}.$$

Choosing $f = 1$, we have

$$\begin{aligned}
B_{m_1,\dots,m_N} f(x_1,\dots,x_N) =& \sum_{i_1=0}^{m_1} \cdots \sum_{i_N=0}^{m_N} f\left(\frac{i_1}{m_1},\dots,\frac{i_N}{m_N}\right) \binom{m_1}{i_1} \cdots \binom{m_N}{i_N} x_1^{i_1} \\
& (1-x_1)^{m_1-i_1} \cdots x_N^{i_N}(1-x_N)^{m_N-i_N} \\
=& \sum_{i_1=0}^{m_1} \binom{m_1}{i_1} x_1^{i_1}(1-x_1)^{m_1-i_1} \cdots \sum_{i_N=0}^{m_N} \binom{m_N}{i_N} x_N^{i_N}(1-x_N)^{m_N-i_N} \\
=& (x_1+1-x_1)^{m_1} \cdots (x_N+1-x_N)^{m_N} = 1.
\end{aligned}$$

It follows that $\|B_{m_1,\dots,m_N}\| \geq 1$. Now for every $f \in \mathcal{C}(I^N)$, we get

$$\begin{aligned}
|B_{m_1,\dots,m_N}(f)(x_1,\dots,x_N)| \leq & \|f\|_\infty \sum_{i_1=0}^{m_1} \cdots \sum_{i_N=0}^{m_N} \binom{m_1}{i_1} \cdots \binom{m_N}{i_N} x_1^{i_1}(1-x_1)^{m_1-i_1} \\
& \cdots x_N^{i_N}(1-x_N)^{m_N-i_N} \\
=& \|f\|_\infty,
\end{aligned}$$

which produces $\|B_{m_1,\dots,m_N}\| \leq 1$. Therefore, we have $\|B_{m_1,\dots,m_N}\| = 1$.

4.1.1 Delineation

The present chapter is organized as follows. In the following section, we laid out some fundamental concepts necessary for the study. Additionally, in this section, we have demonstrated the existence of a one-sided fractal approximation for a multivariate function. In Section 4.3, we have provided an upper bound for the graph of a multivariate α-fractal function. Further, we have hinted construction of dimension preserving α-fractal function. In Section 4.4, we have proved the existence of a Schauder basis composed of multivariate fractal functions for the space $\mathcal{C}(I^N)$. Furthermore, we have demonstrated the existence of a multivariate fractal polynomial that approximates the function $f \in \mathcal{C}(I^N)$.

4.2 Fundamental Results

Lemma 4.3. [88, Lemma 3.1] *Let $A \subset \mathbb{R}^m$ and $f, g : A \to \mathbb{R}^n$ be continuous functions such that f is a Lipschitz function. Then,*

$$\dim_H(\mathcal{G}(f+g)) = \dim_H(\mathcal{G}(g)) \quad \textit{and} \quad \dim_B(\mathcal{G}(f+g)) = \dim_B(\mathcal{G}(g)).$$

Note that $\mathcal{L}ip(I^N)$ is a dense subset of $\mathcal{C}(I^N)$ with respect to the supremum norm.

In view of Lipschitz's invariance property of dimensions, one may conclude that the next theorem holds for both the Hausdorff and the box dimensions.

Theorem 4.4. *Let $\beta \in \mathbb{R}$. Define a set $\mathcal{S}_\beta = \{f \in \mathcal{C}(I^N) : \dim(\mathcal{G}(f)) = \beta\}$. Then, for each $\beta \in [N, N+1]$, $\mathcal{S}_\beta$ is dense in $\mathcal{C}(I^N)$.*

Proof. Let $f \in \mathcal{C}(I^N)$ and $\epsilon > 0$. Since $\mathcal{L}ip(I^N)$ is dense in $\mathcal{C}(I^N)$, then there exists $g \in \mathcal{L}ip(I^N)$ such that

$$\|f - g\|_\infty < \frac{\epsilon}{2}.$$

Further, we consider a non-vanishing function $h \in \mathcal{S}_\beta$. Let $h_* = g + \frac{\epsilon}{2\|h\|_\infty} h$, which immediately gives

$$\|g - h_*\|_\infty \leq \frac{\epsilon}{2}.$$

This together with Lemma 4.3 implies that $\dim(\mathcal{G}(h_*)) = \dim(\mathcal{G}(h)) = \beta$. Hence, we have $h_* \in \mathcal{S}_\beta$ such that

$$\|f - h_*\|_\infty \leq \|f - g\|_\infty + \|g - h_*\|_\infty < \epsilon.$$

This completes the proof. □

The following theorem seems to be widely recognized in the case of one variable. However, when it comes to multiple variables, we are unable to find the proof. To ensure thoroughness, we have included the proof of the theorem.

Theorem 4.5. *Let* $\big(f_k\big)$ *be a sequence of differentiable functions on* I^N*. Assume that for some* $(w_1, \ldots, w_N) \in I^N$*, the sequences* $\big(f_k(w_1, \ldots)\big)$*,* $\big(f_k(\cdot, w_2, \ldots), \ldots$ *and* $\big(f_k(\ldots, w_N)\big)$ *converge uniformly on* I^{N-1}*. If* $(D^{(1,\ldots,1)}f_k)$ *converges uniformly on* I^N*, then* $\big(f_k\big)$ *converges uniformly on* I^N *to a function* f*, and*

$$D^{(1,\ldots,1)}f(x_1, \ldots, x_N) = \lim_{k\to\infty} D^{(1,\ldots,1)}f_k(x_1, \ldots, x_N)$$

for every $(x_1, \ldots, x_N) \in I^N$*.*

Proof. Let $\epsilon > 0$. Since $(D^{(1,\ldots,1)}f_k)$ converges uniformly, there exists $N_1 \in \mathbb{N}$ such that

$$|D^{(1,\ldots,1)}f_k(x_1, \ldots, x_N) - D^{(1,\ldots,1)}f_m(x_1, \ldots, x_N)| < \frac{\epsilon}{2^N}$$

for all $(x_1, \ldots, x_N) \in I^N$, and $k, m \geq N_1$. Using mean-value theorem (see [81, Theorem 9.40]), we have

$$\begin{aligned}
&\Big|\big[f_k(x_1+h_1, \ldots, x_N+h_N) - f_m(x_1+h_1, \ldots, x_N+h_N)\big] \\
&- \sum_{1\leq i\leq N} \big[f_k(x_1+h_1, \ldots, x_{i-1}+h_{i-1}, x_i, x_{i+1}+h_{i+1}, \ldots, x_N+h_N) \\
&\qquad\qquad - f_m(x_1+h_1, \ldots, x_{i-1}+h_{i-1}, x_i, x_{i+1}+h_{i+1}, \ldots, x_N+h_N)\big] \\
&+ \sum_{1\leq i<j\leq N} \big[f_k(x_1+h_1, \ldots, x_{i-1}+h_{i-1}, x_i, x_{i+1}+h_{i+1}, \ldots, x_{j-1}+h_{j-1}, x_j, x_{j+1}+h_{j+1}, \\
&\qquad \ldots, x_N+h_N) - f_m(x_1+h_1, \ldots, x_{i-1}+h_{i-1}, x_i, x_{i+1}+h_{i+1}, \ldots, x_{j-1}+h_{j-1}, x_j, \\
&\qquad\qquad x_{j+1}+h_{j+1}, \ldots, x_N+h_N)\big] + \cdots + (-1)^N\big[f_k(x_1, \ldots, x_N) - f_m(x_1, \ldots, x_N)\big]\Big| \\
=&h_1\cdots h_N \left|D^{(1,\ldots,1)}(f_k - f_m)(t_1, \ldots, t_N)\right| \\
\leq&h_1\cdots h_N \max_{(t_1,t_2,\ldots,t_N)\in I^N} \left|D^{(1,\ldots,1)}f_k(t_1, \ldots, t_N) - D^{(1,\ldots,1)}f_m(t_1, \ldots, t_N)\right| \\
<&\frac{\epsilon}{2^N}h_1\cdots h_N
\end{aligned}$$

$$< \frac{\epsilon}{2^N}. \tag{4.3}$$

With the assumption for $(w_1, \dots, w_N) \in I^N$, we can choose $N_0\ (> N_1) \in \mathbb{N}$ such that

$$\begin{aligned}
&|f_k(w_1, x_2, x_3, \dots, x_N) - f_m(w_1, x_2, x_3, \dots, x_N)| < \frac{\epsilon}{2^N} \text{ for all } k, m \geq N_0,\\
&|f_k(x_1, w_2, x_3, \dots, x_N) - f_m(x_1, w_2, x_3, \dots, x_N)| < \frac{\epsilon}{2^N} \text{ for all } k, m \geq N_0,\\
&\vdots\\
&|f_k(x_1, \dots, x_{N-1}, w_N) - f_m(x_1, \dots, x_{N-1}, w_N)| < \frac{\epsilon}{2^N} \text{ for all } k, m \geq N_0.
\end{aligned} \tag{4.4}$$

Now using (4.3) and (4.4), we have

$$\begin{aligned}
|f_k(x_1, x_2, \dots, x_N) - f_m(x_1, x_2, \dots, x_N)| <& \frac{\epsilon}{2^N} + |f_k(w_1, x_2, \dots, x_N) - f_m(w_1, x_2, \dots, x_N)| + \\
& \cdots + |f_k(x_1, \dots, x_{N-1}, w_N) - f_m(x_1, \dots, x_{N-1}, w_N)|\\
<& \frac{\epsilon}{2^N} + \cdots + \frac{\epsilon}{2^N}\\
<& \epsilon
\end{aligned}$$

for every $(x_1, \dots, x_N) \in I^N$ and $k, m \geq N_0$. This immediately confirms the uniform convergence of (f_k). The rest part follows routine calculations, hence omitted. □

Lemma 4.6. *Let $f : I^{N-1} \to \mathbb{R}$ be a Lipschitz map and $g : I \to \mathbb{R}$ be a continuous function. Consider a mapping $h : I^N \to \mathbb{R}$ be defined by*

$$h(x_1, \dots, x_N) = f(x_1, \dots, x_{N-1}) + g(x_N),$$

then $\dim_H(\mathcal{G}(h)) = \dim_H(\mathcal{G}(g)) + N - 1$.

Proof. Let L be the Lipschitz constant for f. Define a map $\phi : I^{N-1} \times \mathcal{G}(g) \to \mathcal{G}(h)$ such that

$$\phi(x_1, \dots, x_N, g(x_N)) = (x_1, \dots, x_N, h(x_1, \dots, x_N)),$$

where $(x_1,\dots,x_N)\in I^N$.

Claim: ϕ is a bi-Lipschitz mapping. For this, take $M=\max\left\{\sqrt{1+2L^2},\sqrt{2}\right\}$, then for all $(x_1,\dots,x_N),\ (y_1,\dots,y_N)\in I^N$, we have

$$\begin{aligned}
&\|\phi(x_1,\dots,x_N,g(x_N))-\phi(y_1,\dots,y_N,g(y_N))\|_2\\
=&\|(x_1,\dots,x_N,h(x_1,\dots,x_N))-(y_1,\dots,y_N,h(y_1,\dots,y_N))\|_2\\
=&\sqrt{\|(x_1,\dots,x_N)-(y_1,\dots,y_N)\|_2^2+(h(x_1,\dots,x_N)-h(y_1,\dots,y_N))^2}\\
=&\sqrt{\|(x_1,\dots,x_N)-(y_1,\dots,y_N)\|_2^2+(f(x_1,\dots,x_N)+g(x_N)-f(y_1,\dots,y_N)-g(y_N))^2}\\
\le&\sqrt{\|(x_1,\dots,x_N)-(y_1,\dots,y_N)\|_2^2+2(f(x_1,\dots,x_N)-f(y_1,\dots,y_N))^2+2(g(x_N)-g(y_N))^2}\\
\le&\sqrt{\|(x_1,\dots,x_N)-(y_1,\dots,y_N)\|_2^2+2L^2\|(x_1,\dots,x_N)-(y_1,\dots,y_N)\|_2^2+2(g(x_N)-g(y_N))^2}\\
=&\sqrt{(1+2L^2)\|(x_1,\dots,x_N)-(y_1,\dots,y_N)\|_2^2+2(g(x_N)-g(y_N))^2}\\
=&M\sqrt{\|(x_1,\dots,x_N)-(y_1,\dots,y_N)\|_2^2+2(g(x_N)-g(y_N))^2}\\
=&M\|(x_1,\dots,x_N,g(x_N))-(y_1,\dots,y_N,g(y_N))\|_2.
\end{aligned}$$

Hence,

$$\|\phi(x_1,\dots,x_N,g(x_N))-\phi(y_1,\dots,y_N,g(y_N))\|_2\le M\|(x_1,\dots,x_N,g(x_N))-(y_1,\dots,y_N,g(y_N))\|_2.$$

With small manipulation, we can similarly prove that

$$\|\phi(x_1,\dots,x_N,g(x_N))-\phi(y_1,\dots,y_N,g(y_N))\|_2\ge \frac{1}{M}\|(x_1,\dots,x_N,g(x_N))-(y_1,\dots,y_N,g(y_N))\|_2.$$

This proves the claim. Therefore, $\dim_H(\mathcal{G}(h))=\dim_H(I^{N-1}\times\mathcal{G}(g))$, but I^{N-1} and $\mathcal{G}(g)$ both are Borel sets and so

$$\dim_H(I^{N-1}\times\mathcal{G}(g))=\dim_H(I^{N-1})+\dim_H(\mathcal{G}(g))=\dim_H(\mathcal{G}(g))+N-1,$$

which proves the lemma. □

Here, we shall discuss the results obtained for univariate functions concerning dimensional aspects. One important class of functions considered by Mauldin and

Williams [60] is as follows:

$$W_b(x) = \sum_{n=-\infty}^{\infty} b^{-\gamma n} \left[\phi(b^n x + \theta_n) - \phi(\theta_n)\right],$$

where θ_n is an arbitrary real number, ϕ is a periodic function with period one and $b > 1$, $0 < \gamma < 1$. They showed that for a large enough b there exists a constant $C > 0$ such that $\dim_H(\mathcal{G}(W_b))$ is bounded below by $2 - \gamma - (C/\ln b)$. Further, significant progress in the dimension theory of functions is contributed recently by Shen [86] for the following class of functions:

$$f^{\phi}_{\lambda,b}(x) = \sum_{n=0}^{\infty} \lambda^n \phi(b^n x),$$

where $b \geq 2$ and ϕ is a real valued, $\mathbb{Z}$-periodic non-constant, C^2-function defined on $\mathbb{R}$. He proved that there exists a constant K_0 depending on ϕ and b such that if $1 < \lambda b < K_0$, then

$$\dim_H \left(Gr \left(f^{\phi}_{\lambda,b} \right) \right) = 2 + \frac{\log \lambda}{\log b}.$$

For $f \in \mathcal{C}^{1,\dots,1}(I^N)$, we get $\dim(\mathcal{G}(f)) = N$. However, no conclusion can be drawn for the dimensions of its partial derivatives. This is evident from the following example. Let us consider Weierstrass-type nowhere differentiable continuous function $W : I \to \mathbb{R}$ as in [86] with $1 \leq \dim(\mathcal{G}(W)) \leq 2$. Now we define $h : I^N \to \mathbb{R}$ by

$$h(x_1, \dots, x_N) = W(x_N) + x_1 + \dots + x_{N-1}.$$

Here, by Lemma 4.6, we obtain that $N \leq \dim(\mathcal{G}(h)) = \dim(\mathcal{G}(W)) + N - 1 \leq N + 1$. Then, for the function f defined by

$$f(x_1, \dots, x_N) = \int_0^{x_1} \cdots \int_0^{x_N} h(t_1, \dots, t_N)\, dt_1 \cdots dt_N,$$

we have $\dim(\mathcal{G}(f)) = N$ and $N \leq \dim(\mathcal{G}(D^{(1,\dots,1)}f)) = \dim(\mathcal{G}(h)) \leq N+1$.

Theorem 4.7. *Let $f \in \mathcal{C}^{1,\dots,1}(I^N)$ such that $\dim(\mathcal{G}(D^{(1,\dots,1)}f)) = \beta$ for some $N \leq \beta \leq N+1$. Then, we have a sequence (f_k) in $\mathcal{C}^{1,\dots,1}(I^N)$ such that $\dim(\mathcal{G}(D^{(1,\dots,1)}f_k)) = \beta$ and $f_k \to f$ uniformly on I^N.*

Proof. In view of Theorem 4.4, there exists a sequence (g_k) in $\mathcal{C}(I^N)$ such that $\dim(\mathcal{G}(g_k)) = \beta$ and $g_k \to D^{(1,\dots,1)}f$ uniformly on I^N. Further, let us consider a function $f_k : I^N \to \mathbb{R}$ defined by

$$f_k(x_1,\dots,x_N) = \int_0^{x_1} \cdots \int_0^{x_N} g_k(t_1,\dots,t_N)dt_1 \cdots dt_N.$$

Then, $D^{(1,\dots,1)}f_k = g_k$ and $(D^{(1,\dots,1)}f_k) \to D^{(1,\dots,1)}f$ uniformly. Next, we have that the sequences of functions $\big(f_k(0,y_2,\dots,y_N)\big) \to 0,\dots,\big(f_k(y_1,\dots,y_{N-1},0)\big) \to 0$, uniformly on I^{N-1}. Now, Theorem 4.5 completes the proof. □

Theorem 4.8. *Let $f \in \mathcal{C}(I^N)$ with $f(x_1,\dots,x_N) \geq 0$ for all $(x_1,\dots,x_N) \in I^N$. Then, for a given $\epsilon > 0$, there exists $g \in \mathcal{S}_\beta$ satisfying the following:*

$$g(x_1,\dots,x_N) \geq 0 \textit{ for all } (x_1,\dots,x_N) \in I^N \textit{ and } \|f-g\|_\infty < \epsilon.$$

Proof. Let $\epsilon > 0$. Theorem 4.4 yields an element $h \in \mathcal{S}_\beta$ such that

$$\|f-h\|_\infty < \frac{\epsilon}{2}.$$

We define

$$g(x_1,\dots,x_N) = h(x_1,\dots,x_N) + \frac{\epsilon}{2} \text{ for all } (x_1,\dots,x_N) \in I^N.$$

Then, by Lemma 4.3, $g \in \mathcal{S}_\beta$, and by routine calculations, we get

$$\begin{aligned} g(x_1,\dots,x_N) &= h(x_1,\dots,x_N) - f(x_1,\dots,x_N) + f(x_1,\dots,x_N) + \frac{\epsilon}{2} \\ &\geq -\|f-h\|_\infty + f(x_1,\dots,x_N) + \frac{\epsilon}{2} > f(x_1,\dots,x_N) \geq 0. \end{aligned}$$

Furthermore, one has

$$\|f-g\|_\infty \leq \|f-h\|_\infty + \|h-g\|_\infty < \epsilon,$$

hence the proof. □

Theorem 4.9. *Let $f : I^N \to \mathbb{R}$ be a $(n_1,\dots,n_N)$-convex function such that $f(0,x_2,\dots,x_N) = \cdots = f(x_1,\cdots,x_{N-1},0) = 0$, for all $(x_1,\dots,x_N) \in I^N$. Then, for $\epsilon > 0$, there exists $(n_1,\dots,n_N)$-convex function g such that $D^{(n_1,\dots,n_N)}g \in \mathcal{S}_\beta$ and $\|f-g\|_\infty < \epsilon$.*

Proof. Since f is a $(n_1,\dots,n_N)$-convex function, therefore $D^{(n_1,\dots,n_N)}f(x_1,\dots,x_N) \geq 0$ for all $(x_1,\dots,x_N) \in I^N$. Then, using Theorem 4.8, we have, for a given $\epsilon > 0$, there exists $h \in \mathcal{S}_\beta$ such that

$$h(x_1,\dots,x_N) \geq 0 \text{ for all } (x_1,\dots,x_N) \in I^N \text{ and } \left\|D^{(n_1,\dots,n_N)}f - h\right\|_\infty < \epsilon. \quad (4.5)$$

Choose

$$g(x_1,\dots,x_N) = \int_0^{x_1}\cdots\int_0^{x_N}\cdots\int_0^{x_1^{n_1-1}}\cdots\int_0^{x_N^{n_N-1}} h(x_1^{n_1},\dots,x_N^{n_N})\mathrm{d}x_1^{n_1}\cdots\mathrm{d}x_N^{n_N}\cdots\mathrm{d}x_1^1\cdots\mathrm{d}x_N^1.$$

Then, we have $D^{(k_1,\dots,k_N)}g(x_1,\dots,x_N) \geq 0$ for all $(x_1,\dots,x_N) \in I_N$ and $0 \leq k_i \leq n_i$, where $1 \leq i \leq N$, that is, g is a $(n_1,\dots,n_N)$-convex function and $h = D^{(n_1,\dots,n_N)}g \in \mathcal{S}_\beta$. Further, we have

$\|f-g\|_\infty$

$$= \sup \left\{ \left| f - \int_0^{x_1} \cdots \int_0^{x_N} \cdots \int_0^{x_1^{n_1-1}} \cdots \int_0^{x_N^{n_N-1}} h(x_1^{n_1}, \ldots, x_N^{n_N}) \mathrm{d}x_1^{n_1} \cdots \mathrm{d}x_N^{n_N} \cdots \mathrm{d}x_1^1 \cdots \mathrm{d}x_N^1 \right| \right.$$
$$\left. : (x_1, \ldots, x_N) \in I^N \right\}$$
$$= \sup \left\{ \left| \int_0^{x_1} \cdots \int_0^{x_N} \cdots \int_0^{x_1^{n_1-1}} \cdots \int_0^{x_N^{n_N-1}} D^{(n_1,\ldots,n_N)} f(x_1^{n_1}, \ldots, x_N^{n_N}) \mathrm{d}x_1^{n_1} \cdots \mathrm{d}x_N^{n_N} \cdots \mathrm{d}x_1^1 \cdots \mathrm{d}x_N^1 \right. \right.$$
$$\left. - \int_0^{x_1} \cdots \int_0^{x_N} \cdots \int_0^{x_1^{n_1-1}} \cdots \int_0^{x_N^{n_N-1}} h(x_1^{n_1}, \ldots, x_N^{n_N}) \mathrm{d}x_1^{n_1} \cdots \mathrm{d}x_N^{n_N} \cdots \mathrm{d}x_1^1 \cdots \mathrm{d}x_N^1 \right|$$
$$\left. : (x_1, \ldots, x_N) \in I^N \right\}$$
$$\leq \sup \left\{ \int_0^{x_1} \cdots \int_0^{x_N} \cdots \int_0^{x_1^{n_1-1}} \cdots \int_0^{x_N^{n_N-1}} \left| \left(D^{(n_1,\ldots,n_N)} f(x_1^{n_1}, \ldots, x_N^{n_N}) - h(x_1^{n_1}, \ldots, x_N^{n_N}) \right) \right| \right.$$
$$\left. \mathrm{d}x_1^{n_1} \cdots \mathrm{d}x_N^{n_N} \cdots \mathrm{d}x_1^1 \cdots \mathrm{d}x_N^1 : (x_1, \ldots, x_N) \in I^N \right\}$$

using (4.5), we have

$$< \sup \left\{ \int_0^{x_1} \cdots \int_0^{x_N} \cdots \int_0^{x_1^{n_1-1}} \cdots \int_0^{x_N^{n_N-1}} \epsilon \, \mathrm{d}x_1^{n_1} \cdots \mathrm{d}x_N^{n_N} \cdots \mathrm{d}x_1^1 \cdots \mathrm{d}x_N^1 : (x_1, \ldots, x_N) \in I^N \right\}$$
$$< \epsilon.$$

This completes the proof. □

Theorem 4.10. *Let $f \in \mathcal{C}(I^N)$. Then, for $\epsilon > 0$ there exists $g \in \mathcal{S}_\beta$ such that*

$$g(x_1, \ldots, x_N) \leq f(x_1, \ldots, x_N) \text{ for all } (x_1, \ldots, x_N) \in I^N \text{ and } \|f - g\|_\infty < \epsilon.$$

Proof. Since $f \in \mathcal{C}(I^N)$ and $\epsilon > 0$, Theorem 4.4 generates a member $h \in \mathcal{S}_\beta$ such that

$$\|f - h\|_\infty < \frac{\epsilon}{2}.$$

Choose $g(x_1, \ldots, x_N) = h(x_1, \ldots, x_N) - \frac{\epsilon}{2}$ for all $(x_1, \ldots, x_N) \in I^N$. Then,

$$\begin{aligned} g(x_1, \ldots, x_N) &= h(x_1, \ldots, x_N) - f(x_1, \ldots, x_N) + f(x_1, \ldots, x_N) - \frac{\epsilon}{2} \\ &\leq \|f - h\|_\infty + f(x_1, \ldots, x_N) - \frac{\epsilon}{2} < f(x_1, \ldots, x_N). \end{aligned}$$

Furthermore,

$$\|f-g\|_\infty \le \|f-h\|_\infty + \|h-g\|_\infty < \epsilon,$$

establishing the proof. □

4.2.1 One-Sided Approximation

Let f be a continuous function defined on a compact interval J of $\mathbb{R}$ and p be an approximating function which approximates f such that $p(x) \le f(x)$ for all $x \in J$, then p is known as a one-sided approximation of f from below. Similarly, a one-sided approximation from above is defined. Problems of one-sided approximation frequently appear in the analysis. For instance, consider f to be a Riemann integrable function defined on J, then there exist polynomials p and P such that

$$p(x) \le f(x) \le P(x) \text{ for all } x \in J.$$

Different types of approximations have been studied in the literature. For example, in [40], one-sided approximation by trigonometrical polynomials has been studied. In [17], the existence and uniqueness of best one-sided polynomials have been investigated. Some more research can be found in [20, 32, 33, 53] using different types of functions. In this subsection, we aim to demonstrate the existence of the best one-sided approximation using fractal functions. Consider $\beta \in [N, N+1]$, and define

$$\mathcal{C}_\beta(I^N) := \{f \in \mathcal{C}(I^N) : \overline{\dim}_B(\mathcal{G}(f)) \le \beta\}.$$

In view of [38, Proposition 3.4], recall that $\mathcal{C}_\beta(I^N)$ is a normed linear space.

Definition 4.11. *(Best one-sided approximation)* Let $\{g_1, \dots, g_n\}$ be a linearly independent subset of $\mathcal{C}_\beta(I^N)$. For a bounded below and Lebesgue integrable function

$f : I^N \to \mathbb{R}$, we define

$$\mathcal{Y}_n^{\beta}(f) := \Big\{ h \in span\{g_1, \dots, g_n\} : h(x_1, \dots, x_N) \le f(x_1, \dots, x_N) \text{ for all } (x_1, \dots, x_N) \in I^N \Big\},$$

which is nonempty by Theorem 4.10. A function $h_f \in \mathcal{Y}_n^{\beta}(f)$ is said to be a best one-sided approximation from below to f on I^N if

$$\int_{I^N} h_f(x_1, \dots, x_N)\, \mathrm{d}x_1 \cdots \mathrm{d}x_N = \sup \left\{ \int_{I^N} h(x_1, \dots, x_N)\, \mathrm{d}x_1 \cdots \mathrm{d}x_N : h \in \mathcal{Y}_n^{\beta}(f) \right\},$$

where $\displaystyle\int_{I^N} h_m(x_1, \dots, x_N)\, \mathrm{d}x_1 \cdots \mathrm{d}x_N = \underbrace{\int_0^1 \cdots \int_0^1}_{\text{N-times}} h_m(x_1, \dots, x_N)\, \mathrm{d}x_1 \cdots \mathrm{d}x_N.$

In a similar way, the best one-sided approximations from above can be defined.

We state the next theorem for one-sided approximation from below. Though a similar result can be proved in terms of one-sided approximation from above, see, for instance, [33, 91].

Theorem 4.12. *For a bounded below and integrable function $f : I^N \to \mathbb{R}$, there exists a member in $\mathcal{Y}_n^{\beta}(f)$ of best one-sided approximant from below to f on I^N.*

Proof. Let (h_m) be a sequence in $\mathcal{Y}_n^{\beta}(f)$ such that

$$\int_{I^N} h_m(x_1, \dots, x_N)\, \mathrm{d}x_1 \cdots \mathrm{d}x_N \to A \text{ as } m \to \infty, \tag{4.6}$$

where

$$\int_{I^N} h_m(x_1, \dots, x_N)\, \mathrm{d}x_1 \cdots \mathrm{d}x_N = \int_0^1 \cdots \int_0^1 h_m(x_1, \dots, x_N)\, \mathrm{d}x_1 \cdots \mathrm{d}x_N,$$

$$\text{and } A = \sup \left\{ \int_{I^N} h(x_1, \dots, x_N)\, \mathrm{d}x_1 \cdots \mathrm{d}x_N : h \in \mathcal{Y}_n^{\beta}(f) \right\}.$$

With an appropriate constant $M_* > 0$, we have

$$\begin{aligned}\int_{I^N} |h_m(x_1,\dots,x_N)|\, \mathrm{d}x_1\cdots \mathrm{d}x_N \;&\leq\; \int_{I^N} \Big|h_m(x_1,\dots,x_N) - A\Big|\, \mathrm{d}x_1\cdots \mathrm{d}x_N \\ &\quad + \int_{I^N} A\, \mathrm{d}x_1\cdots \mathrm{d}x_N \leq M_*.\end{aligned}$$

Since $\mathcal{Y}_n^\beta(f)$ is a subset of finite-dimensional linear space, the closed set of radius M_* in $\mathcal{Y}_n^\beta(f)$ is compact. Therefore, using the fact that every norm is equivalent on a finite-dimensional linear space, there exists a subsequence (h_{m_k}) and a function h in $\mathcal{Y}_n^\beta(f)$ such that the sequence (h_{m_k}) converges to h in $L^1(I^N)$. Now from the finite-dimensionality of $\mathcal{Y}_n^\beta(f)$, it follows that the sequence (h_{m_k}) also converges to h uniformly. Further, since $h_m(x_1,\dots,x_N) \leq f(x_1,\dots,x_N)$ for all $(x_1,\dots,x_N) \in I^N$, and $h_{m_k} \to h$ uniformly, we get $h(x_1,\dots,x_N) \leq f(x_1,\dots,x_N)$ for all $(x_1,\dots,x_N) \in I^N$. Thus, $h \in \mathcal{Y}_n^\beta(f)$. Now by (4.6), we have

$$\int_{I^N} h(x_1,\dots,x_N)\, \mathrm{d}x_1\cdots \mathrm{d}x_N = \lim_{k\to\infty} \int_{I^N} h_{m_k}(x_1,\dots,x_N)\, \mathrm{d}x_1\cdots \mathrm{d}x_N = A,$$

proving the assertion. □

4.3 Fractal Dimension of Multivariate α-Fractal Function

In this section, we discuss the multivariate α-fractal functions using the idea of Bernstein polynomial, which is motivated by the work of Verma and Viswanathan [90] on rectangular grids.

Let $f \in \mathcal{C}(I^N)$ be a multivariate continuous functions and f^α be the multivariate α-fractal function, as defined in Section 1.3.4, corresponding to f.

Theorem 4.13. *Let f and s be such that*

$$\begin{aligned} |f((x_1,\dots,x_N)) - f((y_1,\dots,y_N))| &\le K_f \|(x_1,\dots,x_N) - (y_1,\dots,y_N)\|_2^\sigma \\ \text{and } |s((x_1,\dots,x_N)) - s((x_1,\dots,x_N))| &\le K_s \|(x_1,\dots,x_N) - (y_1,\dots,y_N)\|_2^\sigma \end{aligned} \tag{4.7}$$

for every $(x_1,\dots,x_N),(y_1,\dots,y_N) \in I^N$, and for fixed $K_f, K_s > 0$. Assume that for some $k_f > 0, \delta_0 > 0$ the following holds: for each $(x_1,\dots,x_N) \in I^N$ and $0 < \delta < \delta_0$, there exists $(y_1,\dots,y_N) \in I^N$ such that $\|(x_1,\dots,x_N) - (y_1,\dots,y_N)\|_2 \le \delta$ and

$$|f((x_1,\dots,x_N)) - f((y_1,\dots,y_N))| \ge k_f \|(x_1,\dots,x_N) - (y_1,\dots,y_N)\|_2^\sigma. \tag{4.8}$$

Furthermore, we suppose $M_1 = \dots = M_N = M,\ x_{k,i_k} - x_{k,i_k-1} = \frac{1}{M}$ for all $i_k \in \Sigma_{M_k},\ k \in \Sigma_N$ and constant scaling function α.
If $|\alpha| < \min\left\{\frac{1}{N}, \frac{k_f}{(K_{f^\alpha}+K_s)M^\sigma}\right\}$, then $\dim_B\left(\mathcal{G}(f^\alpha)\right) = N + 1 - \sigma$.

Proof. Since $M^\sigma|\alpha| < M|\alpha| < 1$, Theorem 3.11 implies that f^α is Hölder continuous with exponent σ. That is,

$$|f^\alpha(x_1,\dots,x_N) - f^\alpha(y_1,\dots,y_N)| \le K_{f^\alpha} \|(x_1,\dots,x_N) - (y_1,\dots,y_N)\|_2^\sigma$$

for some $K_{f^\alpha} > 0$. Let R_{f^α} be the oscillation of f^α over $A \subseteq I^N$ such that

$$\begin{aligned} R_{f^\alpha}[A] &= \sup_{(x_1,\dots,x_N),(y_1,\dots,y_N)\in A} |f(x_1,\dots,x_N) - f(y_1,\dots,y_N)| \\ &= \sup_{(x_1,\dots,x_N)\in A} f(x_1,\dots,x_N) - \inf_{(y_1,\dots,y_N)\in A} f(y_1,\dots,y_N). \end{aligned}$$

For $0 < \delta < 1$, let $N_\delta(\mathcal{G}(f^\alpha))$ be the number of δ-cubes that cover graph of f^α, and $A_{i_1\dots i_N} \subset I^N$ be the $(i_1,\dots,i_N)^{th}$ cell corresponding to the net under consideration, with $\lceil . \rceil$ the ceiling function, we have

$$\begin{aligned}
N_\delta(\mathcal{G}(f^\alpha)) &\leq \sum_{i_1=1}^{\lceil \frac{1}{\delta} \rceil} \cdots \sum_{i_N=1}^{\lceil \frac{1}{\delta} \rceil} \left(1 + \left\lceil \frac{R_{f^\alpha}[A_{i_1 i_2 \ldots i_N}]}{\delta} \right\rceil \right) \\
&\leq \sum_{i_1=1}^{\lceil \frac{1}{\delta} \rceil} \cdots \sum_{i_N=1}^{\lceil \frac{1}{\delta} \rceil} \left(2 + \frac{R_{f^\alpha}[A_{i_1 i_2 \ldots i_N}]}{\delta} \right) \\
&= 2 \left\lceil \frac{1}{\delta} \right\rceil \ldots \left\lceil \frac{1}{\delta} \right\rceil + \sum_{i_1=1}^{\lceil \frac{1}{\delta} \rceil} \cdots \sum_{i_N=1}^{\lceil \frac{1}{\delta} \rceil} \frac{R_{f^\alpha}[A_{i_1 i_2 \ldots i_N}]}{\delta} \\
&\leq 2 \left\lceil \frac{1}{\delta} \right\rceil \ldots \left\lceil \frac{1}{\delta} \right\rceil + \sum_{i_1=1}^{\lceil \frac{1}{\delta} \rceil} \cdots \sum_{i_N=1}^{\lceil \frac{1}{\delta} \rceil} K_{f^\alpha} \delta^{\sigma-1}. \qquad (4.9)
\end{aligned}$$

Consequently, we deduce

$$\overline{\dim}_B\big(\mathcal{G}(f^\alpha)\big) = \overline{\lim_{\delta \to 0}} \frac{\log N_\delta(\mathcal{G}(f^\alpha))}{-\log \delta} \leq N + 1 - \sigma.$$

Therefore, $\underline{\dim}_B\big(\mathcal{G}(f^\alpha)\big) \leq \overline{\dim}_B\big(\mathcal{G}(f^\alpha)\big) \leq N + 1 - \sigma$. It remains to obtain the lower bound. For this, we proceed by recalling the self-referential equation (1.14)

$$f^\alpha(x_1, \ldots, x_N) = f(x_1, \ldots, x_N) + \alpha\big[f^\alpha\big((u_{1,i_1}^{-1}(x_1), \ldots, u_{N,i_N}^{-1}(x_N)\big) - s\big(u_{1,i_1}^{-1}(x_1), \ldots, u_{N,i_N}^{-1}(x_N)\big)\big]$$

for every $(x_1, \ldots, x_N) \in \prod_{k=1}^N I_{k,i_k}$ and $(i_1, \ldots, i_N) \in \prod_{k=1}^N \Sigma_{M_k}$.
For $(x_1, \ldots, x_N), (y_1, \ldots, y_N) \in \prod_{k=1}^N I_{k,i_k}$ (for other cases, we follow Note 3.7), we obtain

$$\begin{aligned}
&|f^\alpha(x_1, \ldots, x_N) - f^\alpha(y_1, \ldots, y_N)| \\
=&\Big| f(x_1, \ldots, x_N) - f(y_1, \ldots, y_N) + \alpha f^\alpha\big(u_{1,i_1}^{-1}(x_1), \ldots, u_{N,i_N}^{-1}(x_N)\big) - \alpha f^\alpha\big(u_{1,i_1}^{-1}(y_1), \ldots, u_{N,i_N}^{-1}(y_N)\big) \\
&\qquad - \alpha s\big(u_{1,i_1}^{-1}(x_1), \ldots, u_{N,i_N}^{-1}(x_N)\big) + \alpha s\big(u_{1,i_1}^{-1}(y_1), \ldots, u_{N,i_N}^{-1}(y_N)\big)\Big| \\
\geq& |f(x_1, \ldots, x_N) - f(y_1, \ldots, y_N)| - |\alpha| \left| f^\alpha\big(u_{1,i_1}^{-1}(x_1), \ldots, u_{N,i_N}^{-1}(x_N)\big) - f^\alpha\big(u_{1,i_1}^{-1}(y_1), \ldots, u_{N,i_N}^{-1}(y_N)\big)\right| \\
&\qquad - |\alpha| \left| s\big(u_{1,i_1}^{-1}(x_1), \ldots, u_{N,i_N}^{-1}(x_N)\big) - s\big(u_{1,i_1}^{-1}(y_1), \ldots, u_{N,i_N}^{-1}(y_N)\big)\right|.
\end{aligned}$$

Using (4.8), we have

$$
\begin{aligned}
&|f^{\alpha}(x_1,\dots,x_N)-f^{\alpha}(y_1,\dots,y_N)|\\
\geq& k_f\ \|(x_1,\dots,x_N)-(y_1,\dots,y_N)\|_2^{\sigma}-|\alpha|K_{f^{\alpha}}d_{I^N}\big((u_{1,i_1}^{-1}(x_1),\dots,u_{N,i_N}^{-1}(x_N)),\\
&\big(u_{1,i_1}^{-1}(y_1),\dots,u_{N,i_N}^{-1}(y_N))\big)^{\sigma}-|\alpha|\ K_s d_{I^N}\big((u_{1,i_1}^{-1}(x_1),\dots,u_{N,i_N}^{-1}(x_N)),(u_{1,i_1}^{-1}(y_1),\dots,u_{N,i_N}^{-1}(y_N))\big)^{\sigma}\\
\geq&\ k_f\ \|(x_1,\dots,x_N)-(y_1,\dots,y_N)\|_2^{\sigma}-|\alpha|\ K_{f^{\alpha}_{\Delta,s}}N^{\sigma}\|(x_1,\dots,x_N)-(y_1,\dots,y_N)\|_2^{\sigma}\\
&-|\alpha|\ K_sN^{\sigma}d_{I^N}((x_1,\dots,x_N),(y_1,\dots,y_N))^{\sigma}\\
=&k_f\ \|(x_1,\dots,x_N)-(y_1,\dots,y_N)\|_2^{\sigma}-|\alpha|\ K_{f^{\alpha}}N^{\sigma}\ \|(x_1,\dots,x_N)-(y_1,\dots,y_N)\|_2^{\sigma}\\
&-|\alpha|\ K_sN^{\sigma}\ \|(x_1,\dots,x_N)-(y_1,\dots,y_N)\|_2^{\sigma}\\
=&\Big(k_f-(K_{f^{\alpha}}+K_s)|\alpha|N^{\sigma}\Big)\|(x_1,\dots,x_N)-(y_1,\dots,y_N)\|_2^{\sigma}.
\end{aligned}
$$

Let $K=k_f-(K_{f^{\alpha}}+K_s)|\alpha|N^{\sigma}$. Now, for $\delta=\frac{1}{N^m}$, we estimate

$$
\begin{aligned}
N_{\delta}(\mathcal{G}(f^{\alpha}))&\geq\sum_{i_1=1}^{N^m}\cdots\sum_{i_N=1}^{N^m}\max\{1,\lceil N^mR_{f^{\alpha}}[A_{i_1\dots i_N}]\rceil\}\\
&\geq\sum_{i_1=1}^{N^m}\cdots\sum_{i_N=1}^{N^m}\lceil N^mR_{f^{\alpha}}[A_{i_1\dots i_N}]\rceil\\
&\geq\sum_{i_1=1}^{N^m}\cdots\sum_{i_N=1}^{N^m}\lceil KN^mN^{-m\sigma}\rceil\\
&\geq N^m\dots N^m.N^mKN^{-m\sigma}\\
&=KN^{-m(\sigma-N-1)}.
\end{aligned}
$$

Using the above bound for $N_{\delta}(\mathcal{G}(f^{\alpha}))$, we obtain

$$
\underline{\dim}_B\big(\mathcal{G}(f^{\alpha})\big)=\underline{\lim}_{\delta\to0}\frac{\log\Big(N_{\delta}(\mathcal{G}(f^{\alpha}))\Big)}{-\log(\delta)}\geq\underline{\lim}_{m\to\infty}\frac{\log\Big(KN^{-m(\sigma-N-1)}\Big)}{m\log N}=N+1-\sigma,
$$

and hence the proof. □

Remark 4.14. With the assumptions in the above theorem, one may construct dimension-preserving approximants for a given function. See, for instance, [88, Theorem 3.16].

In [90, 91], some results have been extended in a bivariate setting. On putting $s = B_{m_1,\dots,m_N}(f)$ in (1.14), we have a unique function $f^{\alpha}_{\Delta,B_{m_1,\dots,m_N}} \in \mathcal{C}(I^N)$ such that

$$\begin{aligned} &f^{\alpha}_{\Delta,B_{m_1,\dots,m_N}}(x_1,\dots,x_N) \\ =&f(x_1,\dots,x_N) + \alpha\left(u^{-1}_{1,i_1}(x_1),\dots,u^{-1}_{N,i_N}(x_N)\right) f^{\alpha}_{\Delta,B_{m_1,\dots,m_N}}\left(u^{-1}_{1,i_1}(x_1),\dots,u^{-1}_{N,i_N}(x_N)\right) \\ &-\alpha\left(u^{-1}_{1,i_1}(x_1),\dots,u^{-1}_{N,i_N}(x_N)\right) B_{m_1,\dots,m_N}(f)\left(u^{-1}_{1,i_1}(x_1),\dots,u^{-1}_{N,i_N}(x_N)\right) \end{aligned} \tag{4.10}$$

for $(x_1,\dots,x_N) \in \prod_{k=1}^{N} I_{k,i_k},\ i_k \in \Sigma_{M_k}$.

Following the work of [90], we define a single-valued fractal operator $\mathcal{F}^{\alpha}_{m_1,\dots,m_N} : \mathcal{C}(I^N) \to \mathcal{C}(I^N)$ by

$$\mathcal{F}^{\alpha}_{m_1,\dots,m_N}(f) := f^{\alpha}_{\Delta,B_{m_1,\dots,m_N}}. \tag{4.11}$$

In [90], several operator theoretic results for bivariate fractal operators are obtained. Following [90, Theorem 3.2], we have $\mathcal{F}^{\alpha}_{m_1,\dots,m_N}$ is a bounded linear operator, see, for instance, [72, Remark 3.8].

Lemma 4.15 ([22], Lemma 1)**.** *Let $(X, \|.\|)$ be a Banach space, and $T : X \to X$ be a linear operator. Suppose there exist constants $\lambda_1, \lambda_2 \in [0, 1)$ such that*

$$\|Tx - x\| \le \lambda_1\|x\| + \lambda_2\|Tx\| \text{ for all } x \in X.$$

Then, T is a topological isomorphism, and

$$\frac{1-\lambda_2}{1+\lambda_1}\|x\| \le \|T^{-1}x\| \le \frac{1+\lambda_2}{1-\lambda_1}\|x\| \text{ for all } x \in X.$$

Theorem 4.16. *The fractal operator* $\mathcal{F}^{\alpha}_{m_1,\dots,m_N} : \mathcal{C}(I^N) \to \mathcal{C}(I^N)$ *is a topological isomorphism.*

Proof. Using (4.10) and Note 4.2, one gets

$$\begin{aligned}\left\|f - \mathcal{F}^{\alpha}_{m_1,\dots,m_N}(f)\right\|_{\infty} &\leq \|\alpha\|_{\infty}\left\|\mathcal{F}^{\alpha}_{m_1,\dots,m_N}(f) - B_{m_1,\dots,m_N}f\right\|_{\infty}\\ &\leq \|\alpha\|_{\infty}\left|\mathcal{F}^{\alpha}_{m_1,\dots,m_N}(f)\right\|_{\infty} + \|\alpha\|_{\infty}\|f\|_{\infty}.\end{aligned}$$

Since $\|\alpha\|_{\infty} < 1$, the previous lemma yields that the fractal operator $\mathcal{F}^{\alpha}_{m_1,\dots,m_N}$ is a topological isomorphism. □

Remark 4.17. For $N = 2$, the above theorem may strengthen item-4 of [90, Theorem 3.2]. To be precise, item-4 tells that $\mathcal{F}^{\alpha}_{m_1,\dots,m_N}$ is a topological isomorphism if $\|\alpha\|_{\infty} < \left(1 + \|I - B_{m_1,\dots,m_N}\|\right)^{-1}$, which is more restricted than the standing assumption considered in the above theorem, that is, $\|\alpha\|_{\infty} < 1$.

Example 4.18. Consider the following function in the domain $[-1, 1] \times [-1, 1]$ as the germ function

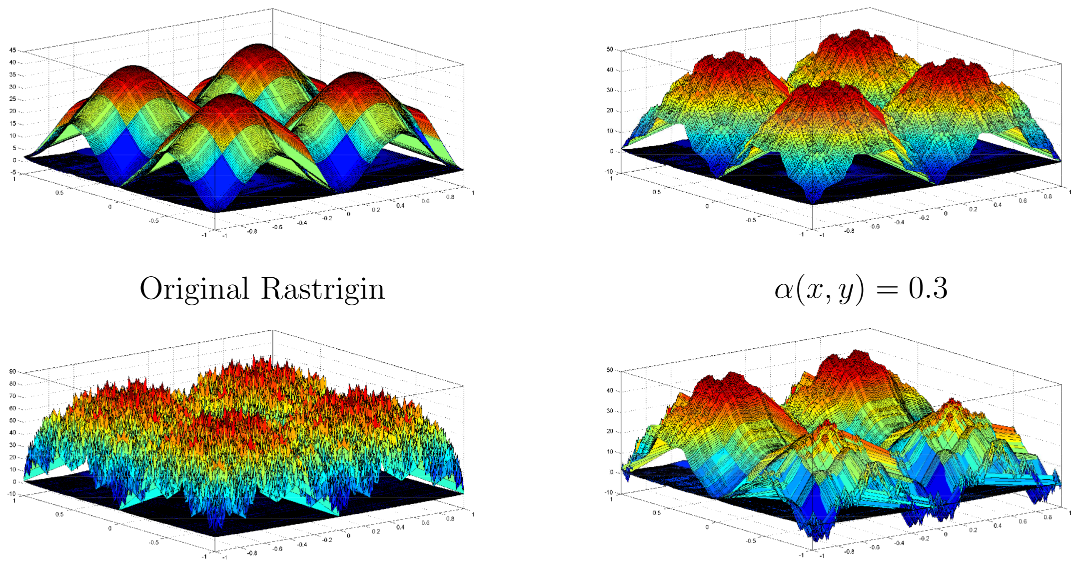
$$f(x, y) = 20 + x^2 + y^2 - 10\cos(2\pi x) - 10\cos(2\pi y),$$

which is shown in Figure 4.1. The aforementioned function is known as the Rastrigin function, and it is one of the typical functions used in the field of global optimization. Let us consider the following parameters:

(i) A net Δ determined by the partition $\{-1, -0.5, 0, 0.5, 1\}$ of $[-1, 1]$.

(ii) Scaling function $\alpha(x, y) = 0.3,\ 0.7,\ \frac{x}{2}$, respectively for all $(x, y) \in [-1, 1] \times [-1, 1]$.

(iii) Base function $s(x, y) = x^2y^2f(x, y)$.



Original Rastrigin $\alpha(x, y) = 0.3$

$\alpha(x, y) = 0.7$ $\alpha(x, y) = \frac{x}{2}$

FIGURE 4.1: The bivariate Rastrigin function and its fractal perturbations behavior with different choices of scaling functions.

The Rastrigin function is a non-convex function that is commonly used in mathematical optimization as a performance test index for optimization techniques. It serves as a classic example of a multi-model non-linear function. Rastrigin presented the function in 1974 and has been developed by Rudolph. Several local minima can be found in the Rastrigin function. The location of the minima is very evenly dispersed.

The graphical representation (Fig 4.1) clearly shows the continuous dependency of scaling function α and base function s, net Δ, and the function f on the fractal version of the well-known Rastrigin function.

As with the Rastrigin function, it is noticed that the "fractalized Rastrigin functions" find a greater number of applications in optimization, but we make no such claim here. Additionally, the degrees of freedom provided by this fractalization method may be advantageous for resolving some issues in conjunction with approximation and optimization.

4.4 Some Approximation Aspects

In this section, we shall return to the multivariate α-fractal functions in the function space $\mathcal{C}(I^N)$. First, let us recall the following well-known definition.

Definition 4.19. A Schauder basis in an infinite dimensional Banach space X is a sequence (e_n) of elements in X satisfying the following condition: for every x in X, there is a unique sequence $(a_n(x))$ of scalars such that

$$x = \sum_{n=1}^{\infty} a_n(x)e_n, \quad \text{i.e.,} \quad \left\| x - \sum_{n=1}^{m} a_n(x)e_n \right\| \to 0 \text{ as } m \to \infty.$$

The coefficients $a_n(x)$ are linear functions of x uniquely determined by the basis referred to as the associated sequence of coefficient functionals.

The existence of the Schauder basis has many practical applications, for instance, finding the best approximation of an element in space if it exists. Schauder basis is especially important for applications in operator equations in Banach spaces. In the previous section, we studied multivariate fractal functions that are close to the prescribed function, at the same time possessing a self-referential structure. In some applications, it is required to maintain the global structure involved in a given problem, and self-referentiality may be beneficial. In contrast to the case $\mathcal{C}([0,1])$ wherein the classical Faber-Schauder system provides a Schauder basis, the situation gets more complicated in the case $\mathcal{C}([0,1]^d)$, $d \geq 2$. The tensor products of Faber-Schauder basis in the copies of $\mathcal{C}([0,1])$ form a basis of $\mathcal{C}([0,1]^d)$. Another different basis is the so-called regular pyramidal and skew pyramidal basis. The reader may refer [85] for a detailed description of various Schauder basis for $\mathcal{C}([0,1]^d)$. In this instance, we find a Schauder basis for $\mathcal{C}(I^N)$ consisting of fractal functions; the maps involved are perturbations of those belonging to a classical basis for $\mathcal{C}(I^N)$.

The central idea is to use the fact that a topological automorphism preserves a Schauder basis. However, we provide the details in the following.

Theorem 4.20. *There exists a Schauder basis consisting of multivariate fractal functions for the space $\mathcal{C}(I^N)$.*

Proof. Let (e_n) be a Schauder basis of $\mathcal{C}(I^N)$, whose existence is hinted at in the last paragraph. Choose α such that $\|\alpha\|_\infty < 1$, so that by Theorem 4.16, the fractal operator $\mathcal{F}^\alpha_{m_1,\dots,m_N}$ is a topological automorphism. If $g \in \mathcal{C}(I^N)$, then $\left(\mathcal{F}^\alpha_{m_1,\dots,m_N}\right)^{-1}(g) \in \mathcal{C}(I^N)$, so that

$$\left(\mathcal{F}^\alpha_{m_1,\dots,m_N}\right)^{-1}(g) = \sum_{n=1}^{\infty} a_n\Big(\left(\mathcal{F}^\alpha_{m_1,\dots,m_N}\right)^{-1}(g)\Big)e_n.$$

By the continuity of the fractal linear operator $\mathcal{F}^\alpha_{m_1,\dots,m_N}$ it follows that

$$g = \mathcal{F}^\alpha_{m_1,\dots,m_N}\left(\mathcal{F}^\alpha_{m_1,\dots,m_N}\right)^{-1}(g) = \sum_{n=1}^{\infty} a_n\Big(\left(\mathcal{F}^\alpha_{m_1,\dots,m_N}\right)^{-1}(g)\Big){e_n}^\alpha,$$

where $e^\alpha_n = \mathcal{F}^\alpha_{m_1,\dots,m_N}(e_n)$. Assume that $g = \sum_{n=1}^\infty b_n {e_n}^\alpha$ is another representation of g. Since $\left(\mathcal{F}^\alpha_{m_1,\dots,m_N}\right)^{-1}$ is also continuous, we have

$$\left(\mathcal{F}^\alpha_{m_1,\dots,m_N}\right)^{-1}(g) = \sum_{n=1}^{\infty} b_n e_n,$$

and hence $b_n = \sum_{n=1}^\infty a_n\left(\left(\mathcal{F}^\alpha_{m_1,\dots,m_N}\right)^{-1}(g)\right)$ for each n. Consequently, (e^α_n) is a Schauder basis for $\mathcal{C}(I^N)$, obtaining the desired conclusion. □

Remark 4.21. For $N = 2$, the above theorem reduces to [90, Theorem 5.1]. Moreover, it is worth noting that the result uses less restrictive conditions on the scale function α than that of [90, Theorem 5.1]. To be precise, [90, Theorem 5.1] is proved using the condition $\|\alpha\|_\infty < (1 + \|Id - B_{m_1,\dots,m_N}\|)^{-1}$, on the other hand, the above theorem uses only the standard assumption on the scale function α, i.e., $\|\alpha\|_\infty < 1$.

Definition 4.22. The fractal operator $\mathcal{F}^\alpha_{m_1,\dots,m_N}$ is a mapping from the space of continuous functions on I^N to itself. It takes a function f and maps it to $f^\alpha_{\Delta,B_{m_1,\dots,m_N}}$. If p is a multivariate polynomial, then $\mathcal{F}^\alpha_{m_1,\dots,m_N}(p)$ is denoted as p^α and is called a multivariate fractal polynomial. The space of all multivariate polynomials on I^N is denoted as $\mathcal{P}(I^N)$, and the image space of $\mathcal{P}(I^N)$ under the operator $\mathcal{F}^\alpha_{m_1,\dots,m_N}$ is denoted as $\mathcal{P}^\alpha(I^N)$.

Theorem 4.23. *Let $\mathcal{C}(I^N)$ be endowed with the uniform norm, $f \in \mathcal{C}(I^N)$, and $B_{m_1,\dots,m_N} : \mathcal{C}(I^N) \to \mathcal{C}(I^N)$, be the Bernstein operator of order $(m_1, \dots, m_N)$. For any $\epsilon > 0$, net Δ of the hypercube I^N, there exists a multivariate fractal polynomial p^α such that*

$$\|f - p^\alpha\|_\infty < \epsilon.$$

Proof. Let $\epsilon > 0$ be given. By the Stone-Weierstrass theorem, there exists a polynomial function p in q-variables such that

$$\|f - p\|_\infty < \frac{\epsilon}{2}.$$

Fix a net Δ of the hyperrectangle I^N, a bounded linear operator $B_{m_1,\dots,m_N} : \mathcal{C}(I^N) \to \mathcal{C}(I^N)$.

Choose $\alpha : I^N \to \mathbb{R}$ as continuous function on I^N with $\|\alpha\|_\infty = \sup\left\{|\alpha(x_1, \dots, x_N)| : (x_1, \dots, x_N) \in I^N\right\} < 1$ such that

$$\|\alpha\|_\infty < \frac{\frac{\epsilon}{2}}{\frac{\epsilon}{2} + \|Id - B_{m_1,\dots,m_N}\|\|p\|_\infty}.$$

Then, we have

$$\|f - p^\alpha\|_\infty \le \|f - p\|_\infty + \|p - p^\alpha\|_\infty \le \|f - p\|_\infty + \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty}\|Id - B_{m_1,\dots,m_N}\|\|p\|_\infty \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

In the above, the first inequality is just the triangle inequality; the second follows from Theorem 4.23, and the third is obvious. □

Remark 4.24. For $N = 2$, the above theorem reduces to [90, Theorem 5.4].

Remark 4.25. In the above proof, we selected $\alpha \in \mathcal{C}(I^N)$, for instance, constants, such that $\|\alpha\|_\infty < \dfrac{\frac{\epsilon}{2}}{\frac{\epsilon}{2} + \|Id - B_{m_1,\dots,m_N}\| \;\; \|p\|_\infty}$. In this case, α may be "close" to 0, and hence p^α may lose self-referentiality and behave as a traditional multivariate polynomial. Alternatively, one can fix $\alpha \in \mathcal{C}(I^N)$ such that $\|\alpha\|_\infty < 1$, but otherwise arbitrary and choose a bounded linear operator $B_{m_1,\dots,m_N} : \mathcal{C}(I^N) \to \mathcal{C}(I^N\mathbb{R})$ such that

$$\|Id - B_{m_1,\dots,m_N}\| < \frac{1 - \|\alpha\|_\infty}{\|\alpha\|_\infty \|p\|_\infty}\frac{\epsilon}{2}.$$

In this case, we expect that the graph of the corresponding fractal polynomial p^α has a box dimension greater than q, thus possessing a "fractality" in it and differs from the traditional multivariate polynomial.

Theorem 4.26. *Let $f \in \mathcal{C}(I^N)$ be such that $f(x_1, \dots, x_N) \ge 0$ for all $(x_1, \dots, x_N) \in I^N$, and let $q \le \beta \le N+1$. Then, for $\epsilon > 0$, and for $\alpha \in \mathcal{C}(I^N)$ satisfying $\|\alpha\|_\infty < 1$, there exists an α-fractal function $g^\alpha_{\Delta,B_{m_1,\dots,m_N}} \in S_\beta$ satisfying*

$$g^\alpha_{\Delta,B_{m_1,\dots,m_N}}(x_1, \dots, x_N) \ge 0 \text{ for all } (x_1, \dots, x_N) \in I^N \text{ and } \|f - g^\alpha_{\Delta,B_{m_1,\dots,m_N}}\|_\infty < \epsilon.$$

Proof. In view of Note 4.2, the Bernstein operator $B_{m_1,\dots,m_N}$ fixes the constant function 1, i.e., $B_{m_1,\dots,m_N}(1) = 1$, where $1(x_1, \dots, x_N) = 1$ on I^N. Consider $\alpha \in \mathcal{C}(I^N)$

such that $\|\alpha\|_\infty < 1$. From (4.10), we deduce

$$\|g^\alpha_{\Delta,B_{m_1,\dots,m_N}} - g\|_\infty \leq \|\alpha\|_\infty \|g^\alpha_{\Delta,B_{m_1,\dots,m_N}} - B_{m_1,\dots,m_N} g\|_\infty \text{ for all } g \in \mathcal{C}(I^N).$$

Choose $g = 1$, then the above inequality gives

$$\|f^\alpha_{\Delta,B_{m_1,\dots,m_N}} - 1\|_\infty \leq \|\alpha\|_\infty \|f^\alpha_{\Delta,B_{m_1,\dots,m_N}} - 1\|_\infty,$$

and this further yields $\|f^\alpha_{\Delta,B_{m_1,\dots,m_N}} - 1\|_\infty = 0$. Therefore, $f^\alpha_{\Delta,B_{m_1,\dots,m_N}} = 1$, i.e, $\mathcal{F}^\alpha_{m_1,\dots,m_N}(1) = 1$.

For $\epsilon > 0$, $\alpha \in \mathcal{C}(I^N)$ and $f \in \mathcal{C}(I^N)$. Using Remark 4.25, there exists a function $h^\alpha_{\Delta,B_{m_1,\dots,m_N}}$ such that

$$\|f - h^\alpha_{\Delta,B_{m_1,\dots,m_N}}\|_\infty < \frac{\epsilon}{2}, \quad \text{where } \mathcal{F}^\alpha_{m_1,\dots,m_N}(h) = h^\alpha_{\Delta,B_{m_1,\dots,m_N}}.$$

Define $g^\alpha_{\Delta,B_{m_1,\dots,m_N}}(x_1,\dots,x_N) = h^\alpha_{\Delta,B_{m_1,\dots,m_N}}(x_1,\dots,x_N) + \frac{\epsilon}{2}$ for all $(x_1,\dots,x_N) \in I^N$.

Since $\mathcal{F}^\alpha_{m_1,\dots,m_N}(1) = 1$,

$$\begin{aligned} g^\alpha_{\Delta,B_{m_1,\dots,m_N}}(x_1,\dots,x_N) &= h^\alpha_{\Delta,B_{m_1,\dots,m_N}}(x_1,\dots,x_N) + \frac{\epsilon}{2} 1(x_1,\dots,x_N) \\ &= h^\alpha_{\Delta,B_{m_1,\dots,m_N}}(x_1,\dots,x_N) + \frac{\epsilon}{2} 1^\alpha(x_1,\dots,x_N). \end{aligned}$$

Further, since $\mathcal{F}^\alpha_{m_1,\dots,m_N}$ is a linear operator

$$g^\alpha_{\Delta,B_{m_1,\dots,m_N}} = h^\alpha_{\Delta,B_{m_1,\dots,m_N}} + \frac{\epsilon}{2} 1^\alpha = \mathcal{F}^\alpha_{m_1,\dots,m_N}\left(h + \frac{\epsilon}{2} 1\right).$$

Moreover,

$$g^\alpha_{\Delta,B_{m_1,\dots,m_N}}(x_1,\dots,x_N) = h^\alpha_{\Delta,B_{m_1,\dots,m_N}}(x_1,\dots,x_N) + \frac{\epsilon}{2}$$

$$= h^{\alpha}_{\Delta,B_{m_1,\dots,m_N}}(x_1,\dots,x_N) + \frac{\epsilon}{2} - f(x_1,\dots,x_N) + f(x_1,\dots,x_N)$$
$$\geq f(x_1,\dots,x_N) + \frac{\epsilon}{2} - \|h^{\alpha}_{\Delta,B_{m_1,\dots,m_N}} - f\|_\infty \geq 0.$$

Further, we get

$$\|f - g^{\alpha}_{\Delta,B_{m_1,\dots,m_N}}\|_\infty \leq \|f - h^{\alpha}_{\Delta,B_{m_1,\dots,m_N}}\|_\infty + \|h^{\alpha}_{\Delta,B_{m_1,\dots,m_N}} - g^{\alpha}_{\Delta,B_{m_1,\dots,m_N}}\|_\infty$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This completes the proof. □

4.5 Conclusion

In this article, we have studied the one-sided approximation of multivariate functions using fractal functions (Subsection 4.2.1). Next, we have constructed multivariate α-fractal function (Equation (1.14)) for a given function $f \in \mathcal{C}(I^N)$. Next, we have given fractal dimensional results for the graph of such fractal functions (Theorem 4.13). Further, we have hinted at the dimension preserving α-fractal function using Bernstein polynomial (Equation (4.10)). Moreover, we have proved the existence of the Schauder basis consisting of multivariate fractal functions for the space $\mathcal{C}(I^N)$(Theorem 4.20). Further, we have defined multivariate fractal polynomial using Bernstein polynomial (Definition 4.22) and proved that every $f \in \mathcal{C}(I^N)$ can be approximated with a multivariate fractal polynomial (Theorem 4.23).

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