Chapter 1

Preliminaries

This chapter provides some basic definitions and results that will be used in this thesis. We begin by giving definitions and properties of certain arithmetical and special functions which are required to state and prove the results appearing in this thesis. Next, we give definitions and basic properties of modular forms, Jacobi forms, Siegel modular forms and associated L-functions.

1.1 Arithmetical and special functions

A complex-valued function $f : \mathbb{N} \to \mathbb{C}$ is said to be an arithmetical function. An arithmetical function f is said to be multiplicative if it satisfies

$$f(mn) = f(m)f(n) \quad \text{whenever} \quad (m, n) = 1. \tag{1.1}$$

Further, if (1.1) holds for all $m, n \in \mathbb{N}$, then f is said to be completely multiplicative.

We now give examples of arithmetical functions which will appear in the forthcoming chapters.

Example 1.1. (Möbius function) *The Möbius function* μ *is defined as follows:* $\mu(1) = 1$ *and for* $n > 1$

$$\mu(n) := \begin{cases} (-1)^r & \text{if } n = p_1 p_2 \cdots p_r, \\ 0 & \text{otherwise,} \end{cases}$$

where $p_1, p_2, \cdots, p_r$ *are distinct primes.*

Observe that $\mu(n)$ is a multiplicative function. Below is the list of first few values of $\mu(n)$:

$n :$	1	2	3	4	5	6	7	8	9	10
$\mu(n) :$	1	-1	-1	0	-1	1	-1	0	0	1

Table 1.1: Particular values of $\mu(n)$.

Let N be a positive integer. A Dirichlet character χ modulo N is a group homomorphism $\chi : \left(\frac{\mathbb{Z}}{N\mathbb{Z}}\right)^* \to \mathbb{C}^*$. A Dirichlet character modulo N is said to be the trivial character if $\chi(n) = 1$ for all n. A Dirichlet character χ modulo N is said to be principal if

$$\chi(n) = \begin{cases} 1 & \text{if } (n, N) = 1, \\ 0 & \text{if } (n, N) > 1. \end{cases}$$

Let χ be a Dirichlet character modulo N and $m|N$, then m is said to be an induced modulus of χ if $\chi(a) = 1$ whenever $(a, N) = 1$ and $a \equiv 1(\bmod\ m)$. The conductor of a character χ is the smallest induced modulus m of χ. A Dirichlet character χ modulo N is said to be a primitive character if the conductor of χ is N.

For a Dirichlet character modulo N, we define an arithmetical function $\tilde{\chi}$ as follows:

$$\tilde{\chi}(n) := \begin{cases} \chi(n) & \text{if } (n, N) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that $\tilde{\chi}(n)$ is a completely multiplicative function. For brevity, from now on we write χ for $\tilde{\chi}$.

Example 1.2. (Generalized divisor function) *For $n \in \mathbb{N}$, the generalized divisor function $d_\ell(n)$ counts the number of ways in which n can be expressed as product of ℓ-factors, i.e., $d_\ell(n) := \sum_{n=n_1\cdots n_\ell} 1$.*

In particular, $d_1(n) = 1$ for all n and $d_2(n) = \sum_{d|n} 1$. $d_2(n)$ counts the number of positive divisors of n. It is also denoted by $d(n)$. It is easy to see that $d(n)$ is a multiplicative function. The first few values of $d(n)$ are listed below:

n :	1	2	3	4	5	6	7	8	9	10
$d(n)$:	1	2	2	3	2	4	2	4	3	4

Table 1.2: Particular values of $d(n)$.

Next we define Lambert series which is the main theme of this thesis.

Definition 1.1. (Lambert Series) For $q \in \mathbb{C}$ with $|q| < 1$, a Lambert series is a series of the form

$$S(q) := \sum_{n=1}^{\infty} a(n) \frac{q^n}{1 - q^n}, \tag{1.2}$$

where $a(n)$ is certain arithmetical function.

The series (1.2) can be expressed in the following form:

$$S(q) = \sum_{n=1}^{\infty} a(n) \sum_{k=1}^{\infty} q^{nk} = \sum_{n=1}^{\infty} b(n) q^n, \tag{1.3}$$

where $b(n) = \sum_{d|n} a(d)$. If we take $q = \exp(-y)$, $y \in \mathbb{R}^+$, then the Lambert series (1.3) can be written as

$$S(q) = \sum_{n=1}^{\infty} b(n) \exp(-ny). \tag{1.4}$$

In this thesis, we consider the above form of Lambert series.

Definition 1.2. (Bernoulli number) The n-th Bernoulli number B_n is defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

It is well-known that $B_n = 0$ for all odd $n > 1$. The first few values of B_n for even n and B_1 are listed below:

n :	0	1	2	4	6	8	10	12	14	16
B_n :	1	$\frac{-1}{2}$	$\frac{1}{6}$	$\frac{-1}{30}$	$\frac{1}{42}$	$\frac{-1}{30}$	$\frac{5}{66}$	$\frac{-691}{2730}$	$\frac{7}{6}$	$\frac{-3617}{510}$

Table 1.3: Particular values of B_n.

Next, we define the well-known Gamma function.

Definition 1.3. (Gamma function) For $s \in \mathbb{C}$ with $\Re(s) > 0$, the Gamma function is defined by the following integral:

$$\Gamma(s) := \int_0^{\infty} x^{s-1} e^{-x} dx.$$

The analytic properties of the Gamma function are given in the next theorem.

Theorem 1.1. *[4, p. 250, Section 12.2] The Gamma function* $\Gamma(s)$ *can be analytically continued to the whole complex plane except for simple poles at negative integers and zero. Also, it satisfies the functional equation* $\Gamma(s+1) = s\Gamma(s)$.

The inverse Mellin transform of the Gamma function is given by

$$e^{-z} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) z^{-s} ds,$$

where $c > 0$ is sufficiently large.

Lemma 1.1. (Stirling's bound for $\Gamma(s)$) *[31, p. 151, A.4.] Let $s = \sigma + iT$ with $p \leq \sigma \leq q$. Then we have*

$$|\Gamma(\sigma + iT)| \sim \sqrt{2\pi} |T|^{\sigma - 1/2} \exp\left(-\frac{\pi |T|}{2}\right) \quad \text{as} \quad |T| \to \infty. \tag{1.5}$$

Next, we define the famous Riemann zeta function.

Definition 1.4. (Riemann zeta function) For $s \in \mathbb{C}$ with $\Re(s) > 1$ the Riemann zeta function is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The Riemann zeta function and Bernoulli numbers are related by the following formula. For even $n \in \mathbb{N}$ we have

$$\zeta(n) = (-1)^{\frac{n}{2}+1} \frac{(2\pi)^n B_n}{2(n!)}. \tag{1.6}$$

Some particular values of $\zeta(n)$ are listed below:

n :	2	4	6	8	10	12	14	16
$\zeta(n)$:	$\frac{\pi^2}{6}$	$\frac{\pi^4}{90}$	$\frac{\pi^6}{945}$	$\frac{\pi^8}{9450}$	$\frac{\pi^{10}}{93555}$	$\frac{691\ \pi^{12}}{638512875}$	$\frac{2\ \pi^{14}}{18243225}$	$\frac{3617\ \pi^{16}}{325641566250}$

TABLE 1.4: Particular values of $\zeta(n)$.

The analytic properties of the Riemann zeta function are given in the next theorem.

Theorem 1.2. *[4, p. 259, Section 12.8] The Riemann zeta function can be analytically continued to the whole complex plane except for a simple pole at $s = 1$ and* *satisfies the functional equation*

$$\Lambda(s) = \Lambda(1-s),$$

where

$$\Lambda(s) := \pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

The Riemann zeta function has zeros at every negative even integers, i.e., $\zeta(-2n) = 0$ for all $n \in \mathbb{N}$. These are called trivial zeros of $\zeta(s)$. The zeros of $\zeta(s)$ other than trivial zeros are called non-trivial zeros. A famous conjecture known as the Riemann hypothesis (RH) concerning the non-trivial zeros of the Riemann zeta function is stated below.

Conjecture: (Riemann Hypothesis) [4, p. 293, Section 13.9] All the non-trivial zeros of $\zeta(s)$ lie on the line $\Re(s) = \frac{1}{2}$.

Theorem 1.3. *[66, p. 2] The Riemann zeta function $\zeta(s)$ has no zeros in the region $\Re(s) > 1$.*

Conjecture: (Simplicity Hypothesis)[32, p. 184] All the non-trivial zeros of $\zeta(s)$ are simple.

We next state a lemma without proof that will be used in the proofs of the main results of Chapters 2 and 3.

Lemma 1.2. *[66, p. 219, Section 9.8] Let T_n be a sequence of positive real numbers such that $|T_n - \Im(\rho)| > \exp\left(-C\frac{\Im(\rho)}{\log(\Im(\rho))}\right)$ for any non-trivial zeros ρ of $\zeta(s)$, where C is some positive constant. Then*

$$\frac{1}{|\zeta(x+iT_n)|} < \exp(A_1 T_n),$$

where $0 < A_1 < \pi/4$.

Definition 1.5. (Dirichlet L-function) Let χ be a Dirichlet character modulo N. The Dirichlet L-function associated to χ is a series of the form

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \Re(s) > 1.$$

In the next theorem, we state the analytic properties of Dirichlet L-function.

Theorem 1.4. *[31, p. 84, Theorem 4.15] Let χ be a primitive Dirichlet character modulo $N \geq 1$. Then $L(s,\chi)$ extends to a meromorphic function on $\mathbb{C}$ which is entire if $\chi \neq 1$ and otherwise admits a unique simple pole at $s = 1$ with residue 1. The completed L-function*

$$\Lambda(s,\chi) = \left(\frac{N}{\pi}\right)^{s/2} \Gamma\left(\frac{s+\delta}{2}\right) L(s,\chi)$$

is entire if $N \neq 1$ and has simple poles with residue 1 at $s = 0$ and $s = 1$ otherwise. Moreover, it satisfies the following functional equation:

$$\Lambda(s,\chi) = i^{-\delta}\, \frac{g(\chi)}{\sqrt{N}} \Lambda(1-s,\bar{\chi}), \qquad \text{where } \delta = \frac{1}{2}(1-\chi(-1)).$$

Conjecture: (Generalized Riemann Hypothesis)[18, p. 124] For a Dirichlet character χ, the non-trivial zeros of $L(s,\chi)$ lie on $\Re(s) = \frac{1}{2}$.

Conjecture: (Grand Simplicity Hypothesis)[63, p. 176] For a primitive Dirichlet character χ, the non-trivial zeros of $L(s,\chi)$ are simple.

Next we define some special functions which will appear in the main results of the thesis.

Definition 1.6. (Meijer G-function)[58, p. 415] Let m, n, p, q be integers with $0 \leq m \leq q$, $0 \leq n \leq p$. Let $a_1, \cdots, a_p$ and $b_1, \cdots, b_q$ be $p+q$ complex numbers such that $a_i - b_j \notin \mathbb{N}$, for $i \in [1, n]$, $j \in [1, m]$. The Meijer G-function is defined by the following line integral:

$$G_{p,q}^{m,n}\left(\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix} \middle| z \right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j + w) \prod_{j=1}^{n} \Gamma(1 - a_j - w) z^{-w}}{\prod_{j=m+1}^{q} \Gamma(1 - b_j - w) \prod_{j=n+1}^{p} \Gamma(a_j + w)} \mathrm{d}w. \quad (1.7)$$

Here, we assume that the line of integration L separates the poles of the factors $\Gamma(b_j + w)$ from the poles of the factors $\Gamma(1 - a_j - w)$. The above integral converges absolutely if $2(m+n) > p + q$ and $|\arg(z)| < (2m + 2n - p - q)\frac{\pi}{2}$.

Definition 1.7. (Confluent hypergeometric function of second kind) For $z \in \mathbb{R}^+$ and $a, b \in \mathbb{Z}$, the confluent hypergeometric function is defined as follows:

$$U(a, b, z) := \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt} t^{a-1} (1+t)^{b-a-1} dt.$$

For arithmetical functions f and g, we write $f(n) = O(g(n))$ if there exists a positive constant K such that $|f(n)| \leq K|g(n)|$ for all n. Another way to represent this is by using the notation $\ll$ and writing $f(n) \ll g(n)$. Further, if $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = 1$ then $f(x)$ is said to be asymptotic to $g(x)$ as $x \to \infty$ and is denoted by $f(x) \sim g(x)$.

Next lemma tells us about the asymptotic behaviour of the confluent hypergeometric function.

Lemma 1.3. *[23, p. 278, Section 6.13.1] For any non-negative integer M and $z \to \infty$ with $|\arg(z)| < \frac{3\pi}{2}$, one has*

$$U(a, b; z) = \sum_{j=0}^{M} (-1)^j \frac{(a)_j (a - b + 1)_j}{j!} z^{-a-j} + O(|z|^{-a-M-1}),$$

where $(a)_j$ *denotes the usual Pochhammer's Symbol defined by*

$$(a)_j = a(a+1)...(a+j-1).$$

Definition 1.8. (Whittaker function) Let κ and μ be complex numbers. The Whittaker function $W_{\kappa,\mu}(z)$ is one of the solutions of the following differential equation:

$$\frac{d^2w}{dz^2} + \left(\frac{1-4\mu^2+4\kappa z - z^2}{4z^2}\right) w = 0.$$

The asymptotic behaviour of the Whittaker function is given in the next lemma.

Lemma 1.4. *[58, p. 335, eq. 13.14.21] As* $z \to \infty$ *with* $|\arg(z)| < \frac{3\pi}{2}$, *one has*

$$W_{\kappa,\mu}(z) \sim \exp(-z/2)z^{\kappa}.$$

A particular case where Meijer G-function reduces to Whittaker function and confluent hypergeometric function for specific values is given in the next lemma.

Lemma 1.5. *[37, p. 58, eq. 2.114, 2.118] For* $|\arg(z)| < \frac{\pi}{2}$, *we have*

$$G^{2,0}_{1,2}\left(\begin{array}{c} a_1 \\ b_1, b_2 \end{array}\middle| z\right) = z^{\frac{b_1+b_2-1}{2}} e^{-z/2} W_{\kappa,\mu}(z), = z^{b_1} e^{-z} U\left(\mu - \kappa + \frac{1}{2}, 1+2\mu; z\right), \quad (1.8)$$

where $\kappa = \frac{1}{2}(b_1+b_2+1) - a_1$ *and* $\mu = \frac{1}{2}(b_1 - b_2)$.

1.2 Modular forms for $SL_2(\mathbb{Z})$

In this section, we give definition and basic properties of modular forms over $SL_2(\mathbb{Z})$. For more details on the theory of modular forms we refer to [30, 39]. The full modular group $SL_2(\mathbb{Z})$ is defined by

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}.$$

The group $SL_2(\mathbb{Z})$ acts on the complex upper half-plane $\mathbb{H} := \{\tau \in \mathbb{H} : \Im(\tau) > 0\}$ as follows:

$$(\gamma, \tau) \mapsto \gamma \cdot \tau := \frac{a\tau + b}{c\tau + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \quad \tau \in \mathbb{H}.$$

Definition 1.9. (Fundamental domain) A fundamental domain for the action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$, denoted by $SL_2(\mathbb{Z}) \setminus \mathbb{H}$, is a closed region in $\mathbb{H}$ such that no two distinct interior points of $SL_2(\mathbb{Z}) \setminus \mathbb{H}$ are equivalent under the action of $SL_2(\mathbb{Z})$ and every point in $\mathbb{H}$ is $SL_2(\mathbb{Z})$-equivalent to some point in $SL_2(\mathbb{Z}) \setminus \mathbb{H}$, i.e., the relation $\gamma \cdot \tau_1 = \tau_2$ does not hold for any interior points $\tau_1, \tau_2 \in SL_2(\mathbb{Z}) \setminus \mathbb{H}$ and $\gamma \in SL_2(\mathbb{Z})$, and for all $\tau \in \mathbb{H}$ there exists $\tau' \in SL_2(\mathbb{Z}) \setminus \mathbb{H}$ and $\gamma \in SL_2(\mathbb{Z})$ such that $\gamma \cdot \tau = \tau'$.

A fundamental domain for the action of $SL_2(\mathbb{Z})$ is given by

$$SL_2(\mathbb{Z}) \setminus \mathbb{H} = \left\{ \tau \in \mathbb{H} \;\middle|\; |\Im(\tau)| \geq 1, \quad |\Re(\tau)| \leq \frac{1}{2} \right\}.$$

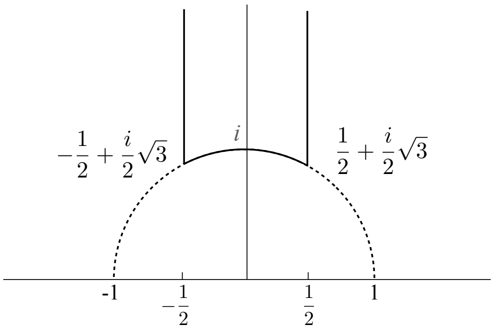

FIGURE 1.1: Fundamental domain for $SL_2(\mathbb{Z})$

Let k be a fixed positive integer. For a complex-valued function $f : \mathbb{H} \to \mathbb{C}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, the slash operator of weight k is defined by

$$(f\,|_k\,\gamma)\,(\tau) := \ (c\tau + d)^{-k} f(\gamma\tau).$$

Now we define modular forms over the full modular group $SL_2(\mathbb{Z})$.

Definition 1.10. (Modular form) A holomorphic function $f : \mathbb{H} \longrightarrow \mathbb{C}$ is said to be a modular form of weight k over $SL_2(\mathbb{Z})$ if it satisfies the following:

1. $f\,|_k\,\gamma = f$ for all $\gamma \in SL_2(\mathbb{Z})$, i.e.,
   $$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \text{ and } \tau \in \mathbb{H}.$$
2. f is holomorphic at the cusp $i\infty$ of $SL_2(\mathbb{Z})$.

Further, f is said to be a cusp form if it vanishes at the cusp $i\infty$ of $SL_2(\mathbb{Z})$.

A modular form f has a Fourier series expansion given by

$$f(\tau) = \sum_{n=0}^{\infty} a_f(n) e(n\tau),$$

where for each n, $a_f(n)$ is called the n-th Fourier coefficient of f. If f is a cusp form, then $a_f(0) = 0$. We denote by $M_k(SL_2(\mathbb{Z}))$ and $S_k(SL_2(\mathbb{Z}))$ the complex vector space of modular forms of weight k over $SL_2(\mathbb{Z})$ and the subspace of cusp forms of weight k over $SL_2(\mathbb{Z})$, respectively. It is well-known that the spaces $M_k(SL_2(\mathbb{Z}))$ and $S_k(SL_2(\mathbb{Z}))$ are finite-dimensional complex vector spaces. In fact, the space $S_k(SL_2(\mathbb{Z}))$ is Hilbert space with respect to the Petersson scalar product defined below.

Let f and $g \in M_k(SL_2(\mathbb{Z}))$ be such that at least one of them is a cusp form. Then the Peterson scalar product $\langle f, g\rangle$ of f and g is defined by

$$\langle f, g\rangle := \int_{SL_2(\mathbb{Z})\backslash\mathbb{H}} f(\tau)\overline{g(\tau)}\Im(\tau)^k d^*\tau,$$

where $d^*\tau := \frac{dudv}{v^2}$ $(\tau = u + iv \in \mathbb{H})$ is an invariant measure under the action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$. The next theorem gives a bound of the Fourier coefficient of a modular form.

Theorem 1.5. *[39, p. 139] Let f be a modular form of weight k with the Fourier series expansion given by $f(\tau) = \sum\limits_{n=0}^{\infty} a_f(n)e(n\tau)$. Then the Fourier coefficients $a_f(n)$ satisfy the following bound:*

$$|a_f(n)| = O(n^{k-1}).$$

Moreover, if f is a cusp form, then

$$|a_f(n)| = O(n^{\frac{k-1}{2}+\epsilon}).$$

Now we give some examples of modular forms.

Example 1.3. *Let k be an even positive integer with $k \geq 4$. The normalized Eisenstein series E_k of weight k is defined as*

$$E_k(\tau) := \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0) \\ (m,n)=1}} (m\tau + n)^{-k}.$$

It is well-known that the E_k is a modular of weight k with Fourier series expansion given by

$$E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e(n\tau),$$

where B_k is the k-th Bernoulli number and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.

In particular, the Fourier series expansion of E_k for $k = 4, 6, 8, 10, 12$ and 14 are given by

$$\begin{aligned}
E_4(\tau) &= 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)e(n\tau) \\
E_6(\tau) &= 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)e(n\tau) \\
E_8(\tau) &= 1 + 480\sum_{n=1}^{\infty}\sigma_7(n)e(n\tau) \\
E_{10}(\tau) &= 1 - 264\sum_{n=1}^{\infty}\sigma_9(n)e(n\tau) \\
E_{12}(\tau) &= 1 + \frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)e(n\tau) \\
E_{14}(\tau) &= 1 - 24\sum_{n=1}^{\infty}\sigma_{13}(n)e(n\tau).
\end{aligned}$$

Next, we give an example of a cusp form.

Example 1.4. *The Ramanujan delta function* Δ *is defined by*

$$\Delta(\tau) := e(\tau)\prod_{n=1}^{\infty}(1-e(\tau))^{24} \;=\; \frac{1}{1728}(E_4(\tau)^3 - E_6(\tau)^2).$$

The function Δ is a cusp form of weight 12 for $SL_2(\mathbb{Z})$ with Fourier series expansion given by

$$\Delta(\tau) = \sum_{n=1}^{\infty}\tau(n)e(n\tau), \tag{1.9}$$

where $\tau(n)$ is the Ramanujan tau function. Some particular values of $\tau(n)$ are listed below:

n :	1	2	3	4	5	6	7	8	9	10
$\tau(n)$:	1	-24	252	-1472	4830	-6048	-16744	84480	-113643	-115920

TABLE 1.5: Particular values of $\tau(n)$.

For each positive integer m, one can define certain linear operator called the m-th Hecke operator T_m (see [5] for the precise definition) on the space $M_k(SL_2(\mathbb{Z}))$. For each m, T_m maps a modular (respectively, cusp) form of weight k to a modular (respectively, cusp) form of weight k. If $f \in M_k(SL_2(\mathbb{Z}))$ has the Fourier series expansion given by $f(\tau) = \sum\limits_{n=0}^{\infty} a_f(n)e(n\tau)$, then the Fourier series expansion of the image of f under T_m is given by $(T_m f)(\tau) = \sum\limits_{n=0}^{\infty} a_{T_m f}(n)e(n\tau)$, where $a_{T_m f}(n) = \sum\limits_{d|(m,n)} d^{k-1}a_f(\frac{mn}{d^2})$. It is well-known that the family $\{T_n : n \in \mathbb{N}\}$ of Hecke operators is commuting (i.e., $T_n \circ T_m = T_m \circ T_n$ for all $m, n \in \mathbb{N}$). Also, each of the operator T_m is a self-adjoint operator on $S_k(SL_2(\mathbb{Z}))$ with respect to the Petersson scalar product.

Definition 1.11. A cusp form $f(\tau) = \sum\limits_{n=1}^{\infty} a_f(n)e(n\tau) \in S_k(SL_2(\mathbb{Z}))$ is said to be a Hecke eigenform if it is a simultaneous eigenfunction for all the Hecke operators. Further, it is said to be normalized if its first Fourier coefficient $a_f(1) = 1$.

1.3 L-function associated to modular forms

In this section, we define various L-functions associated to modular forms and recall some of their properties.

Definition 1.12. (Hecke L-function) Let f be a cusp form of weight k with the Fourier series expansion given by $f(\tau) = \sum_{n=1}^{\infty} a_f(n)e(n\tau)$. The Hecke L-function associated to f is defined by

$$L(f,s) := \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s}. \tag{1.10}$$

The above series converges absolutely for $\Re(s) > \frac{k+1}{2}$. The next theorem gives the analytic properties of $L(f,s)$ for a Hecke eigenform f.

Theorem 1.6. *[31, p.368, Theorem 14.7] Let $f \in S_k(SL_2(\mathbb{Z}))$ be a normalized Hecke eigenform. Then the function $L(f,s)$ can be analytically continued to the whole complex plane and satisfies the functional equation*

$$(2\pi)^{-s}\Gamma(s)L(f,s) = i^k(2\pi)^{-(k-s)}\Gamma(k-s)L(f,k-s). \tag{1.11}$$

Moreover, $L(f,s)$ has the following Euler product:

$$L(f,s) = \prod_{p-\text{prime}} \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\beta_p}{p^s}\right)^{-1}, \tag{1.12}$$

where $\quad \alpha_p + \beta_p = a_f(p) \quad$ *and* $\quad \alpha_p\beta_p = p^{k-1}$.

Next, we define convolution of L-functions associated to cusp forms. Let f and g be cusp forms of weight k with their Fourier series expansion given by

$$f(\tau) = \sum_{n=1}^{\infty} a_f(n)e(n\tau), \qquad g(\tau) = \sum_{n=1}^{\infty} a_g(n)e(n\tau).$$

We define the following convolutions of L-functions associated to f and g :

$$\begin{aligned} L(f \times f, s) &:= \sum_{n=1}^{\infty} \frac{|a_f^2(n)|}{n^s}, && (1.13) \\ L(f \otimes g, s) &:= \zeta(2s) \sum_{n=1}^{\infty} \frac{a_f(n)\overline{a_g(n)}}{n^{s+k-1}}. && (1.14) \end{aligned}$$

Rankin [62] and Selberg [64] studied the analytic properties of the series (1.13) and (1.14), respectively. The precise results are stated below without proof.

Theorem 1.7. *[62, p. 360, Theorem 3] The function $L(f \times f, s)$ has the following properties:*

1. *$L(f \times f, s)$ is absolutely convergent for $\Re(s) > k$.*
2. *$L(f \times f, s)$ can be analytically continued to the whole complex plane with simple poles at $s = k$.*
3. *$L(f \times f, s)$ satisfies the functional equation $R_f^*(s) = R_f^*(2k - 1 - s)$, where*

$$R_f^*(s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s - k + 1)\zeta(2s - 2k + 2)L(f \times f, s).$$

Theorem 1.8. *([13, p. 56], [64, p. 40]) The function $L(f \otimes g, s)$ is entire if $f \neq g$ and can be analytically continued to the whole complex plane except for a simple pole at $s = 1$ if $f = g$. The function $L(f \otimes g, s)$ satisfies the following functional equation*

$$(2\pi^2)^{-s}\Gamma(s)\Gamma(s + k - 1)L(f \otimes g, s) = (2\pi^2)^{s-1}\Gamma(1 - s)\Gamma(k - s)L(f \otimes g, 1 - s).$$

For $f \in S_k(SL_2(\mathbb{Z}))$ with Fourier series expansion $f(\tau) = \sum_{n=1}^{\infty} a_f(n)e(n\tau)$, we normalized the Fourier coefficients of f as follows:

$$\lambda_f(n) := \frac{a_f(n)}{n^{\frac{k-1}{2}}}.$$

From now onwards $\lambda_f(n)$ denotes normalized Fourier coefficents throughout this thesis. The normalized Fourier coefficient $\lambda_f(n)$ of a normalized Hecke eigenform is a multiplicative function and satisfies the following recursive relation [30, eq. 6.83]:

$$\lambda_f(m)\lambda_f(n) = \sum_{d \mid \gcd(m,n)} \lambda_f\left(\frac{mn}{d^2}\right), \tag{1.15}$$

for all positive integers m and n. Now we define the symmetric square L-function associated to f.

Definition 1.13. (Symmetric square L-function) The symmetric square L-function associated to a normalized Hecke eigenform $f \in S_k(SL_2(\mathbb{Z}))$ is defined by

$$\begin{aligned} D_f(s) := L(Sym^2 f, s) &= \sum_{n=1}^{\infty} \frac{\lambda_{sym^2 f}(n)}{n^s} = \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n^s} \\ &= \prod_{p-\text{prime}} (1-\alpha_p^2 p^{-s})^{-1}(1-\alpha_p\beta_p p^{-s})^{-1}(1-\beta_p^2 p^{-s})^{-1}, \end{aligned}$$

where $\alpha_p + \beta_p = \lambda_f(p)$ and $\alpha_p\beta_p = 1$. The above series converges absolutely for $\Re(s) > 1$.

Shimura [65] proved the following relationship between symmetric square L-function and Rankin-Selberg L-function:

$$L(f \times f, s) = \frac{\zeta(s-k+1)}{\zeta(2(s-k+1))} D_f(s).$$

Theorem 1.9. *[65, p. 80, Theorem 1] The completed L-function $R_f(s)$ given by*

$$R_f(s) := \pi^{\frac{-3s}{2}} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s-k+2}{2}\right) D_f(s) \tag{1.16}$$

can be continued to a meromorphic function on the whole s-plane and is holomorphic except for simple poles at $s = k$ and $s = k-1$. Further $R_f(s)$ satisfies the functional equation

$$R_f(s) = R_f(2k - s - 1).$$

1.4 Jacobi forms

For a commutative ring R, we denote by $R^{(n,m)}$ and $M_n(R)$ the set of all $n \times m$ and $n \times n$ matrices with entries in R, respectively. A square matrix $M = (m_{ij}) \in M_n(\mathbb{Z})$ is said to be half-integral if $m_{ii}, 2m_{ij} \in \mathbb{Z}$. We denote by Δ_n (respectively, Δ_n^+) the set of symmetric, positive semi-definite (respectively, positive definite), half-integral $n \times n$ matrices in $M_n(\mathbb{Z})$. For two matrices A and B of appropriate sizes, we define $A[B] := B^t AB$. For $n \geq 1$, we define $J_n := \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$, where 0_n and I_n denote the $n \times n$ zero and identity matrix, respectively. For a positive integer n, the Siegel upper half-plane $\mathbb{H}_n$ of degree n is defined as

$$\mathbb{H}_n := \{Z \in M_n(\mathbb{C}) \mid Z^t = Z,\ \Im(Z) \text{ is positive definite}\}.$$

The symplectic group $Sp_{2n}(\mathbb{R})$ of degree n is defined as

$$Sp_{2n}(\mathbb{R}) := \{M \in M_{2n}(\mathbb{R}) \mid J[M] = J\}.$$

If we write the matrix $M \in Sp_{2n}(\mathbb{R})$ as a block decomposition, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A, B, C, D \in M_n(\mathbb{R})$, then we have

$$\Gamma_n = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| A, B, C, D \in M_n(\mathbb{R}), A^tC = C^tA, B^tD = D^tB, A^tD - B^tC = I_n \right\}.$$

The group $Sp_{2n}(\mathbb{R})$ acts on the Siegel upper half-plane $\mathbb{H}_n$ as follows:

$$M \cdot Z = (AZ + B)(CZ + D)^{-1},\ M \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2n}(\mathbb{R}),\ Z \in \mathbb{H}_n.$$

The Heisenberg group $H_{\mathbb{R}}^{(n,g)}$ is defined as

$$H_{\mathbb{R}}^{(n,g)} := \left\{ [(\lambda, \mu), \kappa] : \lambda, \mu \in \mathbb{R}^{(g,n)}, \kappa \in \mathbb{R}^{(g,g)} \text{ with } \kappa + \lambda^t\mu \text{ symmetric } \right\}$$

where the group operation is defined as

$$[(\lambda, \mu), \kappa][(\lambda', \mu'), \kappa'] := [(\lambda + \lambda', \mu + \mu'), \kappa + \kappa' + \lambda^t\mu' - \mu^t\lambda'].$$

The Jacobi group $G_J^{(n,g)}(\mathbb{R})$ is defined as $G_J^{(n,g)}(\mathbb{R}) := Sp_{2n}(\mathbb{R}) \ltimes H^{(n,g)}$, where the group operation is defined as

$$(M, [(\lambda, \mu), \kappa])(M', [(\lambda', \mu'), \kappa']) := (MM', [(\tilde{\lambda} + \lambda', \tilde{\mu} + \mu'), \kappa + \kappa' + \tilde{\lambda}^t\mu - \tilde{\mu}^t\lambda']),$$

where $M, M' \in Sp_{2n}(\mathbb{R}), [(\lambda, \mu), \kappa], [(\lambda', \mu'), \kappa'] \in H^{(n,g)}$ and $(\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)M'$.

The Jacobi group $G_J^{(n,g)}(\mathbb{R})$ acts on $\mathbb{H}_n \times \mathbb{C}^{(g,n)}$ as follows:

$$(M,[(\lambda,\mu),\kappa])(Z,W) := \left(M\cdot Z,(W+\lambda Z+\mu)(CZ+D)^{-1}\right),$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2n}(\mathbb{R})$.

Let k be a fixed positive integer and $\mathcal{M} \in \Delta_g$. For a complex-valued holomorphic function $\phi : \mathbb{H}_n \times \mathbb{C}^{(g,n)} \longrightarrow \mathbb{C}, M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2n}(\mathbb{R})$ and $X = [(\lambda,\mu),\kappa] \in H^{(n,g)}$, we define

$$\begin{aligned} (\phi|_{k,\mathcal{M}}M)(Z,W) &:= \det(CZ+D)^{-k} e\left(-tr(\mathcal{M}W(CZ+D)^{-1}C^tW)\right)\phi(M\cdot Z, W(CZ+D)^{-1}) \\ (\phi|_{\mathcal{M}}X)(Z,W) &:= e\left(-tr(\mathcal{M}(\lambda Z^t\lambda + 2\lambda^t W + (\kappa+\mu^t\lambda)))\phi(Z,W+\lambda Z+\mu)\right). \end{aligned}$$

Write $\Gamma_n := Sp_{2n}(\mathbb{Z}), H_{\mathbb{Z}}^{(n,g)} := \left\{[(\lambda,\mu),\kappa] \in H_{\mathbb{R}}^{(n,g)} : \lambda,\mu \in \mathbb{Z}^{(g,n)}, \kappa \in \mathbb{Z}^{(g,g)}\right\}$ and $\Gamma_J^{(n,g)}(\mathbb{Z}) := \Gamma_n \ltimes H_{\mathbb{Z}}^{(n,g)}$. We now define Jacobi forms.

Definition 1.14. (Jacobi form) A holomorphic function $\phi : \mathbb{H}_n \times \mathbb{C}^{(g,n)} \to \mathbb{C}$ is said to be a Jacobi form of weight k, index $\mathcal{M}$ for the group $\Gamma_J^{(n,g)}(\mathbb{Z})$ if it satisfies the following:

1. $\phi|_{k,\mathcal{M}}M = \phi$ for every $M \in \Gamma_n$
2. $\phi|_{\mathcal{M}}X = \phi$ for every $X \in H_{\mathbb{Z}}^{(n,g)}$
3. For each $M \in \Gamma_n$, the function $\phi|_{k,\mathcal{M}}$ has a Fourier expansion of the following form

   $$\phi|_{k,\mathcal{M}}(Z,W) = \sum_{T\in\Delta_n}\sum_{R\in\mathbb{Z}^{(n,g)}} c_\phi(T,R)e(tr\left(\frac{1}{\alpha}(TZ)+RW\right)$$

   for some suitable $\alpha \in \mathbb{Z}$.

Analogous to the case of modular forms, one can define an inner product (called Petersson scalar product) on the space of Jacobi cusp forms as follows. Let ϕ and ψ be Jacobi cusp forms of weight k, index $\mathcal{M}$ for the group $\Gamma_J^{(n,g)}(\mathbb{Z})$. Then the Petersson scalar product $\langle \phi, \psi \rangle$ of ϕ and ψ is defined by

$$\langle \phi, \psi \rangle := \int_{\Gamma_J^{(n,g)}(\mathbb{Z})\times\mathbb{C}^{(g,n}} \phi(Z,W)\overline{\psi(Z,W)}\,\det(\Im(Z))^k \exp\left(\frac{-4\pi tr(M[\Im(W)])}{\Im(Z)}\right) dV,$$

where dV is an invariant measure under the action of $\Gamma_J^{(n,g)}(\mathbb{Z})$ on $\mathbb{C}^{(g,n)}$.

For more details on the theory of Jacobi forms, we refer to [22, 74].

1.5 Siegel modular forms

In this section we define Siegel modular forms of degree n, where n is a positive integer.

Let k be a fixed positive integer and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$. For a complex-valued holomorphic function $F : \mathbb{H}_n \longrightarrow \mathbb{C}$, the slash operator of weight k is defined by

$$(F|_k M)(Z) := \det(CZ+D)^{-k} F(M \cdot Z) \quad \text{for} \quad Z \in \mathbb{H}_n. \tag{1.17}$$

Definition 1.15. (Siegel modular form) Let k and n be positive integers. A complex-valued holomorphic function $F : \mathbb{H}_n \to \mathbb{C}$ is said to be a Siegel modular form of weight k and degree n if it satisfies $F|_k M = F$ for all $M \in \mathbb{H}_n$. If $n = 1$, then $F|_k M$ for every $M \in \Gamma_1$ is bounded on each subset $\{u + iv \in \mathbb{C} : v > 0, v > \epsilon\}$ with $\epsilon > 0$.

A Siegel modular form F admits a Fourier series expansion of the form

$$F(Z) = \sum_{T \in \Delta_n} A_F(T) e(tr(TZ)). \tag{1.18}$$

Further, we say F is a Siegel cusp form if in (1.18) the summation runs over $T \in \Delta_n^+$.

The spaces of Siegel modular forms and Siegel cusp forms of weight k and degree n are denoted by $M_k(\Gamma_n)$ and $S_k(\Gamma_n)$, respectively. Analogous to the case of modular forms and Jacobi forms, we define an inner product (called Petersson scalar product) on the space $S_k(\Gamma_n)$ as follows. Let F and G be Siegel cusp forms of weight k and degree n. Then the Petersson scalar product of $\langle F, G \rangle$ of F and G is defined by

$$\langle F, G \rangle := \int\limits_{\Gamma_n \backslash \mathbb{H}_n} F(Z)\overline{G(Z)}(det\ Y)^k dZ,$$

where $Z = X + iY$ and $dZ = (\det\ Y)^{-(2n-1)} dXdY$ is an invariant measure under the action of Γ_n on $\mathbb{H}_n$.

1.6 Fourier-Jacobi expansion

In this section, we describe the Fourier-Jacobi expansion of a Siegel modular form.

Let $Z \in \mathbb{H}_n$ and $T \in \Delta_n$. Write the following block decomposition of the matrices Z and T :

$$Z = \begin{pmatrix} Z_1 & w \\ w^t & z \end{pmatrix}, \qquad T = \begin{pmatrix} T_1 & \frac{1}{2}\lambda \\ \frac{1}{2}\lambda^t & m \end{pmatrix}, \tag{1.19}$$

where Z_1 and T_1 denote the upper left $(n-1) \times (n-1)$ corners of the matrix Z and T, respectively. Let $F \in M_k(\Gamma_n)$ with the Fourier series expansion as in (1.18).

Rewrite the Fourier series expansion of F as follows:

$$\begin{aligned} F(Z) &= \sum_{T\in\Delta_n} A_F(T)e(tr(TZ)) \\ &= \sum_{T=\begin{pmatrix} T_1 & \frac{1}{2}\lambda \\ \frac{1}{2}\lambda^t & m \end{pmatrix}\in\Delta_n} A_F\left(\begin{pmatrix} T_1 & \frac{1}{2}\lambda \\ \frac{1}{2}\lambda^t & m \end{pmatrix}\right) e(tr(T_1Z_1)+\lambda^t w+mz) \\ &= \sum_{m=0}^{\infty}\phi_m(Z_1,w)e(mz), \end{aligned}$$

where for each m, the function $\phi_m(Z_1,w)$ is given by

$$\phi_m(Z_1,w) := \sum_{T_1\geq 0,\lambda\in\mathbb{Z}^{n-1}} A_F\begin{pmatrix} T_1 & \frac{1}{2}\lambda \\ \frac{1}{2}\lambda^t & m \end{pmatrix} e(tr(T_1Z_1)+\lambda^t w).$$

It is called the m-th Fourier-Jacobi coefficient of F. It is well-known that for each m, the function ϕ_m is a Jacobi form of weight k, index m for the group $G_J^{(n-1,n-1)}(\mathbb{Z})$. Further, if $F\in S_k(\Gamma_n)$, then its Fourier-Jacobi expansion is given by $F(Z)=\sum\limits_{m=1}^{\infty}\phi_m(Z_1,w)e(mz)$ and each of the Fourier-Jacobi coefficients is a Jacobi cusp form.

Next lemma gives the bound satisfied by the Petersson scalar product of Fourier-Jacobi coefficients of Siegel cusp forms.

Lemma 1.6. *([46, p. 246],[40, p. 544]) Let $F,G\in S_k(\Gamma_n)$ with their Fourier-Jacobi expansions as follows:*

$$F(Z)=\sum_{m=1}^{\infty}\phi_m(Z_1,w)e(mz), \qquad G(Z)=\sum_{m=1}^{\infty}\psi_m(Z_1,w)e(mz). \tag{1.20}$$

Then

$$\langle\phi_m,\psi_m\rangle = O_{F,G}(m^k).$$

Further, Kohnen [41, p. 134], [44, p. 718] made the following Ramanujan-Petersson conjecture: for any $\epsilon > 0$,

$$\langle \phi_m, \phi_m \rangle = O_{F,\epsilon}\left(m^{k-1+\epsilon}\right). \tag{1.21}$$

Kohnen and Sengupta [45] proved the above conjecture for Hecke eigenform $F \in S_k(\Gamma_2)$ which is a Saito-Kurokawa lift. Recently, Kumar and Paul [47] proved that the above conjecture is true on average for arbitrary degree. The conjecture given in (1.21) is equivalent to

$$\langle \phi_m, \psi_m \rangle = O_{F,\epsilon}\left(m^{k-1+\epsilon}\right). \tag{1.22}$$

For more details on the theory of Siegel modular forms we refer to [3, 12, 38, 52].

1.7 Conclusion

In this chapter, we have presented the essential preliminaries and relevant literature associated with the research conducted. This includes definitions of various mathematical concepts such as arithmetical function, Lambert series, Gamma function, Riemann zeta function, Meijer G-function, Whittaker function, hypergeometric function, modular forms, Jacobi forms, and Siegel modular forms. Further, we have discussed L-function associated to Fourier coefficients of these modular forms and their analytic properties.

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