PREFACE

In a series of articles dating from 1836-37, Sturm and Liouville developed a new theory known as "Sturm-Liouville (SL) Theory". It is observed that SL theory has many applications in the real world. This SL theory deals with linear second-order differential equations subjected to suitable boundary conditions. We all have heard an instrument, such as a guitar. The guitar's sound comes from plucking a string. The motion of the vibrating strings can be expressed by a wave equation which is a typical problem that can be solved using SL theory. SL theory can be used to describe not just the sound of a guitar, but also various crucial physical processes and mechanical systems from classical and quantum physics. Besides this, SL theory has shown wide applications in mathematics as well as in physics. Recent advances in science, engineering, economics, bioengineering, and applied mathematics have demonstrated that fractional derivatives provide a more precise model for various natural phenomena. In the last few decades, Sturm-Liouville problems (SLPs) have been defined using fractional derivatives. Since then researchers have shown great interest in generalizing the theory of classical Sturm-Liouville problems to fractional Sturm-Liouville problems (FSLPs). Replacing ordinary derivatives with fractional derivatives in a classical SLP leads to the formation of a class of FSLPs.

The main objectives of this thesis are to develop the numerical algorithms to solve FSLPs, generalized fractional Sturm-Liouville problems (GFSLPs), and tempered fractional Sturm-Liouville problems (TFSLPs).

The first chapter of this thesis describes the SLP and the historical background of the problem. This chapter includes definitions and some basic properties of fractional integrals and derivatives such as R-L, fractional integrals/derivatives. Definition of generalized fractional integrals/derivatives, tempered fractional integrals/derivatives, and their properties, utilized in this thesis, are given in the further sections of the chapter. Moreover, the model problem and literature review of the problem are discussed.

In subsequent chapters, different forms of GFSLPs are defined, and numerical algorithms to solve the defined GFSLPs are proposed.

Chapter 2 describes a numerical scheme for the GFSLP with mixed boundary conditions. We prove the well-posedness of the proposed GFSLP. Further, the approximated eigenvalues of GFSLP are obtained. The obtained eigenvalues are real and corresponding eigenfunctions are orthogonal. The theoretical and numerical convergence orders of eigenvalues and eigenvectors are also discussed. Further, the numerically obtained eigenvalues and eigenfunctions are used to construct an approximate solution of the one-dimensional fractional diffusion equation defined in a bounded domain.

Further, in chapter 3, we explore to develop a numerical algorithm for solving the generalized fractional Sturm-Liouville differential equation. We define the GFSLP in terms of Caputo generalized fractional derivatives. Firstly, the GFSLP is transformed into an integral form using a weighted Laplace transform. Next, we consider the GFSLP under different cases of boundary conditions. Further, we approximate the generalized fractional integral using Lagrange interpolating polynomial and form an eigenvalue problem for different types of boundary conditions. The bound for the truncation error in the approximation is proved. Finally, the test cases of GFSLPs are considered for numerical simulations of the proposed numerical scheme and examine the theoretical claims.

Chapter 4 presents a numerical algorithm for the FSLP defined in terms of Caputo generalized fractional derivative. Firstly, the well-posedness of the considered FSLP is discussed. Further, we have divided the provided interval into non-uniform node points and varied the weight and scale functions to compute the estimated eigenvalues and corresponding eigenfunctions. Eigenvalues thus obtained are real and eigenfunctions are orthogonal. Furthermore, we have found the bound for the solution and discussed the convergence order numerically. After this, an approximate solution of the fractional diffusion equation with generalized fractional derivatives is formed using approximated eigenvalues and eigenfunctions.

In chapter 5, a TFSLP with mixed boundary conditions is considered. Firstly, the TFSLP is transformed into its corresponding integral form. Next, to present the numerical solution of integral form, we use linear interpolation to determine the approximation of the tempered integral operator. The discretized version of the integral form is presented for two cases: $\alpha \in (0, 1]$ and $\alpha \in (1, 2]$. In the final part of this chapter, some examples of numerical solutions of these equations are shown. Also, we discuss the convergence order of numerical schemes.

Finally, the thesis is concluded in chapter 6, and remarks on future work are given.