# PREFACE

*Optimization* is a process we engage in our daily lives to accomplish tasks or achieve goals, often without even realizing it. To formalize this process, mathematics defines optimization as the method of identifying the best choice or choices (decision variables) from a set of all possible options (feasible region) with the aim of achieving (minimizing/maximizing) one or multiple objectives.

The early works that laid the foundation of optimization theory are linear programming, nonlinear programming, integer programming, dynamic programming, and convex optimization, etc. However, it soon became apparent that the conventional techniques of these topics were insufficient for modeling real-life optimization problems, which inherently contain uncertainty in their parameters, constraints, and objective functions. This uncertainty often arises from noise, measurement errors, or the stochastic and vague nature of the problems themselves. Consequently, this realization has spurred the development of modern optimization theory, which now includes areas such as interval optimization, stochastic optimization, robust optimization, fuzzy optimization, multi-objective optimization and set optimization. Among these, the two promising fields are interval optimization and set optimization. Interval optimization addresses uncertainty in problems by using intervals, whereas set optimization by set-valued functions. This thesis aims to advance these two subfields of optimization through both theoretical and methodical (algorithmic) contributions.

In an optimization problem, a crucial aspect to consider is the nature of the objective function, which can be smooth or nonsmooth, and convex or nonconvex. Smooth functions are comparatively easier to manage, as the gradient of the function at an optimal point is zero. However, for nonsmooth functions, the derivative at the optimal point $x^*$ may not exist, rendering the conventional optimality condition $\nabla f(x^*) = 0$ inapplicable. In such scenarios, concepts such as the normal cone, tangent cone, subdifferential, and generalized subdifferential play crucial roles. The tangent cone aids in identifying feasible directions essential for deriving optimality conditions such as the Karush-Kuhn-Tucker (KKT) conditions. The normal cone characterizes constraints relevant to these optimality conditions. The subdifferential replaces the gradient in handling nonsmooth but convex functions. Recognizing the limitations of the subdifferential for nonsmooth and nonconvex functions, various generalizations such as the weak subdifferential [41], Dini-Hadamard subdifferential [17], and Clarke subdifferential [45], etc. are employed.

Naturally, in the context of interval optimization, interval-valued objective functions can exhibit smoothness or nonsmoothness, as well as convexity or non-convexity. Consequently, in order to establish KKT-type optimality conditions and to substitute gradients, it becomes necessary to define and consolidate the concepts of normal and tangent cones, and generalized subdifferentials, respectively, for interval optimization. However, the current literature lacks these crucial concepts essential for advancing the field. Therefore, this thesis aims to address this significant gap by (1) developing the theory of normal and tangent cones for sets of intervals in the case of nonsmooth convex interval-valued functions, and (2) defining weak-subdifferentials, Dini-Hadamard subdifferentials, and Clarke-subdifferentials for nonsmooth non-convex interval-valued functions.

In addition to advancing theory in any practical field, it is crucial to ensure that numerical algorithmic developments keep pace with theoretical progress. Therefore, following our theoretical contributions to interval optimization, the natural progression involves the development of algorithms specifically tailored for this field. However, such algorithms would exclusively apply to interval objects, defined as connected and ordered subsets of real numbers, and cannot be generalized to other mathematical entities (e.g., vectors, set of scalars, set of vectors, set of intervals, etc.) Consequently, this thesis shifts focus beyond interval optimization to the development of numerical algorithms within the broader and more general framework of set optimization, which encompasses interval optimization as a specialized instance.

Under the umbrella of numerical and algorithmic approaches to solve set optimization problem, recent years have witnessed an increase in research efforts. These ideas, largely drawing from the fields of single and multi-objective optimization, have proven highly effective in the context of set optimization. One notable class of optimization algorithms is the trust-region method. Initially introduced and developed in the 1970s, trust-region methods have emerged as fundamental tools in numerical optimization due to their robustness, superior convergence properties, and efficacy in addressing complex nonlinear problems—particularly those that pose challenges for traditional line search methods. Given the empirical success observed in single and multi-objective optimization, it is reasonable to expect that trust-region methods will similarly demonstrate superior performance in the context of set optimization. Because of the nascent stage of set optimization research itself, trust-region methods have not yet been developed and studied. Therefore, a major contribution of this thesis is the introduction of a trust-region algorithm tailored specifically for set optimization, validated through extensive numerical experiments to benchmark its performance. Additionally, as further contributions, we improve the performance of our trust-region method by proposing two novel non-monotonic modifications: the max-type and average-type non-monotone trust-region methods for set optimization.

We hope that the contributions presented in this thesis will incrementally advance the fields of interval and set optimization. Moreover, we aim for these contributions to serve as both a motivation and a foundational resource for future research endeavors.