# Chapter 1

# Introduction, Preliminaries and Plan of the Thesis

## 1.1 Introduction

Zadeh[122] introduced fuzzy sets to deal with classes of objects which do not have precisely defined criteria of membership. Mathematically, a fuzzy set in a set $X$ is a function from $X$ to the unit interval [0,1]. The theory of fuzzy sets had been extensively applied to widespread disciplines. Several books and monographs are available to provide good coverage of the applications of fuzzy set theory e.g., Dubois and Prade[35], Zimmermann[124], Klir and Yuan[61], Liu and Luo[68], Ross[90], Hanss[47], Smithson and Verkuilen[103], Rodabaugh and Klement[89](eds.), etc.

Chang[24] introduced a fuzzy topology on a set $X$, by replacing 'subsets' by 'fuzzy sets' in the usual definition of a topology on $X$. He also introduced the basic concepts in fuzzy topology, e.g., fuzzy continuity, fuzzy compactness, etc., as obvious generalizations of the corresponding concepts in topology. Subsequently, Goguen[44] generalized this concept and introduced $L-$fuzzy topology using $L-$fuzzy sets, where $L$ is an arbitrary bounded lattice. Lowen[69] observed that in Chang's fuzzy topological spaces, some basic desirable properties are not satisfied e.g., a constant map between two such spaces is not necessarily continuous. So he modified Chang's definition. He defined a fuzzy topology on a set $X$ as a collection of fuzzy sets in $X$ which contains all constant fuzzy sets and is closed under arbitrary suprema and finite infima. Höhle[49] introduced 'fuzzifying topology'. Šostak[104] defined $I-$ fuzzy topology on a set $X$. There are other approaches e.g., due to Hutton[51], Rodabaugh[88], Wang[118], etc.

The theory of fuzzy sets had progressed rapidly but there is difficulty regarding assignment of membership function characterizing a fuzzy set, it extremely depends upon the individual. Molodtsov[78] observed that the possible reason for this difficulty was the inadequacy of the parametric tool in the theory of fuzzy sets. Keeping this in mind, he introduced soft sets. He defined a soft set over $X$ as a mapping from a set of parameters $E$ to the set of all subsets of $X$. He applied the theory of soft sets to stability and regularization, game theory, operations research, Riemann integration, Perron integration, etc. Maji et al.[74] presented an application of soft sets in decision making problems. Chen[25] proposed a reasonable definition of soft set parametrization reduction and compared it with attributes reduction in rough set theory. Pie and Mio[85] showed that soft sets are a class of special information systems. Zou and Xiao[125] presented a data analysis approach of soft sets under incomplete information. Kharal and Ahmad[60] defined 'soft image' and 'inverse image' of a soft set and they applied these notions to the problems of medical diagnosis in medical systems.

Algebraic structure of soft sets has been studied by many researchers, e.g., [3, 4, 39, 57, 58, 80, 81, 99], etc.

Topological structure of soft sets has been studied by several authors. Shabir and Naz[100] introduced soft topological spaces by replacing 'sets' by 'soft sets' in the usual definition of topology. They gave the notions of basic concepts e.g., soft open, soft closed, soft interior, soft closure, soft subspace, soft neighborhood of a point, soft separation axioms $T_i$, $i = 0, 1, 2, 3, 4$, soft regular and soft normal spaces etc. and studied their basic properties. Soft Hausdorff topological spaces were further studied by Varol and Aygün[117]. They also introduced the concepts of convergence of a sequence and homeomorphism in soft topological spaces. They have investigated the relation between convergence of a sequence and Hausdorffness in soft topological spaces. Hussain[50] studied soft connected spaces. Aygünoğlu and Aygün[9] introduced enriched soft topology and defined soft continuity, soft product topology, soft compactness, proved Alexander's subbase theorem and Tychonoff theorem in these spaces. They observed that constant soft mappings between two non-enriched soft topological spaces are not generally soft continuous, however they are soft continuous between two enriched soft topological spaces. Several other applications of soft sets have been presented by many researchers(cf.[15, 16, 45, 48, 56, 64], etc.).

Maji et al.[73] introduced the notion of a fuzzy soft set. Since then, many researchers have been working in constructing theoretical and applicative background of fuzzy soft sets. Some related works are as follows:

1. Roy and Maji[92] presented a novel fuzzy soft set theoretic approach to decision making problems.

2. Kharal and Ahmad[59] introduced the concept of a mapping on the classes of fuzzy soft sets and studied properties of fuzzy soft images and fuzzy soft inverse images of fuzzy soft sets. They[2] studied some more properties of fuzzy soft sets, which were introduced and studied by Maji et al.[73], Roy et al.[92] and Yang et al.[120] and defined arbitrary fuzzy soft union and fuzzy soft intersection. Further, they proved De Morgan's laws in fuzzy soft set theory.

3. Aygünoğlu and Aygün[8] introduced the concept of fuzzy soft groups, using a t-norm.

4. Cağman et al.[17] proposed fuzzy soft aggregation operator and used it in constructing an efficient decision making method.

5. Kong et al.[63] proposed a new algorithm based on grey relational analysis and applied it to decision making problems.

Topological structure of fuzzy soft sets was introduced by Tanay and Kandemir[113]. It was further studied by Varol and Aygün[115], Çetkin and Aygün[20], Roy and Samanta[93], etc. Varol et al.[115] presented the notion of fuzzy soft topological spaces in both Chang's sense(which is similar to Tanay and Kandemir's definition) and Lowen's sense and showed that a fuzzy soft topological space gives rise to a parametrized family of fuzzy topological spaces. In [115], a fuzzy soft topology in Lowen's sense is called an enriched fuzzy soft topology. They had also introduced fuzzy soft continuity of fuzzy soft mappings and shown that a constant mapping between fuzzy soft topological spaces is not fuzzy soft continuous, in general. However, a constant mapping between enriched fuzzy soft topological spaces is always fuzzy soft continuous. Further, they had introduced and studied the notions of fuzzy soft closure operator, fuzzy soft interior operator and initial fuzzy soft topology. Mahanta and Das[72] studied fuzzy soft topological spaces, which was introduced by Tanay et al.[113]. They have also introduced and studied the notions of a fuzzy soft point, fuzzy soft closure, fuzzy soft interior, separation axioms and connectedness in fuzzy soft topological spaces. Varol et al.[116] defined soft topology using a new approach. They defined soft topology over a set $X$ as a soft set over $2^{2^X}$. They also introduced $L-$soft topology, $L-$fuzzifying soft topology and $L-$fuzzy soft topology on a set $X$ with respect to a parameter set $E$ and studied soft compactness and $L-$fuzzy soft compactness in soft topological spaces and $L-$fuzzy soft topological spaces respectively. Several other researchers have been working in this area(cf. [43, 83, 105], etc.).

Binary relations are an important tool to express preferences, but they are not sufficient to deal with real life preferences. To tackle those cases, $L-$relations were introduced and studied by Salii[96]. Fuzzy relations are special cases of $L-$relations, when $L$=[0,1]. They have been studied in detail by several authors, e.g., [21–23, 40, 55, 65], etc., and used in different areas, such as linguistic[26], decision making[66], clustering[12], etc. It is a well known fact by now that topological structure on a set is closely related to an ordered structure on that set. Some of the well known results are as follows:

As mentioned in [38]:

1. Given a preordered set $(X, \leq)$, the family of all upper sets in $X$ forms a topology on $X$, called the Alexandrov topology induced by $(X, \leq)$.
2. On the other hand, given a topological space $(X, \tau)$, the relation $\leq$ defined by the following, is a preorder on $X$:

$$x \leq y \text{ if } x \in cl\{y\},$$

where $cl\{y\}$ denotes the closure of the set $\{y\}$ in $\tau$.

In literature, similar studies in fuzzy context have been done by several authors. Some of these are as follows:

1. In [54], the authors have introduced fuzzy and metric topologies induced by $T$-indistinguisibility operators(or $T$-fuzzy equivalence relations) and studied their relationship.

2. In [87], the authors have established a one to one correspondence between the family of all fuzzy preorders on $X$ and the family of all fuzzy topologies on $X$ which satisfy the (TC) axiom.

3. In [121], the authors have established a one to one correspondence between the family of all fuzzy preorders on $X$ and the family of all fuzzy topologies $\tau$ on $X$, in the sense of Lowen[69] such that $\tau$ is closed under arbitrary intersection and whose associated closure operators(say, $c$) satisfy the condition: $c(\alpha \wedge \lambda) = \alpha \wedge c(\lambda)$, $\forall \lambda \in I^X, \alpha \in I$, where $I^X$ is the family of all fuzzy sets in $X$.

4. In [67], the authors have shown that there is a one to one correspondence between the family of all fuzzy preorders on a set $X$ and the family of all fuzzy (*)-Alexandrov spaces(here a fuzzy (*)-Alexandrov space means a fuzzy topological space in the sense of Lowen[69] in which the fuzzy topology is closed under arbitrary intersection and satisfies the additional conditions: $(i)\alpha * \lambda \in \tau$ and $(ii)\alpha \to \lambda \in \tau$, $\forall \alpha \in I$, $\lambda \in \tau$, where '$\to$' denotes the implication operator with respect to the left continuous t-norm *).

5. In [38], the authors have discussed how fuzzifying topologies are induced from fuzzy preorder relations on a set $X$ and conversely how a specialization order can be induced by a fuzzifying topology on a set $X$. In [37], the authors have established a one to one correspondence between the family of all fuzzifying topologies and that of fuzzy preorders on a set $X$.

6. In [114], the authors have considered fuzzy topologies given by Lai and Zhang[67], Qin and Pei[87] and Yeung et al.[121] and shown that they are either dual or same and hence the corresponding results in these papers are essentially equivalent.

The study of topologies induced by binary relations was initiated by Smithson[102]. Since then, many researchers have been working in this area (cf. [6, 75, 95, 111, 112]). Topologies induced by different kinds of fuzzy relations have also been studied in literature(cf. [18, 27], etc.).

Guttman[46] had given the idea of representability of a binary relation $\mathcal{R}$ between two non empty sets $A$ and $X$, by proposing two functions $f : A \to \mathbb{R}$ and $g : X \to \mathbb{R}$ such that

$$a\mathcal{R}x \Leftrightarrow f(a) > g(x)$$

for each $a \in A$ and $x \in X$.

It was further studied by Ducamp and Falmagne[36], Doigon et al.[34] etc. Representations of different types of ordering have also been studied by several authors(see e.g., [19, 82, 86]).

In fuzzy set theory, the representability of a fuzzy total preorder additive fuzzy preference structure without incomparability and a compatible fuzzy semiorder, in terms of the $\alpha-$cuts of their corresponding fuzzy weak preference relation have been respectively studied by Agud et al.[1] and Induráin et al.[52].

Fuzzy weak orders with respect to a left continuous t-norm $T$ and their representability by means of the residual implication operator associated with $T$ (called $T-$representable fuzzy weak orders) have been studied by Baets et al.[11] and Sali et al.[98]. Baets et al.[11] have obtained characterizations for a $T_M-$representable (also called Gödel representable) fuzzy weak orders and for the fuzzy relation which can be written as the union or intersection of a finite family of $T_M-$representable(or Gödel representable) fuzzy weak orders. Sali et al.[98] have obtained characterizations for a $T_P-$representable fuzzy weak order and for finite intersections of fuzzy weak orders with respect to any left continuous t-norm $T$.

In the present thesis, chapters two and three are on separation axioms of fuzzy soft topological spaces. In chapter four, we have introduced and studied the notion of compactness in fuzzy soft topological spaces. In chapters five and six, we have introduced and studied fuzzy topologies generated by fuzzy relations. Chapter seven is devoted to a study of representability of fuzzy biorders and fuzzy weak orders.

## 1.2 Preliminaries

In this section, we mention the definitions, notations and basic results which will be used throughout the thesis. Here we use symbols $A$, $X$ and $Y$ to denote non empty sets.

Mathematically, a *fuzzy set* $f$ in $X$ is a function $f : X \to [0, 1]$, where $f(x)$ gives the degree of membership of $x$ in $f$(cf.[122]). Standard fuzzy set operations, which are obtained as a natural extension of the corresponding operations in set theory, are as follows:

Let $f$ and $g$ be fuzzy sets in $X$. Then

1. $f$ and $g$ are said to be *equal* if $f(x) = g(x)$, for each $x \in X$.
2. $f$ is said to be *subset* of $g$ if $f(x) \leqslant g(x)$, for each $x \in X$.
3. *Union* of $f$ and $g$ is the fuzzy set in $X$, given by $(f \cup g)(x) = \max\{f(x), g(x)\}$, for each $x \in X$.
4. *Intersection* of $f$ and $g$ is the fuzzy set in $X$, given by $(f \cap g)(x) = \min\{f(x), g(x)\}$, for each $x \in X$.
5. *Complement* of $f$ is the fuzzy set in $X$, given by $f^c(x) = 1 - f(x)$, for each $x \in X$.

A *constant fuzzy set* $f$ in $X$, taking value $\alpha \in [0, 1]$, is given by $f(x) = \alpha, \forall x \in X$ and denoted by $\alpha_X$. The fuzzy sets $0_X$ and $1_X$ are usually denoted by $\phi$ and $X$, respectively.

The definitions (3) and (4) of the union and intersection of fuzzy sets respectively can be extended for an arbitrary family of fuzzy sets(cf. [76]). For an arbitrary family $\{f_i : i \in \Omega\}$ of fuzzy sets in $X$, the *union* $\bigcup_{i \in \Omega} f_i$ and the *intersection* $\bigcap_{i \in \Omega} f_i$, both are fuzzy sets in $X$, defined as follows:

1. $(\bigcup_{i \in \Omega} f_i)(x) = \sup\{f_i(x) : i \in \Omega\}$, for each $x \in X$.
2. $(\bigcap_{i \in \Omega} f_i)(x) = \inf\{f_i(x) : i \in \Omega\}$, for each $x \in X$.

*Support* of a fuzzy set $f$ in $X$, denoted by supp$f$, is the crisp subset of $X$, given by(cf. [76]):

$$\text{supp} f = \{x \in X : f(x) > 0\}.$$

We use the following definition of fuzzy points, given by Srivastava et al.[106].

**Definition 1.1.** A *fuzzy point* $x_\lambda$ $(0 < \lambda < 1)$ in $X$ is a fuzzy set in $X$ given by

$$x_\lambda(x') = \begin{cases} \lambda, & \text{if } x' = x \\ 0, & \text{otherwise.} \end{cases}$$

Here $x$ and $\lambda$ are respectively called the support and value of $x_\lambda$.

The intersection of two fuzzy sets $f$ and $g$ is in general, defined in terms of a binary operation $i : [0,1] \times [0,1] \to [0,1]$ as follows:

$$(f \cap g)(x) = i(f(x), g(x)),\ \forall x \in X, \tag{1.1}$$

which gives the degree of membership of $x$ in the intersection $f \cap g$, in terms of degrees of membership of $x$ in $f$ and $g$ both(cf.[61]). It is required that this binary operation $i$ must possess the properties of t-norms, given in the following definition.

**Definition 1.2.** [61] A *triangular norm* or a *t-norm* is a mapping $T : [0,1] \times [0,1] \to [0,1]$ such that the following conditions are satisfied:

1. commutativity: $T(x, y) = T(y, x)$, for each $x, y \in [0, 1]$;
2. monotonicity: $y \leq z$ implies that $T(x, y) \leq T(x, z)$, for each $x, y, z \in [0, 1]$;
3. associativity: $T(T(x, y), z) = T(x, T(y, z))$, for each $x, y, z \in [0, 1]$;
4. boundary condition: $T(x, 1) = x$, for each $x \in [0, 1]$.

Some examples of t-norms are as follows:

1. Nilpotent minimum $T_{nM}$:

$$T_{nM}(x,y) = \begin{cases} \min\{x, y\}, & \text{if } x + y > 1 \\ 0, & \text{otherwise,} \end{cases}$$

   for each $(x, y) \in [0, 1] \times [0, 1]$.

2. Standard intersection $T_M$:

$$T_M(x, y) = \min\{x, y\}, \text{ for each } (x, y) \in [0, 1] \times [0, 1].$$

3. Algebraic product $T_P$:

$$T_P(x, y) = xy, \text{ for each } (x, y) \in [0, 1] \times [0, 1].$$

4. Lukasiewicz $T_L$:

$$T_L(x, y) = \max\{0, x + y - 1\}, \text{for each } (x, y) \in [0, 1] \times [0, 1].$$

Note that the definition (4) of fuzzy intersection can be obtained by replacing $i$ by $T_M$ in 1.1.

A value $x \in (0,1)$ is said to be a *zero divisor* of a t-norm $T$ if there exists $y \in (0,1)$ such that $T(x,y) = 0$. In that case $T$ is said to admit or to have a zero divisor. If for $T$, there is no $x \in (0,1)$ such that it is a zero divisor of $T$, then $T$ is said to be a *t-norm without zero divisors*(cf. [32]).

Similarly, as in case of fuzzy intersections, the union of two fuzzy sets $f$ and $g$ is in general, defined in terms of a binary operation $u : [0,1] \times [0,1] \to [0,1]$ as follows:

$$(f \cup g)(x) = u(f(x), g(x)),\ \forall x \in X, \tag{1.2}$$

which gives the degree of membership of $x$ in the union $f \cup g$, in terms of degrees of membership of $x$ in $f$ and $g$ both(cf.[61]). It is required that this binary operation $u$ must possess the properties of t-conorms, given in the following definition.

**Definition 1.3.** [61] A *triangular conorm* or a *t-conorm* is a mapping $S : [0,1] \times [0,1] \to [0,1]$ such that the following conditions are satisfied:

1. commutativity: $S(x,y) = S(y,x)$, for each $x, y \in [0,1]$;
2. monotonicity: $y \leq z$ implies that $S(x,y) \leq S(x,z)$, for each $x, y, z \in [0,1]$;
3. associativity: $S(S(x,y),z) = S(x,S(y,z))$, for each $x, y, z \in [0,1]$;
4. boundary condition: $S(x,0) = x$, for each $x \in [0,1]$.

Some examples of t-conorms are as follows:

1. Nilpotent maximum $S_{nM}$:

$$S_{nM}(x,y) = \begin{cases} \max\{x,y\}, & \text{if } x + y < 1 \\ 1, & \text{otherwise,} \end{cases}$$

   for each $(x,y) \in [0,1] \times [0,1]$.

2. Maximum $S_M$:

$$S_M(x,y) = \max\{x,y\}, \text{ for each } (x,y) \in [0,1] \times [0,1].$$

3. Algebraic sum $S_P$:

$$S_P(x,y) = x + y - xy, \text{ for each } (x,y) \in [0,1] \times [0,1].$$

4. Bounded sum $S_L$:

$$S_L(x,y) = \min\{x + y, 1\}, \text{ for each } (x,y) \in [0,1] \times [0,1].$$

Note that the definition (3) of fuzzy union can be obtained by replacing $u$ by $S_M$ in 1.2. Further, for any t-conorm $S$, $S(a,b) \geq S_M(a,b)$, for each $(a,b) \in [0,1] \times [0,1]$.

Next, t-norms and t-conorms are closely related. For any t-norm $T$, the mapping $S : [0,1] \times [0,1] \to [0,1]$ defined as $S(x,y) = 1 - T(1-x, 1-y)$, for each $(x,y) \in [0,1] \times [0,1]$ is a t-conorm. It is called dual t-conorm of $T$. Conversely, corresponding to a t-conorm $S$, the mapping defined as $T(x,y) = 1 - S(1-x, 1-y)$, for each $(x,y) \in [0,1] \times [0,1]$ is a t-norm.

Now we recall the definitions of a binary relation and some related concepts.

**Definition 1.4.** [34] A *binary relation* $\mathcal{R}$ between $A$ and $X$ is a subset of $A \times X$, i.e., $\mathcal{R} \subseteq A \times X$. If $(a,b) \in \mathcal{R}$, then it is customary to write this as $a\mathcal{R}b$. The *transpose* $\mathcal{R}^t$, *complement* $\mathcal{R}^c$ and *dual* $\mathcal{R}^d$ of $\mathcal{R}$ are binary relations between $X$ and $A$, $A$ and $X$, $X$ and $A$ respectively and are defined as follows:

1. $\mathcal{R}^t = \{(x,a) : (a,x) \in \mathcal{R}\}$.
2. $\mathcal{R}^c = \{(a,x) : (a,x) \notin \mathcal{R}\}$.
3. $\mathcal{R}^d = \{(x,a) : (a,x) \notin \mathcal{R}\}$.

If we take $A = X$ i.e., $\mathcal{R} \subseteq A \times A$, then $\mathcal{R}$ is said to be a binary relation on $A$.

Next, we mention the definitions of a fuzzy relation and of its different types.

**Definition 1.5.** [122] A *fuzzy relation* $\mathcal{R}$ between $A$ and $X$ is a fuzzy subset of $A \times X$ i.e. $\mathcal{R}$ is a mapping from $A \times X$ to $[0,1]$. The *transpose* $\mathcal{R}^t$, *complement* $\mathcal{R}^c$ and *dual* $\mathcal{R}^d$ of $\mathcal{R}$ are respectively the fuzzy relations between $X$ and $A$, $A$ and $X$ and $X$ and $A$(cf.[61],[123]) and are defined as follows:

1. $\mathcal{R}^t(x,a) = \mathcal{R}(a,x)$, for each $(x,a) \in X \times A$;

2. $\mathcal{R}^c(a,x) = 1 - \mathcal{R}(a,x)$, for each $(a,x) \in A \times X$;

3. $\mathcal{R}^d(x,a) = 1 - \mathcal{R}(a,x)$, for each $(x,a) \in X \times A$.

If $X = A$, then $\mathcal{R}$ is called a fuzzy relation on $A$.

**Definition 1.6.** [41] Let $T$ be a t-norm and $S$ be its dual t-conorm. Then a fuzzy relation $\mathcal{R}$ on $A$ is said to be:

1. *reflexive* if $\mathcal{R}(a,a) = 1$, for each $a \in A$;

2. *irreflexive* if $\mathcal{R}(a,a) = 0$, for each $a \in A$;

3. $T-$*transitive* if $\mathcal{R}(a,b) \geq T\{\mathcal{R}(a,c), \mathcal{R}(c,b)\}$, for each $a,b,c \in A$;

4. *negatively* $S-$*transitive* if $\mathcal{R}(a,b) \leq S\{\mathcal{R}(a,c), \mathcal{R}(c,b)\}$, for each $a,b,c \in A$;

5. *strongly* $S-$*complete* if $S\{\mathcal{R}(a,b), \mathcal{R}(b,a)\} = 1$, for each $a,b \in A$;

6. $S-$*complete* if $S\{\mathcal{R}(a,b), \mathcal{R}(b,a)\} = 1$, for each $a,b \in A, a \neq b$;

7. $T-$*asymmetric* if $T\{\mathcal{R}(a,b), \mathcal{R}(b,a)\} = 0$, for each $a,b \in A$.

The following two definitions of a fuzzy topological space are given by Chang([24]) and Lowen([69]), respectively.

**Definition 1.7.** [24] A *fuzzy topological space* is a pair $(X,\tau)$ consisting of a non empty set $X$ and a family $\tau$ of fuzzy sets in $X$ satisfying the following conditions:

1. $0_X, 1_X \in \tau$.

2. If $\{f_i : i \in \Omega\}$ is an arbitrary family of fuzzy sets in $\tau$, then $\bigcup_{i\in\Omega} f_i \in \tau$.

3. If $f, g \in \tau$, then $f \cap g \in \tau$.

**Definition 1.8.** [69] A *fuzzy topological space* is a pair $(X,\tau)$, where $X$ is a non empty set and $\tau$ is a family of fuzzy sets in $X$ such that the following conditions are satisfied:

1. $\alpha_X \in \tau, \forall \alpha \in [0,1]$.

2. If $\{f_i : i \in \Omega\}$ is an arbitrary family of fuzzy sets in $\tau$, then $\bigcup_{i\in\Omega} f_i \in \tau$.

3. If $f, g \in \tau$, then $f \cap g \in \tau$.

In both the definitions 1.7 and 1.8, $\tau$ is called a *fuzzy topology* on $X$ and members of $\tau$ are called *fuzzy open sets*. A fuzzy set in $X$ is called *fuzzy closed* if $f^c \in \tau$.

Let $(X, \tau)$ be a fuzzy topological space. Then a subfamily $\mathcal{B}$ of $\tau$ is called a *base* for $\tau$ if every member of $\tau$ can be written as a union of members of $\mathcal{B}$ and a subfamily $\mathcal{S}$ of $\tau$ is called a *subbase* for $\tau$ if the family of finite intersections of its members forms a base for $\tau$(cf.[76]). A fuzzy topology $\tau$ is said to be *generated* by a family $\mathcal{S}$ of fuzzy sets in $X$ if every member of $\tau$ is a union of finite intersections of members of $\mathcal{S}$(cf. [108]).

Now we give the definitions of separation axioms and compactness in fuzzy topological spaces.

Let $(X, \tau)$ be a fuzzy topological space. Then:

**Definition 1.9.** [71] $(X, \tau)$ is said to be *fuzzy* $T_0$ if for $x, y \in X, x \neq y$, there exists a fuzzy open set $U$ such that $U(x) \neq U(y)$.

**Definition 1.10.** [107] $(X, \tau)$ is said to be *fuzzy* $T_1$ if for two distinct fuzzy points $x_r, y_s$ in $X$, there exist two fuzzy open sets $U, V$ such that $x_r \in U$, $x_r \notin V$, $y_s \notin U$, $y_s \in V$.

**Definition 1.11.** [110] $(X, \tau)$ is said to be *fuzzy* $T_1$ if for $x$, $y \in X$ such that $x \neq y$, there exist two fuzzy open sets $U, V$ such that $U(x) = 1$, $U(y) = 0$, $V(x) = 0$, $V(y) = 1$.

**Definition 1.12.** [106] $(X, \tau)$ is said to be *fuzzy* $T_2$ or *Hausdorff* if for two distinct fuzzy points $x_r, y_s$ in $X$, there exist two fuzzy open sets $U, V$ such that $x_r \in U$, $y_s \in V$ and $U \cap V = 0_X$.

Next, let $(X, \tau)$ be a fuzzy topological space in the sense of Lowen[69]. Then:

**Definition 1.13.** [69] A fuzzy set $f$ in $X$ is said to be *fuzzy compact* if for any family $\beta \subseteq \tau$ such that $\bigcup_{\mu\in\beta} \mu \supseteq f$ and for all $\epsilon > 0$, there exists a finite subfamily $\beta_o \subseteq \beta$ such that $\bigcup_{\mu\in\beta_o} \mu \supseteq f - \epsilon_X$.

**Definition 1.14.** [69] $(X, \tau)$ is said to be *fuzzy compact* if each constant fuzzy set in $X$ is fuzzy compact.

**Definition 1.15.** [55] A collection $\mathcal{P}$ of fuzzy sets in $X$ is called a *fuzzy partition* of $X$ if the following conditions are satisfied:

1. For all $U \in \mathcal{P}$, there is some $x \in X$ such that $U(x) = 1$.

2. For all $x \in X$, there is exactly one $U \in \mathcal{P}$ such that $U(x) = 1$.

3. If $U, V \in \mathcal{P}$ such that $U(x) = V(y) = 1$ for some $x$, $y \in X$, then $U(y) = V(x)$.

Now we recall some definitions associated with fuzzy soft sets and with its topological structure, which can be found in [113, 115] and [8].

**Definition 1.16.** [115] A *fuzzy soft set* $f_A$ over $X$ is a mapping from $E$ to $I^X$ i.e., $f_A : E \to I^X$ such that $f_A(e) \neq 0_X$, if $e \in A \subseteq E$ and $f_A(e) = 0_X$, otherwise. Here $E$ is called the parameters set.

The set of all fuzzy soft sets over $X$ will denote by $\mathcal{F}(X, E)$.

**Definition 1.17.** [115] A *constant fuzzy soft set* $\alpha_E$ over $X$ is given by $\alpha_E(e) = \alpha_X, \forall e \in E$.

**Definition 1.18.** [115] Let $f_A, g_B \in \mathcal{F}(X, E)$.Then

1. $f_A$ is said to be a *fuzzy soft subset* of $g_B$, denoted by $f_A \sqsubseteq g_B$, if $f_A(e) \subseteq g_B(e),\ \forall e \in E$.

2. $f_A$ and $g_B$ are said to be *equal*, denoted by $f_A = g_B$, if $f_A \sqsubseteq g_B$ and $g_B \sqsubseteq f_A$.

3. The *union* of $f_A$ and $g_B$, denoted by $f_A \sqcup g_B$, is the fuzzy soft set over $X$ defined by
$$(f_A \sqcup g_B)(e) = f_A(e) \cup g_B(e),\ \forall e \in E.$$

4. The *intersection* of $f_A$ and $g_B$, denoted by $f_A \sqcap g_B$, is the fuzzy soft set over $X$ defined by
$$(f_A \sqcap g_B)(e) = f_A(e) \cap g_B(e),\ \forall e \in E.$$

   Two fuzzy soft sets $f_A$ and $g_B$ over $X$ are said to be *disjoint* if $f_A \sqcap g_B = 0_E$.

5. Let $\Omega$ be an index set and $\{f_{A_i} : i \in \Omega\}$ be a family of fuzzy soft sets over $X$. Then their *union* $\bigsqcup_{i\in\Omega} f_{A_i}$ and *intersection* $\sqcap_{i\in\Omega} f_{A_i}$ are defined, respectively as follows:

   (a) $(\bigsqcup_{i\in\Omega} f_{A_i})(e) = \bigcup_{i\in\Omega} f_{A_i}(e), \ \forall e \in E.$

   (b) $(\sqcap_{i\in\Omega} f_{A_i})(e) = \bigcap_{i\in\Omega} f_{A_i}(e), \ \forall e \in E.$

6. The *complement* of $f_A$, denoted by $f_A^c$, is the fuzzy soft set over $X$, defined by

$$f_A^c(e) = 1_X - f_A(e), \ \forall e \in E.$$

**Definition 1.19.** [8] Let $\mathcal{F}(X, E)$ and $\mathcal{F}(Y, K)$ be the collection of all the fuzzy soft sets over $X$ and $Y$ respectively and $E$, $K$ be parameters sets for the universe $X$ and $Y$ respectively. Let $\varphi : X \to Y$ and $\psi : E \to K$ be two maps. Then a *fuzzy soft mapping* from $X$ to $Y$ is the pair $(\varphi, \psi)$ where,

$$(\varphi, \psi) : \mathcal{F}(X, E) \to \mathcal{F}(Y, K)$$

and the image and inverse image of a fuzzy soft set are defined as follows:

1. Let $f_A \in \mathcal{F}(X, E)$. Then the *image* of $f_A$ under the fuzzy soft mapping $(\varphi, \psi)$ is the fuzzy soft set over $Y$, denoted by $(\varphi, \psi)f_A$ and is defined as:

$$((\varphi, \psi)f_A)(k)(y) = \begin{cases} \sup\limits_{\varphi(x)=y} \sup\limits_{\psi(e)=k} f_A(e)(x), & \text{if } \varphi^{-1}(y) \neq \phi \text{ and } \psi^{-1}(k) \neq \phi \\ 0, & \text{otherwise,} \end{cases}$$

$$\forall y \in Y, \forall k \in K.$$

2. Let $g_B \in \mathcal{F}(Y, K)$. Then the *inverse image* of $g_B$ under the fuzzy soft mapping $(\varphi, \psi)$ is the fuzzy soft set over $X$, denoted by $(\varphi, \psi)^{-1} g_B$ and is defined as:

$$((\varphi, \psi)^{-1} g_B)(e)(x) = g_B(\psi(e))(\varphi(x)), \ \forall e \in E, \forall x \in X.$$

A fuzzy soft mapping $(\phi, \psi)$ is said to be *injective* if $\phi$ and $\psi$ both are injective and *surjective* if $\phi$ and $\psi$ both are surjective. Further, $(\phi, \psi)$ is said to be *constant* if $\phi$ and $\psi$ both are constant[115].

**Definition 1.20.** [115] Let $f_A \in \mathcal{F}(X, E)$ and $g_B \in \mathcal{F}(Y, K)$. Then the *fuzzy soft product* of $f_A$ and $g_B$, denoted by $f_A \times g_B$, is the fuzzy soft set over $X \times Y$ and is defined by

$$(f_A \times g_B)(e, k) = f_A(e) \times g_B(k), \forall (e, k) \in E \times K$$

and for $(x, y) \in X \times Y$,

$$(f_A(e) \times g_B(k))(x, y) = \min\{f_A(e)(x), g_B(k)(y)\}.$$

**Definition 1.21.** ([113],[115]) A *fuzzy soft topological space* relative to the parameters set $E$ is a pair $(X, \tau)$ consisting of a non empty set $X$ and a family $\tau$ of fuzzy soft sets over $X$ satisfying the following conditions :

1. $0_E, 1_E \in \tau$.
2. If $f_A, g_B \in \tau$, then $f_A \sqcap g_B \in \tau$.
3. If $(f_A)_j \in \tau, \forall j \in \Omega$ , where $\Omega$ is some index set, then $\bigsqcup_{j\in\Omega} (f_A)_j \in \tau$.

**Definition 1.22.** ([113], [115]) A *fuzzy soft topological space* relative to the parameters set $E$ is a pair $(X, \tau)$ consisting of a non empty set $X$ and a family $\tau$ of fuzzy soft sets over $X$ satisfying the following conditions:

1. $\alpha_E \in \tau$, $\forall \alpha \in [0, 1]$.
2. If $f_A, g_B \in \tau$, then $f_A \sqcap g_B \in \tau$.
3. If $f_{A_j} \in \tau, \forall j \in \Omega$, where $\Omega$ is some index set, then $\bigsqcup_{j\in\Omega} f_{A_j} \in \tau$.

In both the definitions 1.21 and 1.22, $\tau$ is called a *fuzzy soft topology* over $X$, members of $\tau$ are called *fuzzy soft open sets*. A fuzzy soft set $g_B$ over X is called *fuzzy soft closed* if $(g_B)^c \in \tau$.

We mention here that the fuzzy soft topology given in the definition 1.22, has been called *'enriched fuzzy soft topology'*, in [115].

**Definition 1.23.** [115] A fuzzy soft topology $\tau_1$ is called *finer* than a fuzzy soft topology $\tau_2$ if $\tau_2 \subseteq \tau_1$ and then $\tau_2$ is called coarser than $\tau_1$.

**Definition 1.24.** ([115]) Let $(X, \tau)$ be a fuzzy soft topological space. Then a subfamily $\mathcal{B}$ of $\tau$ is called a *base* for $\tau$ if every member of $\tau$ can be written as a union of members of $\mathcal{B}$.

**Definition 1.25.** ([115]) Let $(X, \tau)$ be a fuzzy soft topological space. Then a subfamily $\mathcal{S}$ of $\tau$ is called a *subbase* for $\tau$ if the family of finite intersection of its members forms a base for $\tau$.

**Definition 1.26.** ([115]) A fuzzy soft topology $\tau$ over $X$ is said to be *generated* by a family $\mathcal{S}$ of fuzzy soft sets over $X$ if every member of $\tau$ is a union of finite intersections of members of $\mathcal{S}$.

**Definition 1.27.** ([115]) Let $\{(X_i, \tau_i)\}_{i\in\Omega}$ be a family of fuzzy soft topological spaces relative to the parameters sets $E_i$ and for each $i \in \Omega$, we have a fuzzy soft mapping

$$(\varphi, \psi)_i : (X, E) \to (X_i, \tau_i).$$

Then the fuzzy soft topology $\tau$ over $X$ is said to be *initial* with respect to the family $\{(\varphi, \psi)_i\}_{i\in\Omega}$ if $\tau$ has as subbase the set

$$\mathcal{S} = \{(\varphi, \psi)_i^{-1}(f_{A_i}) : i \in \Omega,\ f_{A_i} \in \tau_i\}$$

i.e., the fuzzy soft topology $\tau$ over $X$ is generated by $\mathcal{S}$.

**Definition 1.28.** ([115]) Let $\{(X_i, \tau_i)\}_{i\in\Omega}$ be a family of fuzzy soft topological spaces relative to the parameters sets $E_i$. Then their *product* is defined as the fuzzy soft topological space $(X, \tau)$ relative to the parameters set $E$, where $X = \prod\limits_i X_i$, $E = \prod\limits_i E_i$ and $\tau$ is the fuzzy soft topology over $X$ which is initial with respect to the family $\{(p_{X_i}, q_{E_i})\}_{i\in\Omega}$, $p_{X_i} : \prod\limits_i X_i \to X_i$ and $q_{E_i} : \prod\limits_i E_i \to E_i$, $i \in \Omega$ are the projection maps i.e., $\tau$ is generated by the family

$$\{(p_{X_i}, q_{E_i})^{-1}(f_{A_i}) : i \in \Omega, f_{A_i} \in \tau_i\}.$$

In particular, let $(X_1, \tau_1)$ and $(X_2, \tau_2)$ be two fuzzy soft topological spaces relative to the parameters sets $E_1$ and $E_2$, respectively, then their product is the fuzzy soft topological space $(X_1 \times X_2, \tau)$ relative to the parameters set $E_1 \times E_2$, where $\tau$ is generated by the set,

$$\mathcal{S} = \{(p_{X_1}, q_{E_1})^{-1} f_{A_1}, (p_{X_2}, q_{E_2})^{-1} g_{A_2} : f_{A_1} \in \tau_1,\ g_{A_2} \in \tau_2\}.$$

Note that

$$\begin{aligned}((p_{X_1}, q_{E_1})^{-1} f_{A_1})(e_1, e_2)(x, y) &= f_{A_1}(q_{E_1}(e_1, e_2))(p_{X_1}(x, y)) \\ &= f_{A_1}(e_1)(x) \\ &= (f_{A_1} \times 1_{E_2})(e_1, e_2)(x, y), \ \forall (x, y) \in X_1 \times X_2.\end{aligned}$$

Therefore, $(p_{X_1}, q_{E_1})^{-1} f_{A_1} = f_{A_1} \times 1_{E_2}$.
Similarly, $(p_{X_2}, q_{E_2})^{-1} g_{A_2} = 1_{E_1} \times g_{A_2}$.

So, $\mathcal{S}$ has the following form

$$\mathcal{S} = \{f_{A_1} \times 1_{E_2},\, 1_{E_1} \times g_{A_2} \,:\, f_{A_1} \in \tau_1,\, g_{A_2} \in \tau_2\}$$

and $\tau$ has a base $\mathcal{B}$ of the form

$$\mathcal{B} = \{f_{A_1} \times g_{A_2} : f_{A_1} \in \tau_1,\, g_{A_2} \in \tau_2\}.$$

**Definition 1.29.** [115] Let $(X_1, \tau_1)$ and $(X_2, \tau_2)$ be two fuzzy soft topological spaces. Then a fuzzy soft mapping

$$(\varphi, \psi) : (X_1, \tau_1) \to (X_2, \tau_2)$$

is said to be *fuzzy soft continuous* if $(\varphi, \psi)^{-1} f_B \in \tau_1, \forall f_B \in \tau_2$.

## 1.3 Plan of the thesis

The plan of the thesis is as follows:

It consists of 7 chapters. Chapter 1 is introductory and contains necessary preliminaries and plan of the thesis.

In chapter 2, we have introduced and studied Hausdorff separation axiom in fuzzy soft topological spaces. We have shown that Hausdorff fuzzy soft topological spaces satisfy productive, projective and hereditary properties. Further, we have obtained a characterization of a Hausdorff fuzzy soft topological space.

In chapter 3, we have introduced the notions of $T_0$ and $T_1$ separation axioms in fuzzy soft topological spaces. We have given a complete comparison of our definitions with those given by Mahanta and Das[72]. We have also shown that $T_0$ and $T_1$ fuzzy soft topological spaces satisfy productive, projective and hereditary properties and obtained a characterization for a $T_1$ fuzzy soft topological space.

In chapter 4, we have introduced and studied compactness in fuzzy soft topological spaces as an extension of the fuzzy compactness in a fuzzy topological space given by Lowen[69]. We have proved the counterparts of the Alexander's subbase lemma and the Tychonoff theorem for fuzzy soft topological spaces.

In chapter 5, we have introduced the notion of fuzzy topologies generated by fuzzy relations as a generalization of the corresponding concept given by Knoblauch[62]. We have introduced the notions of preorderble and orderable fuzzy topologies and obtained characterizations of a fuzzy topology generated by a fuzzy relation, fuzzy topology generated by a fuzzy interval order, preorderable and orderable fuzzy topologies. We have also introduced the notion of fuzzy bitopological spaces generated by a fuzzy relation and obtained some related results. In particular, we have obtained a characterization of a fuzzy bitopological space generated by a fuzzy relation and it has also been proved that if $(X, \tau_1, \tau_2)$ is the fuzzy bitopological space generated by a fuzzy relation $\mathcal{R}$, then the fuzzy topology $\tau_1$ is fuzzy $T_i$ iff $\tau_2$ is fuzzy $T_i$, $i = 0, 1$.

In chapter 6, we have introduced the notion of fuzzy topologies generated by fuzzy relations, as a generalization of the corresponding concept given by Smithson[102]. We have obtained sufficient conditions under which this generated fuzzy topology becomes fuzzy $T_0$, fuzzy $T_1$ and fuzzy $T_2$. We have also introduced the notion of 'finite intersection property' in fuzzy topological spaces and established a characterization of the Lowen's fuzzy compactness in terms of this property. Using this result, we have obtained a sufficient condition under which the fuzzy topology generated by a fuzzy relation, becomes fuzzy compact.

In chapter 7, we have studied representability of fuzzy biorders and fuzzy weak orders. Several results have been obtained. In particular, we have shown that union of a finite family of fuzzy weak orders with respect to a t-norm $T$ is a fuzzy quasi-transitive relation with respect to $T$. Further, we have obtained a characterization for a $T_L-$representable fuzzy weak order.