Chapter 1

Introduction

> “The greatest glory in living lies not in never
> falling, but in rising every time we fall.”
>
> –Nelson Mandela.

1.1 Background

1.1.1 Linear and Nonlinear Waves

In most of the physical problems, the nonlinearity present inherently, gives rise to some interesting phenomena like shock formation, wave interaction, evolutionary processes of the non-linear waves etc. In recent years, there have been a noticeable increase in research and development activities on non-linear waves. Violent disturbances which result from detonation of explosives, the flow through rocket nozzles, supersonic flight of projectiles, or impact on solids differ greatly from the “linear” phenomena of sound, light, or electromagnetic signals. In contrast to the latter, their propagation is governed by non-linear differential equations, and as a consequence, the familiar laws of superposition, reflection, and refraction ceases to be valid; but even more novel features appear, among which the occurrence of shock fronts is most conspicuous. Shock waves in air are small transition layers of very rapid changes of physical quantities such as pressure, density and temperature. Understanding and analysis of nonlinear wave motion is a matter of obvious importance. During the period of beginning, more than hundred years ago, many mathematician and physicist like Stokes, Earnshaw, Riemann, Rankine, Hugoniot, Lord Rayleigh and later Hadamard, Von Neumann, Courant, Friedrichs, G. B. Whitham, and others developed the fundamental concepts and wrote research papers and books initiating this field of research. During the last few decades, however, when the barriers between applied and pure science were forced down, a wide-spread interest arose in nonlinear wave motion, particularly in shock waves and expansion waves.

The most familiar physical notion of a wave is the one involving the propagation of a disturbance or variation that transfers energy progressively from point to point in a medium, and that may take the form of elastic deformation or of a variation of pressure, electric or magnetic intensity, electric potential, or temperature. This form of wave is thus inherently connected with the motion of some kind involving the space $\mathbb{R}^n$ and the time t so that it gives rise naturally to problems of an evolutionary nature with respect to time. For this reason, the time variable will always need to be distinguished from the other independent variables. In linear wave motion, as, for example, in the transmission of sound, disturbances are always propagated with a definite speed (relative to the medium) which may vary with in the medium. This "sound speed" is a local property of the medium itself and remains the same for every conceivable linear wave motion in the medium. Such a sound speed also plays a role in nonlinear wave motion. Small disturbances or "wavelets," slightly modifying a given primary wave motion, are propagated with a certain speed, again called sound speed, though in this case the sound speed depends not only on the position within the medium but on the state of the medium induced by primary motion. The distinctive feature of nonlinear waves concerns with the disturbances or discontinuities which are not necessarily small.

Waves occur in most scientific and engineering disciplines, for example, fluid mechanics, optics, electromagnetism, solid mechanics, structural mechanics, quantum mechanics, etc. The waves for all these applications are described by solutions to either linear or nonlinear partial differential equations. The homogeneous linear partial differential equations satisfy the principle of superposition, i.e., the linear combination of two solutions will also be the solution. Due to this fact, the solution space of the homogeneous linear partial differential equation forms vector space, and the linear structure of that space can be used with an advantage in constructing solutions with desired properties that can meet diverse boundary and initial conditions. If the governing system of equations is nonlinear, it is not possible to apply the principle of superposition of solutions as in the case of linear partial differential equations. The present work is concerned with the solutions to the problems involving hyperbolic system of partial differential equations in gas dynamics. Therefore we only focus on nonlinear hyperbolic waves and on the discontinuities produced during their propagation. Under certain conditions such waves can be represented by a quasilinear system of first order equations, which are linear in the first derivatives of the dependent variables, but the coefficients may be functions of dependent variables. In fact, when the effects of viscosity and heat conduction are neglected, the equations reduce to hyperbolic system of equations (Euler's equations).

1.1.2 Hyperbolic system of PDEs

Hyperbolic PDEs arise as basic model in many applications, and especially in various branches of fluid dynamics in which conservation laws are involved. The one-dimensional hyperbolic system of first order PDEs covers wide range of areas of scientific and technological interest. In particular, it has wide range of applications in gas dynamics, fluid dynamics, aerodynamics, multi phase flows, astrophysics, and plasma physics etc. The most prominent feature of the quasilinear hyperbolic system of PDEs lies in the fact that a smooth solution breaks down after a finite time. The breaking of these smooth solutions gives rise to one of the most interesting nonlinear phenomena that occur in nature, i.e., the appearance of shock, which contains sudden jumps in density, pressure and velocity. Another interesting feature of quasilinear hyperbolic system of PDEs is interaction of nonlinear waves. The detailed study of mathematical properties and applications of nonlinear wave propagation problems in the context of hyperbolic systems of PDEs may be found in the books and monographs by Courant and Friedrichs [4] , Lax [5], Jeffrey [6], Zheng [7], Bressan [8], Dafermos [9], Sharma [2] and Smoller [10] etc.

In order to present the mathematical description, let us consider first order partial differential equation of the form

$$u_{i,t} + \sum_{j=1}^{m} a_{ij}(x,t,u_i)u_{i,x} + b_i(x,t,u_i) = 0 \tag{1.1}$$

for $i = 1, ..., m$. This is a system of m equations in m unknowns u_i that depend on space x and a time–like variable t. Here u_i are the dependent variables and x, t are the independent variables; this is expressed via the notation $u_i = u_i(x,t)$; $u_{i,t}$ denotes the partial derivative of u_i with respect to t; similarly $u_{i,x}$ denotes the partial derivative of u_i with respect to x. The system (1.1) can also be written in matrix form as

$$U_t + AU_x + B = 0, \tag{1.2}$$

where

$$U = \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ u_m \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{bmatrix}, A = \begin{bmatrix} a_{11} & . & . & a_{1m} \\ a_{21} & . & . & a_{2m} \\ . & . & . & . \\ . & . & . & . \\ a_{m1} & . & . & a_{mm} \end{bmatrix}.$$

If the entries a_{ij} of the matrix A are all constant and the components b_j of the vector B are also constant then system (1.2) is linear with constant coefficients. If $a_{ij} = a_{ij}(x,t)$ and $b_i = b_i(x,t)$, the system is linear with variable coefficients.The system is still linear if B depends linearly on U and is called quasi–linear if the coefficient matrix A is a function of the vector U, that is A=$A(U)$. Note that quasi–linear systems are in general systems of non–linear equations. In system (1.2), $B = 0$ corresponds to the homogeneous system.

Definition 1.1.1. (**Hyperbolic system**) The system (1.2) is called a first-order hyperbolic system of partial differential equations if the matrix $A(U)$ admits m real eigenvalues

$$\lambda_1(U) \leq \lambda_2(U) \leq \lambda_3(U) \leq \ldots.., \leq \lambda_m(U),$$

together with set of m linearly independent eigenvectors L_1, L_2, L_3.....,L_m. The eigenvalues are also called the characteristic speeds or wave speeds associated with (1.2). The system is said to be strictly hyperbolic if its eigenvalues are distinct:

$$\lambda_1(U) < \lambda_2(U) < \lambda_3(U) < \ldots.., < \lambda_m(U).$$

We now introduce some notions of linearity and nonlinearity for each j-wave family.

Definition 1.1.2. For each $i, j = 1, ..., m$, we say that the j-characteristic field corresponding to j-characteristic of (1.2) is **genuinely nonlinear** when

$$\nabla \lambda_j(U).L_j(U) \neq 0,$$

and **linearly degenerate** when

$$\nabla \lambda_j(U).L_j(U) = 0,$$

where $\nabla = \left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, ...\frac{\partial}{\partial u_m}\right)$.

1.1.3 The Riemann Problem

The Riemann problem is one of the most fundamental problem in the field of hyperbolic conservation laws. In comparison with the Cauchy problem, it is easier to study but still reveals some basic properties of the Cauchy problem. Furthermore, the solutions of Riemann problem constitute the basic building blocks for the construction of solution to the Cauchy problem by using the random choice method. The Riemann problem was initiated and solved for one-dimensional Euler equations of isentropic flows in gas dynamics by Riemann [11] in his pioneering work on the mathematical theory of shock waves. It is an initial value problem with the simplest discontinuous initial data which is scale invariant and piecewise constant with one arbitrary discontinuity. There are the concept of weak solution and the method of phase plane analysis for Riemann's solution which reveals the elementary waves of isentropic flows: shock waves and rarefaction waves.

The Riemann problem for the one–dimensional time–dependent Euler equations is the Initial Value Problem for the conservation laws

$$U_t + F(U)_x = 0. \tag{1.3}$$

Here, $U = \begin{bmatrix} \rho \\ \rho u \end{bmatrix}, \quad F(U) = \begin{bmatrix} \rho u \\ \rho u^2 + p \end{bmatrix},$

where $(x, t) \in R \times R_+$.

The space and time coordinates are represented by x and t respectively. $u(x, t)$ and $\rho(x, t) > 0$ stand for velocity and density respectively. The initial conditions of the Riemann problem for the system (1.3) is given by

$$U(x, 0) = \begin{cases} \rho_l, \rho_l u_l, & \text{if } x < 0 \\ \rho_r, \rho_r u_r, & \text{if } x > 0 \end{cases}. \tag{1.4}$$

Here, U_l and U_r denote the left and right constant state respectively which is separated by the jump discontinuity at $x = 0$.

Generalized Riemann problem is characterized by non-constant initial data while a Riemann problem is determined by initial constant data which are, in fact, equilibrium states of the governing system. The Generalized Riemann problem for the linear hyperbolic system provides a smooth solution for the system. The Generalized Riemann problem for non-linear hyperbolic system provides a bounded discontinuous solution depending on its characteristics. Usually, Generalized Riemann problem is considered as a perturbation of a Riemann problem, and it has been proved that the solution obtained in Generalized Riemann problem has the same behavior as the classical solution of the Riemann problem in the neighborhood of the origin [12, 13]. The Generalized Riemann problem for governing system (1.3) with initial boundary condition

$$U(x,0) = \begin{cases} (\rho_l(x), \rho_l(x)u_l(x)), & \text{if } x < 0, \\ (\rho_r(x), \rho_r(x)u_r(x)), & \text{if } x > 0. \end{cases} \tag{1.5}$$

Here $\rho_l(x)$, $\rho_r(x)$, $u_l(x)$ and $u_r(x)$ are smooth arbitrary functions such that $\rho_l(0) \neq \rho_r(0)$, $u_l(0) \neq u_r(0)$.

For a scalar nonlinear case where $\lambda_1(U) = F'(U)$ is strictly convex, the weak solution to this problem will be either a shock wave or a continuous rarefaction wave. Generally, bounded, piecewise smooth weak solution to a system of conservation laws satisfy the Rankine-Hugoniot condition at discontinuities. The Rankine-Hugoniot condition is

$$S[U] = [F],$$

where [.] represents the jump in a quantity across a shock of speed S, giving a relation between the speed S of the discontinuity and the constant states U_l and U_r on each side of the discontinuity. An additional condition, entropy condition, is required to pick out the physically correct weak solution. One such condition is the Lax entropy condition, which states that if $F(U)$ is convex, a shock wave would satisfy the entropy condition

$$\lambda_1(U_l) > S > \lambda_1(U_r).$$

More general weak solutions for (1.3) are the nonlinear wave solutions briefly described below:

- **Shock Wave:** If $\lambda_1(U_l) > \lambda_1(U_r)$, the entropy satisfying weak solution is a shock wave given by

  $$U(x,t) = \begin{cases} U_l \quad for \quad \frac{x}{t} < S, \\ \\ U_r \quad for \quad \frac{x}{t} \geq S, \end{cases}$$

  where S is the shock speed given by the Rankine-Hugoniot condition.

- **Rarefaction Wave:** If $\lambda_1(U_l) < \lambda_1(U_r)$, then the correct entropy solution is a rarefaction wave given by

  $$U(x,t) = \begin{cases} U_l \quad for \quad \lambda_1(U_l) > \frac{x}{t}, \\ \\ U(\frac{x}{t}) \quad for \quad \lambda_1(U_l) \leq \frac{x}{t} \leq \lambda_1(U_r), \\ \\ U_r \quad for \quad \lambda_1(U_r) \leq \frac{x}{t}, \end{cases}$$

  where $U(\frac{x}{t})$ is the solution of $F'(U(\frac{x}{t})) = \frac{x}{t}$.

These elementary wave solutions will also be the key elements to describe the structure of the solution of the Riemann problem for non-linear systems. The solution of the Riemann problem (1.3) together with (1.4) is composed of $n+1$ constant states separated by n waves corresponding to the different characteristic fields. The structure of the solution in the $x-t$ plane is depicted in figure 1.1.

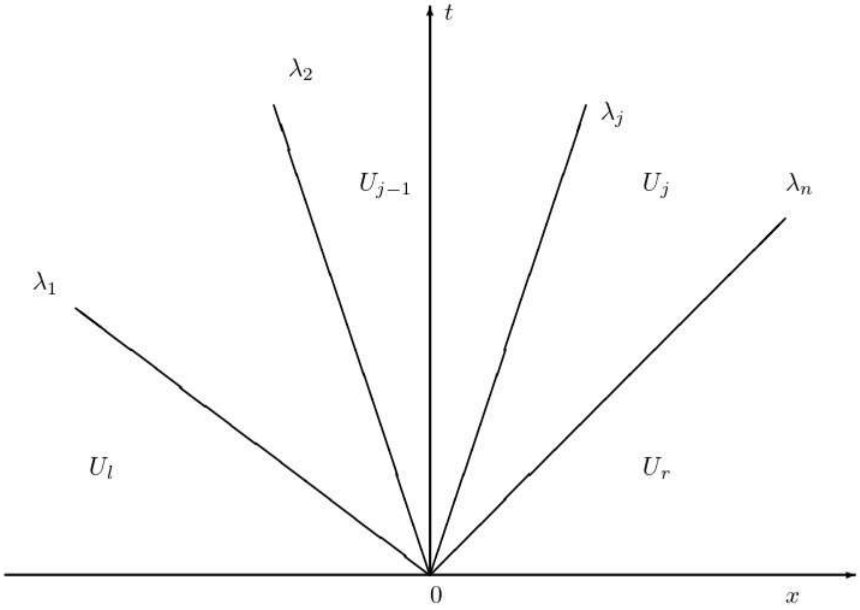

FIGURE 1.1: Structure of the Riemann solution for a system of conservation laws

1.1.4 Ideal and Non-ideal Gases

The equation of state of an ideal gas is written as

$$PV = nRT,$$

where n is the number of molecules of the gas, R is the gas constant, T is the absolute temperature, P is the pressure and V is the volume of the gas. It is a good description of most gases in the low density regime, where on average molecules are far apart.

The equation of real gases is $\lim\limits_{P\to 0} PV/RT = 1$ with compressibility factor Z defined as

$$Z(P,T) = PV/RT.$$

It may be remarked that the deviation from unity indicates the degree of departure from ideal behavior. In theoretical derivations of the ideal gas law it is necessary to make two assumptions i.e. the gas molecules are very small (they have no volume) and the molecules are non-interacting.

However, if the temperature of the gas is very high and density is too low the assumption that the gas is ideal, is no longer valid. The popular alternative to the ideal gas is a simplified van der Waals model. Dutch Physicist van der Waals derived an equation of state without the assumptions of ideal gas, which is known as the van der Waals equation of state, written as

$$\left(P + \frac{na}{V^2}\right)(V - nb) = nRT,$$

where a and b are constants and determined experimentally and n is the number of molecules of the gas. Since the specific volume of a gas is reciprocal of the density of the gas. So, van der Waals equation of state, in terms of density ρ, may be given as

$$\left(P + na\rho^2\right)(1 - nb\rho) = n\rho RT.$$

The van der Waals model is very near to the behavior of real gases for wide range of temperature and pressure. In case of compressible flows, the pressure of gas is taken to be very high consequently the term $a\varrho^2$ is very small compared to the gas pressure P.

1.1.5 Dusty Gas

The study of a two-phase flow of gas and dust particles has been of great interest because of many applications to different engineering problems. Gas flows, which carry an appreciable amount of solid particles, may exhibit significant relaxation effects as a result of particles being unable to follow rapid changes of velocity and temperature of the gas. When the mass concentration of the particles is comparable with that of the gas, the flow properties become significantly different from that of a pure gas. Here, we consider a mixture of a perfect gas and a large number of small dust particles of uniform spherical shape.

Dusty gas is considered to be mixture of gas and small solid dust particles where these dust particles attain less than five percent of total volume [14]. At very high speed of fluid, these small solid particles behave as a pseudo fluid [15]. We consider the mixture as the mixture of two fluids: one is gas and the other is the pseudo fluid of solid particles. The solid particles are spheres of identical mass m_{sp}, radius r_{sp} and specific heat c_{sp}. We consider an element of mixture of gas and solid particles (dusty gas) with total mass $M = M_g + M_{sp}$ and with total volume $V = V_g + V_{sp}$, where subscript g refers to the value for the gas and subscript sp refers to that of the solid particles. The volume of solid particles in mixture is obtained as:

$$V_{sp} = n_{sp}.V.\tau_{sp},$$

where τ_{sp} and n_{sp} is the volume of a solid particle and the number of solid dust particles per unit volume of dusty gas respectively. The mass of solid particles in the volume V of the mixture is written as:

$$M_{sp} = n_{sp}.V.m_{sp}.$$

The species density of the solid particles is defined as:

$$\rho_{sp} = \frac{M_{sp}}{V_{sp}} = \frac{m_{sp}}{\tau_{sp}}.$$

Also, the partial density of the pseudo-fluid of solid particles is defined as:

$$\overline{\rho}_{sp} = \frac{M_{sp}}{V_{sp}} = n_{sp}.m_{sp} = Z\rho_{sp} = n_{sp}.\rho_{sp}.\tau_{sp},$$

where Z represents the fraction of volume of solid particles in the mixture. Further, volume fraction of solid particles is given as:

$$Z = \frac{V_{sp}}{V} = n_{sp}.\tau_{sp}.$$

The species density of the gas or the fluid is defined as:

$$\rho_g = \frac{M_g}{V_g}.$$

Similarly, the partial density of a gas is defined as:

$$\overline{\rho}_g = \frac{M_g}{V} = (1 - Z)\rho_g.$$

Let us consider the thermodynamic equilibrium condition such as:

$$T_{sp} = T_g = T.$$

The density of the mixture is obtained as

$$\rho = Z\rho_{sp} + (1 - Z)\rho_g = \overline{\rho}_{sp} + \overline{\rho}_g.$$

The mass concentration of the pseudo fluid of the solid particles is obtained as:

$$k_p = \frac{\overline{\rho}_{sp}}{\rho} = \frac{Z\rho_{sp}}{\rho}.$$

The pressure of the mixture is written as:

$$p = p_{sp} + p_g.$$

The total pressure of the mixture is p which is obtained from the perfect gas law as:

$$p = R\rho_g T_g.$$

With the help of above analysis, the pressure of the mixture as a whole is obtained as:

$$p_m = p = R\rho_g T_g = R\left(\frac{\rho_m - Z\rho_{sp}}{1-Z}\right)T_g = R\rho_m\left(\frac{1-k_{sp}}{1-Z}\right)T.$$

Therefore, $p_m = \frac{\rho_m R_m T}{1-Z}$ where, $R_m = (1-k_p)R$. Here, R may be considered as an effective gas constant of the mixture and subscript m refers to the value of the gas constant in the mixture as a whole.

1.2 Motivation and Overview

Any mathematical model of a continuum is given by a system of partial differential equations (PDEs). In continuum mechanics, the conservation laws of mass, momentum and energy form a common starting point, and each medium is then characterized by its constitutive laws. The conservation laws and constitutive equations for the field variables, under quite natural assumptions, reduce to field equations, i.e., partial differential equations, which, in general, are nonlinear and nonhomogeneous. The nonlinear PDEs are classified as elliptic, parabolic and hyperbolic. Out of these, hyperbolic system of conservation laws are one of the most important class of nonlinear PDEs. A large number of physical phenomena are modeled by system of quasilinear first order partial differential equations that result from the balance laws of continuum physics. These equations, expressed in terms of divergence, are commonly called conservation laws. One of the most common example of hyperbolic system of PDEs is the Euler's equations, which consist of the equations describing the conservation of mass, momentum and energy. The Euler equations arise from the compressible Navier-Stokes equation by neglecting the viscosity and heat conduction. The theoretical foundation of gasdynamics is formed by the application of the basic conservation laws of mechanics and the second law of thermodynamics to a moving volume of a compressible gas. Mathematically, shock is a discontinuous solution of a hyperbolic system of conservation laws which satisfies the Rankine-Hugoniot conditions as well as entropy condition. Shock waves are basically the waves traveling with velocity greater than the speed of sound. Across the shock wave, there is a rapid change in pressure, density, velocity and temperature. The motivation behind the study of shock waves is their appearance in explosions, implosions, detonations, tsunami, earthquakes, supersonic movement of the bodies, powerful electric discharges, traffic flow etc.

The nonlinear hyperbolic conservation laws are one of the major challenge to contemporary mathematical research. When we deal with the hyperbolic system of PDEs, it is very difficult to determine the exact solution of the problem. The main difficulty is the fact that discontinuities, which are shocks, slip surfaces, etc. always appear in their solutions, even though they are very smooth at initial time. There are two ways to determine the solution of the hyperbolic system of PDEs, one is the analytical approach and other is numerical approach. During the period of seventy eighty years, many analytical and numerical methods have been developed to study the non-linear waves described by the hyperbolic system of PDEs. Several analytical methods such as method of characteristics, progressive wave approach, method of wave front analysis, perturbation method, differential constraint method and many other have been developed to understand the physical properties of the waves governed by quasilinear hyperbolic system of PDEs. Also, various numerical methods such as finite element method, finite volume method and many other numerical techniques have been made to generalize the study of hyperbolic PDEs. The detailed study of mathematical properties, analytical and numerical methods, and applications of the nonlinear wave propagation problems in the context of hyperbolic systems of PDEs may be found in the books and monographs by Courant and Friedrichs [4], Jeffrey [6], Whitham [16], Zheng [7], Bressan [8], Dafermos [9], Sharma [2], Smoller [17], Holden and Risebro [18], LeVeque [19] etc. Solving a Cauchy problem for hyperbolic conservation laws, we find two important challenges. The first one is that the solution may not be continuous after a finite time and the second one is how to choose an admissible solution from several weak solutions. In order to obtain a unique solution to the Cauchy problem, terms such as admissible wave, stable solution, physically relevant solution and so on are introduced. The complete theoretical basis of conservation laws and the admissibility of the weak solution are given by Lax [5].

In recent decades, the Riemann Problem got significant attention due to its applications in various flow phenomena such as granular flow, shallow water flow, traffic flow etc. It is well known fact that in the solution of the hyperbolic system of conservation laws, discontinuities, which are shocks, slip surfaces, etc. always appear in their solutions. Therefore it is natural to consider discontinuous initial data instead of smooth ones. The Riemann Problem is a initial value problem with the simplest discontinuous initial data. It was initiated and solved for one-dimensional isentropic flow by Riemann himself in 1860 [11]. This result is pioneer of mathematical theory of shock waves. It is said that this work is the most important contribution to mathematical physics Riemann [20]. Courant and Friedrichs [4] extended Riemann's result to one-dimensional adiabatic flow. Since then, many pieces of work have been contributed to one-dimensional Riemann problem for various conservation laws. A theory has been established for appropriate amplitude of Riemann data [5, 21] for strictly hyperbolic systems and general Riemann data [22, 23, 24, 17, 25] for the compressible Euler equations. It is fully recognized that the Riemann problem is the most fundamental problem in the entire field of nonlinear hyperbolic conservation laws. Thus, it serves as a touchstone for numerical schemes due to the explicit structure of Riemann solution. For nonlinear hyperbolic system of conservation laws, we refer to Ruggeri and Simic [26]; Liu [27]; Godlewski [28]; Jeffrey [6]; Lax [5] and references cited therein. The theory of nonlinear hyperbolic system of conservation laws in one space dimension usually assume that the system to be strictly hyperbolic with genuinely nonlinear or linearly degenerate characteristic fields. Moreover, general results on the existence of entropy weak solutions to these systems are established only for initial values with small total variation (see Glimm [29] and Lax [5]). However, it is recognized that most of the physical systems do not fit into the standard theory of hyperbolic systems of conservation laws. A natural question is whether these results remain true for a non-strictly hyperbolic system, or even for strictly hyperbolic systems for initial data with large total variation. Keyfitz and Kranzer [30] considered the Riemann problem for one strictly hyperbolic system whose characteristic fields are genuinely nonlinear. They found that for large initial data, the Riemann problem may admit no classical wave solution (which consists of contact discontinuities, shock, and/or rarefaction waves). In contrast to Lax and Glimm's findings, during the last fifty years, several studies have been done on the non-classical solutions such as measure solutions, singular shocks, and so on. The discovery of a new type of nonlinear non-classical wave known as a delta shock wave has attracted mathematicians and physicists worldwide.

1.3 Literature Review

The phenomenon of shock waves is mainly associated with aerospace engineering and in particular with the supersonic flight. The development of this particular branch of physics began, in 1746 when a mathematician Robins determined velocity of the bullet by ballistic pendulum and noticed a growth in aerodynamic drag as velocity tends to the sound speed. However, in the 19th century, the phenomenon of the shock wave was still a mystery to many researchers. In 1759, Euler wrote a letter to Lagrange in which he stated that the velocity of sound depends on the amplitude of sound (size of disturbance), this is not the case when the amplitude of the sound is infinitesimal. Nevertheless, his assumption that velocity would diminish with increasing amplitude was incorrect. Though the time had come for scientists to go through the experimental study and observation of the behavior of shock waves but due to the lack of apparatus and insufficient literature in that discipline it was impossible for them to study the occurrence of shock wave; meanwhile in 1897 the challenge was taken by Hugoniot [31]. In 1808, Poisson [32] was the first researcher to solve the Euler equation for the one-dimensional unsteady fluid-flow and got the exact solutions. In 1823, Poisson created a milestone in non-linear wave theory by constructing isentropic gas law for the sound wave with infinitesimal amplitude. In 1848, Stokes [33] used the term “surface of discontinuity”. He further extended the theory of acoustic wave, having a finite amplitude, by considering the problem of wave steepening. Stokes was not sure about the possibility of discontinuous motion, in this confusion he used isentropic relation which decides the role of dissipation and energy conservation in shock formation, instead of the correct energy equation. He derived the conservation laws for mass and momentum which are used very frequently in modern days. In 1860, a revolution came in the history of the non-linear wave when Riemann [11] evolved the theory of wave of finite amplitude. He studied two dimensional flow of a gas in Eulerian coordinates under the assumption that pressure is the function of density only i.e., $p = p(\rho)$. He solved the PDEs by Monge's method and demonstrated the propagation of the wave of finite amplitude; his outcomes became a trademark in the direction of mathematical treatment of shock wave formation. He showed that actual wave divides into two parts rarefaction wave and shock wave. The rarefaction wave rises thicker so when a gas passes through it became cool and expanded while shock wave rises thinner so when the gas passes through it became hot and compressed. He wrongly made the assumption that entropy remains unchanged across a shock wave which undoubtedly implies that it is a reversible process (second law of thermodynamics) whereas entropy is an irreversible process. It was first Rankine [34] in 1870 and later Hugoniot [31] in 1887 who derived wellknown jump conditions over a discontinuity by considering conservation laws of mass, momentum and energy which are known as Rankine-Hugoniot (R-H) conditions. They are also the first to recognize that entropy is an irreversible process. In 1899, Chapman [35] implemented the theory of Riemann on the detonation wave. During this time Villie constructed shock tube for measuring the speed of shock waves. He observed that the shock waves propagate with a speed greater than that of sound. He also described the analogy between shock and detonation and compared the wave speed with the theory of Hugoniot [36].

When there is a relative motion between a body and a fluid, the disturbance caused by body is propagated through the fluid with the speed of sound. The speed of sound in such cases is the speed with which rarefaction and compression waves of very small amplitude propagate. When the compressions in the flow are of finite amplitude, there usually occurs a discontinuity in the pressure leading to a shock wave. Gas compressions, which have finite amplitude, travel faster than the speed of sound, as in the case of strong explosions. The phenomena of formation of the shock wave was largely studied by various physicists and mathematicians. First investigations of supersonic free air jets using high-speed photography reveal a “lyre” pattern of reflected shock waves, later called “shock diamonds” [37]. These studies prompt the idea of using a supersonic blow-down wind tunnel. It is recognized that a supersonic flow passing around a sharp corner expands through a “fan” of Mach lines centered at the corner, later called the “Meyer-Prandtl expansion fan” [38]. In 1910, Taylor [39] have determined the necessary conditions for the discontinuous motion in gases. In the late 1920s, investigations originating from the practical needs of aeroballistics (minimization of wave drag), aeronautics (high-speed propellers) and steam turbine development (optimal Laval nozzle geometry) eventually lead to the establishment of gas dynamics, a new branch of fluid dynamics. A dimensionless parameter characterizing the flow velocity with respect to the sound velocity of the surrounding (quiescent) medium is introduced; this is later called the “Mach number” [40]. In the period 1940-1945 underwater explosions became a hot topic among the researchers of Wood’s Hole Institution USA and England. They generated some theoretical shock wave models on water and high explosives. The work was summarized in “Black Books” by Cole [41]. In 1949, Broderick [42] have considered the general equation for isentropic, irrotational, axially symmetrical flow of a gas, neglecting viscosity and conductivity, past a thin body of revolution placed in a uniform, supersonic stream, with its axis lying in the direction of the undisturbed stream. Courant and Friedrichs [4] have described the properties of supersonic flows and the non-linear wave in the book entitled as “Supersonic flow and shock waves”. One of the interesting properties of the shock waves is the problem of determining the differential effects of the shock fronts on the rear flow field. To this problem Thomas [43] developed a tensorial approach which was further extended by Kanwal [44] for three dimensional shocks in stationary, pseudo-stationary and unsteady flows of non-conducting gases. The problem of vorticity generation by a shock has also been solved by several authors like Truesdell [45], Hayes [46], Kanwal [47]. Taniuti and Wei [48] presented a class of nonlinear partial differential equations which admit a reduction to tractable nonlinear equations such as the Burgers and the Kortweg-deVries equation and discussed the applications to the hydrodynamics and the plasma physics. This method of reduction is based on a singular perturbation expansion. Varley and Cumberbatch [49] have introduced the method of relatively undistorted waves, which is based on a scheme of successive approximations to system of hyperbolic equations, to discuss high frequency waves governed by non-linear equations. Ambika et al. [50] have generalized the theory of progressive method to study the evolutionary process of the waves of finite and small amplitude. Nath et al.[51] further analyzed the progressive wave solution in dusty medium. The essential ideas underlying the theory of progressive waves may be found in [49, 52, 53, 54]. Ram [55] studied the propagation of acceleration waves along the characteristic path by using the characteristics of the governing quasilinear system as the reference coordinate system. Shankar [56] have also investigated the propagation of acceleration waves in radiation-magneto gasdynamics. For the general behavior and the properties of acceleration waves, the reader is referred to the following research articles [57, 58, 59, 60, 61, 62]. The study of fluid flow containing solid particles has been the interest of many engineering and scientific research such as fluidized beds, centrifugal separation of particular matter from fluids, many chemical processes, solid particle motion in rocket exhaust and dust flow in geophysical and astrophysical problems [63, 64, 15, 65]. Carrier [66] has studied the feature of shock waves in dusty gases in which the plane steady decelerated flow of a dusty gas mixture is analyzed in an appropriate manner. Singh et al. [67] used the wavefront analysis method to study the shock wave formation in a two-dimensional steady supersonic flow of a radiating gas past plane and axisymmetric bodies such as a beak and sharp-edged ring. For details about the wavefront method, one may refer to [6].

The study of Riemann problem started with the work " theory of waves of finite amplitude" by great mathematician G. F. B. Riemann (1859), which was not limited to a single progressive wave and suited to calculate the propagation of planar waves of finite amplitude proceeding in both directions. In 1860, Riemann [11] introduced the Riemann problem for a system of conservation laws in gas dynamics. Lax determined the solution of the Riemann problem for the condition when the initial data of the problem consists of two constant states U_*^1 and U_*^2, where U_*^1 and U_*^2 are respectively the vector of conserved variables to the left and right of $x = 0$ such that $\|U_*^1 - U_*^2\|$ is appropriately small and left and right constant states are divided by jump discontinuity at $x = 0$. In the consideration of Euler equations, the Riemann problem consists of the shock tube problem and for detailed discussion of shock tube problem and other physical problems in form of conservation laws of gasdynamics, the readers are recommended to study the book by Courant and Friedrichs [4].

The exact solution to the Riemann problem is of great significance. For instance, it constitutes the basic building block for the construction of solutions to general initial value problems using the well known random choice method proposed by Glimm [29]. The exact solutions of the Riemann problem for hyperbolic system of conservation laws are proposed by many authors, for detailed methodologies, the reader is referred to the book by [10, 68, 69, 70]. Also, Chorin in [71, 72] proposed the new approach to obtain the exact solution to the Riemann problem. Another improvement to the Godunov's first Riemann solver was presented in [73].

For an illuminating treatment on Riemann problem, we also refer to an article by Liu [27] and the books of Dafermos [9], Bressan [8] and LeVeque [19]. The special solution of Euler equations in which one of the Riemann invariants remains constant throughout the flow field is called a simple wave. In simple wave solutions, waves break and the solution has to be complemented by the introduction of shock waves.

When the shock strength is small and even moderate, jumps in entropy and the Riemann invariants are surprisingly small, see Whitham [74]; this, indeed, formed the basis for an approximate theory for shocks of weak or moderate strength developed by Friedrichs [4], where the actual shock conditions are replaced by the transition through a corresponding simple compression wave.

1.4 Aims and Thesis Objectives

The aim of the present thesis is to study some non-linear wave propagation problems governed by the quasilinear hyperbolic system of partial differential equations. The main objective of this work is to determine the analytical solution of the Euler system of PDEs and examine the behavior of propagating wave in certain mediums. In this investigation, we have used some analytical techniques such as progressive wave method, method of characteristics, method of wavefront analysis to determine the solution of the one dimensional and two dimensional system of PDEs. We also constructed the solution of one-dimensional Riemann Problem under the influence of the external force. We are motivated to solve the problem for non-homogeneous hyperbolic system which is modified into homogeneous hyperbolic system of conservation laws to study the solution of Riemann problem with constant initial data by introducing new variable for the velocity. The purpose of this thesis is to generalize the process of wave propagation described by hyperbolic system PDEs by using some analytical approach, and examine the solution of the one-dimensional Riemann problem with the presence of external force. In order to address the entire work of the thesis, we have been motivated to work on the following objectives:

1. Investigation of the propagation of finite and small amplitude waves in non-ideal gas in the presence of dust particles.

2. Study of the effect of small solid dust particles on the growth and decay behavior of non-linear waves for generalized geometry of flows in non-ideal dusty gasdynamics.
3. Generalization of theory of acceleration discontinuities and the study of various parameter effects on the propagating waves in non-ideal dusty magnetogasdynamics.
4. To examine the behavior of shock wave in two dimensional planar and axisymmetric non-ideal radiating gas flow under the influence of magnetic field.
5. To determine the structure of solution of the Riemann problem for the one-dimensional compressible hyperbolic system under the influence of external force.

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