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# Abstract

Fractal interpolation and fractal dimension are two rapidly growing and fascinating areas of research. Fractal interpolation techniques are useful to approximate non-smooth data sets. In 1986, Barnsley [2] introduced the concept of fractal interpolation functions (FIFs) using the idea of an iterated function system (IFS). This interpolation technique exhibits some level of self-similarity. Classical interpolation methods described in existing literature may not capture the self-similarity and non-smoothness in the data set. Barnsley's contribution sparked significant interest in fractal interpolation, leading to numerous publications on theory and applications by various researchers, see, for instance, [1, 5, 6, 7, 8, 9, 12] and references therein.

Numerous types of FIFs have been constructed, and their properties have been studied in the literature. For instance, in [4], Barnsley et al. studied the concept of Hidden variable fractal interpolation functions. In [13], Navascues et al. have introduced a new type of fractal interpolation function associated with a continuous real-valued univariate function, known as  $\alpha$ -fractal function. In [11], Massopust has introduced the concept of fractal functions and fractal surfaces. In [14], Ruan et al. have explored the study of fractal interpolation function on a rectangular grid.

Estimating the fractal dimension is a growing area of research in fractal theory, similar to fractal interpolation. It involves determining the statistical ratio of the complexity of fractal patterns and how it changes with the scale of measurement. Various concepts of fractal dimension have been introduced in the literature, such as Hausdorff dimension, box dimension, and packing dimension [3, 10, 11], etc. Unlike topological or algebraic dimensions, fractal dimensions can take on noninteger values. In this thesis, we focus on two notions of fractal dimension: Hausdorff dimension and box (Minkowski) dimension.

In this thesis, the author explores the fractal interpolation and fractal dimension of various types of functions and their fractional integrals. The thesis is composed of seven chapters, including an introduction and a conclusion. Chapter 1 provides an overview of fractal interpolation functions and the theory of fractal dimensions, specifically the Hausdorff dimension and the box dimension. It also includes a review of the existing literature on fractal interpolation and fractal dimension. This chapter lays the groundwork for the subsequent chapters by presenting the foundational theories.

Chapter 2 presents some initial concepts regarding the dimension of graphs of vector-valued functions. Specifically, a fresh attempt is made to determine the fractal dimension of the graph of the Katugampola fractional integral of a vector-valued continuous function with bounded variation, which is defined on a closed bounded interval in  $\mathbb{R}$ . It is proved that the dimension of the graph of a continuous vector-valued function with bounded variation is 1, and the same applies to the dimension of the graph of its Katugampola fractional integral. Additionally, for an Hölder continuous function, an upper limit for the dimension of the graph of each coordinate function of the Katugampola fractional integral of the function is established.

Chapter 3 delves into the concept of multivariate fractal interpolation and function approximation using a fractal function called  $\alpha$ -fractal function. In this chapter, the author has worked on the fractal interpolation function for a multivariate continuous function defined on the interval  $[0, 1] \times \cdots \times [0, 1]$  ( $N$ -times), where  $N$  is a natural number. It is demonstrated that the multivariate  $\alpha$ -fractal function retains certain properties of the original function. Further, the behavior of the  $\alpha$ -fractal function, when restricted to the coordinate axis, is examined. Furthermore, the author has investigated the box dimension and the Hausdorff dimension of the graph of the multivariate  $\alpha$ -fractal function and its restriction. Finally, the author has proved that the mixed Riemann-Liouville fractional integral of a fractal function satisfies a self-referential equation.

Chapter 4 focuses on the approximation of continuous multivariate functions defined on the domain  $[0, 1] \times \cdots \times [0, 1]$  ( $N$ -times), where  $N$  is a natural number. The chapter presents various well-known results on multivariate constrained approximation using dimension-preserving approximants. The construction of multivariate dimension-preserving approximants using  $\alpha$ -fractal interpolation functions is discussed, along with the existence of one-sided approximations using fractal functions. An upper bound for the fractal dimension of the graph of the  $\alpha$ -fractal function is provided. The chapter also explores the approximation properties of  $\alpha$ -fractal functions. It establishes the existence of a Schauder basis consisting of multivariate fractal functions for the space of all real-valued continuous functions defined on  $[0, 1]^N$ . Finally, the existence of multivariate fractal polynomials for approximation is proven.

Chapter 5 advances the concept of  $\alpha$ -fractal functions in the direction of set-valued maps. The chapter aims to demonstrate how set-valued continuous functions defined on a closed and bounded interval of real numbers can be approximated by

$\alpha$ -fractal functions. It also presents the perturbation error between the given set-valued continuous function and its  $\alpha$ -fractal function. Furthermore, a new graph for set-valued functions is defined, which differs from the standard graph in the existing literature, and the bounds on the fractal dimension of this newly defined graph for certain classes of set-valued functions are established. The need for this new graph is explained through examples. Finally, it is proven that the new graph of an  $\alpha$ -fractal function serves as an attractor for an iterated function system.

Chapter 6 delves into the measure theoretic aspects of FIFs generated using Barnsley's method (1986). The chapter examines the properties of the invariant measures that are supported on the graph of FIF and compares these findings with previously established theories. Additionally, the chapter explores the study of fractal transformations between two FIFs that satisfy the same interpolation data. Furthermore, the chapter introduces function spaces and presents several results that provide conditions under which the FIF can be considered an element of these spaces. These spaces are then utilized to estimate the fractal dimension of the FIFs. The chapter also includes various remarks, notes, and examples to support the study.

Finally, in Chapter 7, the author concludes the thesis with some future work suggestions.

# Bibliography

- [1] M. Barnsley and A. Vince. Developments in fractal geometry. *Bulletin of Mathematical Sciences*, 3:299–348, 2013.
- [2] M. F. Barnsley. Fractal functions and interpolation. *Constructive Approximation*, 2(1):303–329, 1986.
- [3] M. F. Barnsley. *Fractals Everywhere*. Academic Press, 2014.
- [4] M. F. Barnsley, J. Elton, D. Hardin, and P. Massopust. Hidden variable fractal interpolation functions. *SIAM Journal on Mathematical Analysis*, 20(5):1218–1242, 1989.
- [5] M. F. Barnsley, B. Harding, A. Vince, and P. Viswanathan. Approximation of rough functions. *Journal of Approximation Theory*, 209:23–43, 2016.
- [6] M. F. Barnsley and A. N. Harrington. The calculus of fractal interpolation functions. *Journal of Approximation Theory*, 57(1):14–34, 1989.
- [7] P. Bouboulis, L. Dalla, and V. Drakopoulos. Construction of recurrent bivariate fractal interpolation surfaces and computation of their box-counting dimension. *Journal of Approximation Theory*, 141(2):99–117, 2006.
- [8] A. Chand and G. Kapoor. Stability of affine coalescence hidden variable fractal interpolation functions. *Nonlinear Analysis: Theory, Methods & Applications*, 68(12):3757–3770, 2008.
- [9] A. Chand and P. Viswanathan. A constructive approach to cubic Hermite fractal interpolation function and its constrained aspects. *BIT Numerical Mathematics*, 53:841–865, 2013.
- [10] K. Falconer. *Fractal Geometry: Mathematical Foundations and Applications*. John Wiley & Sons, 2004.

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- [11] P. R. Massopust. *Fractal Functions, Fractal Surfaces, and Wavelets*. Academic Press, 2016.
- [12] M. Navascués and M. Sebastián. Smooth fractal interpolation. *Journal of Inequalities and Applications*, 2006:1–20, 2006.
- [13] M. A. Navascués. Fractal polynomial interpolation. *Zeitschrift für Analysis und ihre Anwendungen*, 24(2):401–418, 2005.
- [14] H.-J. Ruan and Q. Xu. Fractal interpolation surfaces on rectangular grids. *Bulletin of the Australian Mathematical Society*, 91(3):435–446, 2015.