Chapter 1

Introduction

1.1 Spectral Graph Theory

Graphs are mathematical structures that depict the pairwise relationship between objects. "Spectral graph theory" is a branch of graph theory in which linear algebra, particularly, matrix theory plays an important role. It offers the pleasure of seeing many useful and unexpected connections between two beautiful, apparently unrelated, parts of mathematics: matrix theory and graphs. There are several graph matrices which can be used for this purpose, for example, adjacency matrix, Laplacian matrix, and signless Laplacian matrix to name a few. The eigenvalues are strongly connected to almost all key invariants of a graph and hold a wealth of information. Spectral graph theory focuses on extracting this information from graphs.

Researchers have defined different matrices corresponding to a graph according to their requirements. For example, the adjacency matrix works in a wonderful way in studying the energy of a molecule. An interesting quantity in Hückel theory is the total π-electron energy E_π, which is the sum of the energies of all the electrons in a molecule. The π-electron energy of a molecule having $n = 2k$ atoms is $E_\pi = 2\sum_{i=1}^{k} \lambda_i$, where λ_i, $i = 1, 2, \ldots, k$, are the k-largest eigenvalues of adjacency matrix [4]. For a bipartite graph, $E_\pi = \sum_{i=1}^{n} |\lambda_i|$, due to the symmetry of the adjacency spectrum λ_i.

The least eigenvalue of the Laplacian of a graph is always zero. The second least eigenvalue of the Laplacian of a graph is known as the algebraic connectivity of the graph, and the eigenvector corresponding to this eigenvalue is known as the Fiedler vector in honor of the mathematician Miroslav Fiedler [17]. Fiedler first elucidated the properties of the vector that bears his name with an interest in understanding how such vectors relate back to the graph from which they were derived. Identifying clusters is an important aspect in the field of electrical network connections [40]. The key point of interest is that algebraic connectivity and the Fiedler vector gives the clusters of vertices in the graph. Besides, algebraic connectivity is important due to the fact that it is a good parameter to measure, to a certain extent, how well a graph is connected. For example, it is well-known that a graph is connected if and only if its algebraic connectivity is different from zero [4].

In spectral graph theory, a very well-known problem is determining the graphs by their spectrum. It is proved in [22, 39] that signless Laplacian performs better way in this case. However, the signless Laplacian spectrum does not help us to determine whether a graph is connected or not. Hence, it is clear that any type of matrix that is associated with a graph disseminates different information. Thus, the study of graphs and associated matrices has received wide attention.

Some other motivating factors of this topic are: firstly, this theory is useful to study the structural property of DNA [16, 33]. Secondly, the concept of graph energy has a chemical origin and a chemical interpretation [20], and the energy of a graph has a direct connection with its eigenvalues. There are several other useful applications one can talk about the graph matrices. Besides, the interconnection between these matrices according to their spectral as well as graph theoretical properties is a part of this research. Here, we present a remark that will serve as a useful tool for proving various results throughout the thesis.

Remark 1.1.1. For a monic polynomial $\mathcal{P}(z)$ in the complex variable z, it holds that $\mathcal{P}(z) \to \infty$ as $z \to \infty$. Consequently, if $\mathcal{P}(r) < 0$ for some $r \in \mathbb{R}$, then $\mathcal{P}(z)$ must have at least one root greater than r.

The intention of this thesis is to build such a spectral theory by means of the signless Laplacian matrix. We mainly discuss the structural properties of the graph. Before that, we begin with some basic definitions and concepts that are required for further study of this thesis.

1.2 Matrix-theoretical Concepts

For $m, n \in \mathbb{N}$, we use $M_{m\times n}$ to denote a matrix M of order $m \times n$, and M_m to denote a square matrix M of order $m \times m$. We use O and J to represent matrices of appropriate orders with all entries equal to 0 and 1, respectively. The identity matrix is denoted by I, and the determinant of a square matrix M is denoted by $det(M)$.

The multiset of all the eigenvalues of the square matrix M together with their multiplicities is called the *spectrum*, $Spec(M)$, of the matrix M, and $\lambda^{(r)}$ denotes the eigenvalue λ with algebraic multiplicity r. The transpose of a matrix M is denoted by M^t. The *spectral radius* of a square matrix M is $\rho(M) = \max\{|\beta| \mid \beta \in Spec(M)\}$. The eigenvalues of a real symmetric matrix are always real. The *least,* *second smallest*, and the *second largest eigenvalues* of a real symmetric matrix M are denoted by $\lambda_{\min}(M), \lambda_{s2}(M)$, and $\lambda_{l2}(M)$, respectively.

We call a matrix as non-negative (respectively, positive) if all its entries are non-negative (respectively, positive). A non-negative matrix M_m is said to be irreducible if and only if $(I+M)^{m-1} > 0$ ([25]). For any two real matrices $M_{m\times n} = (m_{ij})$ and $N_{m\times n} = (n_{ij})$, we write $M \leq N$ (respectively, $M < N$) to indicate that $m_{ij} \leq n_{ij}$ (respectively, $m_{ij} < n_{ij}$) for all $i = 1, \ldots, m$, and $j = 1, \ldots, n$. For any two non-negative matrices M_m and N_m, we say M is dominated by N if $M \leq N$. Note that, if M_m is dominated by N_m, then $\rho(M) \leq \rho(N)$. Now, we give a result on the determinant of the block matrix.

Theorem 1.1. *([13], Lemma 2.2). Let* $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ *be a square matrix, where* M_{22} *is invertible. Then* $det(M) = det(M_{22}) det(M_{11} - M_{12}M_{22}^{-1}M_{21})$.

In this thesis, we study different results by virtue of well known theorems, namely the Perron-Frobenius Theorem and Interlacing Theorem on eigenvalues, and using equitable quotient matrix. We will numerically estimate the eigenvalues and characteristic polynomial of a square matrix M using eig(M) and charpoly(M) function, respectively, in MATLAB.

Remark 1.2. In this thesis, we use MATLAB to calculate the eigenvalues of various matrices to support several mathematical results. Now, a question may arise at this stage whether these eigenvalues obtained from MATLAB can be reliably used in our mathematical proof. Well, it's true that MATLAB can only estimate eigenvalues in finite precision to within a certain tolerance. However, this does not affect the validity of our conclusions. In our case, we have used it to show that a certain eigenvalue, say λ, of a real symmetric square matrix B, is either greater than or less than or equal to some particular integer, say k. Hence, for the cases $\lambda > k$ and

$\lambda < k$, MATLAB based results justify our conclusion. Besides, we have also verified such results from the behavior of the respective characteristic polynomial in the intervals $[k-1,k]$ and $[k,k+1]$. If we are concluding $\lambda = k$ using MATLAB, then we have cross-checked it from the characteristic polynomial $\mathcal{P}_B(z)$ of B by showing that $\mathcal{P}_B(k) = 0$. Additionally, in some instances, the characteristic polynomial in Remark 1.1.1 is used to show that $\lambda > k$.

1.2.1 Perron-Frobenius Theorem

Theorem 1.3. *(Perron-Frobenius Theorem, [[25],Theorem 8.4.3]). Let M be an irreducible non-negative square matrix. Then the following will hold.*

(a) *$\rho(M)$ is a simple eigenvalue of M.*

(b) *$Mx = \rho(M)x$ for some real vector $x > 0$.*

(c) *Let N be a matrix such that $O \leq N \leq M$. Then $\rho(N) \leq \rho(M)$. If $\rho(N) = \rho(M)$, then $N = M$.*

1.2.2 Interlacing Theorem

Let $m \geq n$, and N_n be a submatrix of a matrix M_m such that both M and N are diagonalizable and both $Spec(M), Spec(N)$ contains only real numbers. Suppose the eigenvalues of M and N are $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_m(M)$ and $\lambda_1(N) \geq \lambda_2(N) \geq \cdots \geq \lambda_n(N)$, respectively, then we say that the eigenvalues of N interlace the eigenvalues of M if for all $i = 1, 2, ..., n$,

$$\lambda_i(M) \geq \lambda_i(N) \geq \lambda_{m-n+i}(M)$$

holds.

Theorem 1.4. *(Interlacing Theorem,[[25], Theorem 4.3.17]). Let N be a principal submatrix of a real symmetric matrix M, then the eigenvalues of N interlace the eigenvalues of M.*

1.2.3 Equitable Quotient Matrix

Let M be a complex matrix of order m described in the following block form

$$M = \begin{pmatrix} M_{11} & \dots & M_{1t} \\ \vdots & \ddots & \vdots \\ M_{t1} & \dots & M_{tt} \end{pmatrix}$$

where the blocks M_{ij} are $m_i \times m_j$ matrices for $1 \leq i, j \leq t$ and $m = m_1 + \cdots + m_t$. For $1 \leq i, j \leq t$, let r_{ij} be the average row sum of M_{ij}, i.e., r_{ij} is the sum of all entries in M_{ij} divided by the number of rows. Then the square matrix $\mathcal{E} = (r_{ij})$ of order t is called the *quotient matrix* of M. If, in addition, for each pair i, j, each of the m_i rows of M_{ij} has the same sum, then $\mathcal{E}$ is called the *equitable quotient matrix* of M.

Theorem 1.5. *([41], Theorem 2.3). Let $\mathcal{E}$ be the equitable quotient matrix of a complex square matrix M. Then $Spec(\mathcal{E}) \subseteq Spec(M)$.*

Theorem 1.6. *([41], Theorem 2.5). Let $\mathcal{E}$ be the equitable quotient matrix of a non-negative square matrix M. Then $\rho(\mathcal{E}) = \rho(M)$.*

1.3 Graph-theoretical Concepts

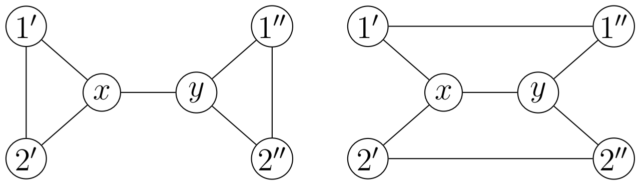
All the graphs considered in this thesis are undirected and without multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. In Chapters 2-3, we consider only simple graphs, and in Chapter 4, we added self-loops to a simple graph G. We use G^H to denote the graph after adding a self-loop to every vertex of a subset H of $V(G)$.

We call G as H-*free* if H is not a subgraph of G. $K_{q,p}$ is a complete bipartite graph with q and p vertices in the first and second partite sets, respectively. C_n, P_n, and K_n are the cycle, path, and complete graph, respectively, of order n. The subdivision of G is the graph obtained from G by replacing all the edges by P_3, and is denoted by $S(G)$. The union of G_1 and G_2 is the graph $G_1 \cup G_2$, whose vertex set is $V(G_1) \cup V(G_2)$ and whose edge set is $E(G_1) \cup E(G_2)$.

For a vertex $x \in V(G)$, the *degree*, $d_G(x)$, is the number of vertices adjacent to x in G, and $d_G^{\max}$ is the *maximum degree of* G. We use $N(x)$ to denote the *neighborhood of* x, and $e' = xy \in E(G)$ to denote an edge with incident vertices x, y. The *edge-degree* e-$deg_G(e')$ of an edge $e' = xy$ is $|N(x) \cup N(y)| - 2$. If for every $e' \in E(G)$, e-$deg_G(e')$ is equal, then we call G as an *edge-regular* graph, and the edge-degree of G is denoted by e-deg_G. If G is not edge-regular, we call it as *edge-non-regular graph*, and denote the *maximum edge-degree* of G by e-$deg_G^{\max}$.

The complement $\overline{G}$ of the graph G is a graph with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{xy | xy \notin E(G)\}$. For an edge $e' \in E(G)$, we use $G - e'$ to denote the graph after removing the edge e' from G. Later, we also refer to the graph $\overline{K_{3,3} - e'}$, as shown in Figure 1.1(a). The *Cartesian product* $G_1 \Box G_2$ of two graphs G_1 and G_2 is a graph with $V(G_1 \Box G_2) = V(G_1) \times V(G_2)$ and $(x_1, y_1)(x_2, y_2) \in E(G_1 \Box G_2)$ if and only if $x_1 = x_2$ and $y_1 y_2 \in E(G_2)$ or, $y_1 = y_2$ and $x_1 x_2 \in E(G_1)$. For example, see Figure

1.1(b) for $K_{1,2}\Box K_2$. The subgraph induced by a subset H of vertices in G is given by $G[H]$.

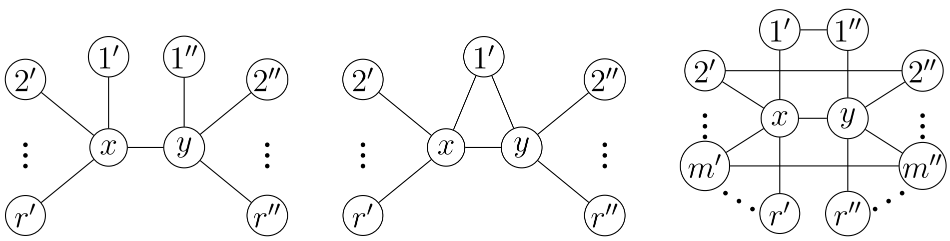


(a) $\overline{K_{3,3}-e'}$ (b) $K_{1,2}\Box K_2$

FIGURE 1.1: Examples of complement and cartesian product of graphs.

We define a *double star* $\mathcal{S}_r^2$ obtained by taking two disjoint copies of star graph $K_{1,r}$ and adding an edge between the vertices of degree r, see Figure 1.2(a). Consider $\mathcal{S}_r^{2,1}$ as a *variation of* $\mathcal{S}_r^2$, where the vertices of degree $r+1$ in S_r^2 has exactly one common neighbor, see Figure 1.2(b). We call the vertices of degree $r+1$ in $\mathcal{S}_r^2$ and $\mathcal{S}_r^{2,1}$ as the central vertices.

Let $\mathcal{S}_r^2(m)$ be a graph obtained from $\mathcal{S}_r^2$ by adding $m(\leq r)$ disjoint edges between the pendant vertices which are at distance 3 in $\mathcal{S}_r^2$ (see Figure 1.2(c)).



(a) $\mathcal{S}_r^2$ (b) $\mathcal{S}_r^{2,1}$ (c) $\mathcal{S}_r^2(m)$

FIGURE 1.2: Double star and its variations.

1.4 Signless Laplacian of Graph

Let the adjacency matrix of G be $A(G) = (a_{uv})$, where $a_{uv} = 1$ if $uv \in E(G)$ and 0 otherwise. Let $D(G)$ be the diagonal matrix with $D(G)_{xx} = d_G(x)$, for any $x \in V(G)$. The Laplacian $L(G)$ of a graph is defined by $D(G) - A(G)$, while the matrix $D(G) + A(G)$ is called the *signless Laplacian* of G and denoted by $Q(G)$. The matrix $Q(G)$ is positive semidefinite for a simple graph G. If G is connected, then $Q(G)$ is irreducible. The eigenvalues of $Q(G)$ are referred as the Q-*eigenvalues* of G, and are all non-negative when G is a simple graph. The Q-*spectral radius* $q(G)$ of G is the largest Q-eigenvalue of G.

In this thesis, we focus on the signless Laplacian of graphs. In order to give motivation for such a choice of matrix, we introduce some notions and present some relevant computational results.

We call the graphs having the same spectrum for an associated matrix M as cospectral graphs with respect to M or M-cospectral graphs. A cospectral mate of G is a graph which is cospectral with G, but not isomorphic to G. Let $\mathcal{G}$ be a finite set of graphs, and $\mathcal{G}'$ be the set of graphs in $\mathcal{G}$ which have a M-cospectral mate in $\mathcal{G}$. The spectral uncertainty of $\mathcal{G}$ with respect to M is $|\mathcal{G}'|/|\mathcal{G}|$.

For the sets of all graphs on n vertices, we denote the spectral uncertainties with respect to the adjacency matrix, Laplacian matrix, and signless Laplacian matrix by a_n, l_n, and s_n, respectively. The papers [22, 39] provide the values of a_n, l_n, and s_n for the sets of all graphs on n vertices where $n \leq 11$:

n	4	5	6	7	8	9	10	11
a_n	0	0.059	0.064	0.105	0.139	0.186	0.213	0.211
l_n	0	0	0.026	0.125	0.143	0.155	0.118	0.090
s_n	0.182	0.118	0.103	0.098	0.097	0.069	0.053	0.038

TABLE 1.1

Observe that $s_n \leq a_n, l_n$ for $n \geq 7$. Also, the sequence s_n is decreasing, for $n \leq 11$ while the sequence a_n is increasing for $n \leq 10$.

This provides a strong basis for believing that studying graphs via their Q-spectra is more efficient than studying them by their adjacency spectra, particularly for identifying graphs based on their spectrum. Moreover, the signless Laplacian spectrum performs better in determining graphs by their spectrum (known as DS graphs) compared to the spectra of other commonly used graph matrices, such as the Laplacian and the Seidel matrix [39]. Hence, an idea expressed in [39] suggests that the signless Laplacian seems to be the most convenient matrix for studying graph properties among those associated with a graph. This suggestion was widely accepted in 2005 [7], at a time when there were almost no results in the literature on the spectra of the signless Laplacian. Since then, the signless Laplacian has attracted considerable attention from researchers. We present a few relevant and interesting results here.

In [10], the authors proved that G is a simple regular graph if and only if $Q(G)$ has an eigenvector having all coordinates equal to 1. Besides, they also proved that the number of distinct Q-eigenvalues of a simple graph G is at least $d+1$ if G is a connected graph with diameter d.

Suppose G is a simple connected graph having n vertices and m edges. In [11], the authors showed that

$$q(G) \leq \frac{2m}{n-1} + n - 2$$

with equality if and only if G is star graph $K_{1,n-1}$ or complete graph K_n. If minimum vertex degree of G is δ, then $\lambda_{\min}(Q(G)) < \delta$ [14].

Next, we present some results on the Q-spectrum required for this thesis.

Proposition 1.7. *([5], Proposition 1.3.9). The number of connected bipartite components of a simple graph G is equal to the multiplicity of the Q-eigenvalue 0 in G.*

Proposition 1.8. *([32], Proposition 2.7). A simple connected graph G has $d_G(v) \leq \lceil q(G) - 1 \rceil$, for any $v \in V(G)$, where $q(G)$ is the Q-spectral radius of G. If G has a vertex v having $d_G(v) = q(G) - 1$, then $G = K_{1,q(G)-1}$.*

Theorem 1.9. *([9],Theorem 4.7). Let G be a simple finite graph. Then*

$$\min(d_G(x) + d_G(y)) \leq q(G) \leq \max(d_G(x) + d_G(y))$$

where (x, y) runs over all pairs of adjacent vertices of G. For connected graph G, equality holds in either of these inequalities if and only if G is regular or semiregular bipartite.

For more information on signless Laplacian of graphs, see [6, 10, 11, 24].

We use $Q_p(H)$ to denote the principal submatrix of $Q(G)$ corresponding to the vertices of $H \subseteq V(G)$. The principal submatrix $Q_p(H)$ of the signless Laplacian $Q(G)$, corresponding to a subset $H \subseteq V(G)$, is defined by

$$Q_p(H)_{uv} = \begin{cases} d_G(u) & \text{if } \ u = v \\ 1 & \text{if } \ uv \in E(G) \\ 0 & \text{if } \ uv \notin E(G). \end{cases}$$

For two distinct vertices $u, v \in V(G)$, the (u, v)-th entry of $A(G)$ is denoted by a_{uv}, and thus the (u, v)-th entry of $Q_p(H)$ is a_{uv} as well. We use $a_{\cdot\cdot}$ and $d_G(\cdot)$ in place of a_{uv} and $d_G(z)$ when u, v and z are suitable vertices within the context.

1.5 Q-integral Graph

A graph G is said to be M-integral if the matrix M has only integer eigenvalues. Originally, A-integral graphs have been studied, and the question about which graphs are integral dates back to Harary and Schwenk (1974) [23], who remarked that the general problem appeared intractable. For some results on graphs and integral graphs, we refer the readers to [3, 12, 18]. A-integral graphs are very rare. However, other kinds of integral graphs could be more frequent. For example, out of 112 simple connected graphs on 6 vertices, only 6 of them are A-integral [3], while 37 are L-integral [28]; and just 13 are Q-integral [8]. Since the complement of a simple L-integral graph is also L-integral, the number of simple L-integral graphs becomes high. Besides, there is no known corresponding relation between the characteristic polynomial of $A(G)$ and characteristic polynomial of $Q(G)$, which would preserve the property of being integral. However, it is well known that a simple graph is Q-integral if and only if its line graph is A-integral. A graph is said to be ALQ-integral if it is A-integral, L-integral, and Q-integral. If a graph is regular and A-integral, then it is also ALQ-integral. It is established by a computer search [38] that there are exactly 172 simple connected Q-integral graphs up to 10 vertices. Among them there exists exactly two ALQ-integral graphs; both are non-regular graphs on 10 vertices, however, one is bipartite and another one is non-bipartite. For every simple ALQ-integral graph G, the product $G\Box K_2$ is a bipartite ALQ-integral graph [6].

The wheel graph on n vertices is Q-integral if and only if $n = 4, 7$ [15]. For a simple connected Q-integral graph G with at most two vertices of degree larger than 2, we have $\lambda_{\min}(Q(G)) = 1$ [15]. Moreover, if G is a simple connected graph on n vertices having exactly one vertex of degree larger than 2, then G is Q-integral if and only if $G = K_{1,n-1}$ [15].

In this thesis, we focus our attention on Q-integral graphs. It is of interest for researchers to explore the structure of a graph under certain conditions or restrictions on edge-degree, and to identify a special graph class as an induced subgraph.

In 2008, Simić and Stanić [35] studied simple connected Q-integral graphs with e-$deg_G^{\max} \leq 5$. They gave a classification of simple connected Q-integral graphs with e-$deg_G^{\max} \leq 4$, and also gave results on possible parameters for the signless Laplacian spectra of edge-regular Q-integral graphs when the e-$deg_G = 5$. However, the complete characterization of edge-non-regular simple connected Q-integral graphs with e-$deg_G^{\max} = 5$ is still an unsolved problem. In 2019, the simple connected Q-integral graphs with e-$deg_G^{\max} \leq 6$ was studied by Park and Sano [32]. They gave a structural classification for such graphs G when $q(G) = 6$. It is interesting that a simple connected Q-integral graph with e-$deg_G^{\max} = q(G) = 6$ always contains a double star $\mathcal{S}_3^2$ as a subgraph. However, it was proved in [31] that there is no simple connected Q-integral bipartite graph having $\mathcal{S}_3^2$ as an induced subgraph.

It is quite surprising to observe that the double star $\mathcal{S}_r^2$ is always a subgraph of simple connected Q-integral graph with e-$deg_G^{\max} = 2r$, where $r = q(G) - 3$; $q(G) \in \{5, 6\}$, see [32, 35]. Eventually, a question arises about the existence of the double star $\mathcal{S}_r^2$ in a simple connected Q-integral graph with $e - \deg_G^{\max} = 2r$, where $r = q(G) - 3$.

Apart from simple graphs, graphs with self-loops (representing heteroatoms) are of evident chemical significance and were much studied in the past, including their spectral properties [1, 19, 34]. Moreover, graph energy has a direct connection with graph eigenvalues, and the graphs with integral Q-spectrum are of high interest now, which motivates us to observe how the addition of self-loops is going to affect the Q-integrality of a Q-integral graph.

1.6 Outline of the Thesis

We start with Q-integral graphs and then move our attention to graphs having integer-valued Q-spectral Radius. Finally, we end this thesis with the graph obtained by adding self-loops to a subset of vertices of a Q-integral graph. The rest of the thesis is organized as follows.

In ***Chapter 2***, we study simple connected Q-integral graphs G with e-$deg_G^{\max} \leq 8$ and give a structural classification when $q(G) = 7$. We have shown that $G \neq K_{1,4}\square K_2$ is bipartite containing one of the four special subgraphs $\mathcal{S}_4^2(m)(m = 0, 1, 2)$ (see Figure 1.2(c)) as an induced subgraph, if $q(G) = 7$ and e-$deg_G^{\max} = 8$.

In addition, we give a bound for e-$deg_G^{\max}$ in terms of $q(G)$ for simple connected graphs with integer-valued Q-spectral radius. Further, as an application, we show that the Q-spectral radius of any simple connected edge-non-regular graph G with $q(G) \in \mathbb{Z}$ and e-$deg_G^{\max} = 5$ is always 6 (which was unsolved earlier), and that there does not exist any simple connected edge-non-regular graph with integer-valued Q-spectral radius $q(G) \leq 4$.

In ***Chapter 3***, we study simple connected graphs with integral Q-spectral radius $q(G)$ and e-$deg_G^{\max} = 2q(G) - 6$. We give a necessary and sufficient condition for such a simple connected graph G to contain $\mathcal{S}_r^2, \mathcal{S}_r^{2,1}$, where $r = q(G) - 3$, as a subgraph. Using this condition, we also characterize simple connected graphs having $\lambda_{\min}(Q(G)) \notin (0,1)$ to contain only double star $\mathcal{S}^2_{q(G)-3}$ as a subgraph for $q(G) \in \{4, \ldots, 9\}$. In 2008, Simić and Stanić [35] showed that the only simple connected edge-non-regular Q-integral graph with e-$deg_G^{\max} = 4$ is $\mathcal{H}^*$ and $K_{1,2}\Box K_2$, see Figure 3.5. In this chapter, we extend this result by characterizing all such edge-non-regular simple connected graphs when G may not be Q-integral and instead have only $\lambda_{\min}(Q(G)) \notin (0,1)$, $q(G) \in \mathbb{Z}$.

In ***Chapter 4***, we prove that for a non-empty proper subset H of $V(K_{p,p})$, $K_{p,p}^H$ is Q-integral if and only if $p = 4$ and self-loops are added in exactly two vertices of each partite set, or $p = 2$ and self-loops are added to all the vertices of one partite set. Besides, we prove that if H is a non-empty proper subset of $V(K_{1,p})$, then $K_{1,p}^H$ is Q-integral if and only if $p = 4$ and self-loops are added to all the pendant vertices. Using this results, we also answer when $Spec(M+3D') \subseteq \mathbb{Z}$ where M is permutation similar to either $\begin{pmatrix} \mathrm{O}_p & \mathrm{J} \\ \mathrm{J} & \mathrm{O}_p \end{pmatrix}$ or $\begin{pmatrix} p & \mathrm{J} \\ \mathrm{J} & I_p \end{pmatrix}$, and D' is any $(0,1)$-diagonal matrix.

Finally, in ***Chapter 5***, we provide a summary and conclusions for the entire thesis, along with a discussion of the scope for future work.

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