

# Chapter 4

## Inductive algebras for the motion group of the plane

In this chapter we show that *each irreducible representation of the motion group of the plane has a unique maximal inductive algebra, and it is self-adjoint.*

In Section 4.1, we recall the structure and representations of the motion group of the plane and set up notation. The main result is proved in the Section 4.2.

### 4.1 Formulation

Recall (see [13, p155]) that the motion group of the plane is the semidirect product

$$M(2) = SO(2) \ltimes \mathbb{R}^2,$$

with the standard action of  $SO(2)$  on  $\mathbb{R}^2$ . The group  $M(2)$  may be identified with the group

$$\left\{ \left[ \begin{array}{cc} a & b \\ 0 & 1 \end{array} \right] \mid a, b \in \mathbb{C}, |a| = 1 \right\}.$$

Let  $G = M(2)$ . For each  $n \in \mathbb{Z}$ , let

$$\rho_n \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} = a^n,$$

then each  $\rho_n$  is a character of  $G$ . These, of course, have the trivial inductive algebra  $\mathbb{C}$ , which is self adjoint. Let  $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ . Recall (see [13, p157]), the irreducible representations other than the characters of  $G$  are parameterized by a nonzero real number  $\xi$  and act on the Hilbert space  $L^2(\mathbb{T})$  by the formula:

$$(\pi_\xi \left( \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \right) F)(z) = e^{i\xi \operatorname{Re}(b\bar{z})} F(a^{-1}z).$$

For each  $\varphi \in L^\infty(\mathbb{T})$ , let  $m_\varphi : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})$  be defined by  $m_\varphi(F) = \varphi F$ . Let

$$\mathcal{B} = \{m_\varphi \mid \varphi \in L^\infty(\mathbb{T})\}.$$

Then  $\mathcal{B}$  is a maximal-abelian subalgebra of  $\mathcal{B}(L^2(\mathbb{T}))$  (see [21, Prop 4.7.6]), and  $\mathcal{B}$  is  $\pi_\xi$ -inductive. Therefore  $\mathcal{B}$  is a maximal  $\pi_\xi$ -inductive algebra. Moreover, it is self-adjoint. Our main result is the following theorem.

**Theorem 4.1.1.**  *$\mathcal{B}$  is the only maximal  $\pi_\xi$ -inductive algebra.*

## 4.2 The proof

Let  $\mathcal{H} = L^2(\mathbb{T})$ . Define  $S : \mathcal{H} \rightarrow \mathcal{H}$  by

$$S(f)(z) = zf(z), \quad z \in \mathbb{C}.$$

We will show that the algebra  $\mathcal{A}$  is generated by  $S$  and  $S^{-1}$ . Since the same is true for  $\mathcal{B}$ , it follows that  $\mathcal{A} = \mathcal{B}$ .

Let  $\mathfrak{g}$  be the Lie algebra of  $G$ . Let  $\mathcal{H}^\infty$  be the space of all smooth vectors in  $\mathcal{H}$  for the representation  $\pi_\xi$ . Recall that  $v \in \mathcal{H}$  is smooth if the map  $g \mapsto \pi_\xi(g)v$  is a smooth mapping on  $G$ . The Lie algebra  $\mathfrak{g}$  acts on  $\mathcal{H}^\infty$  [2]. Observe that  $\mathfrak{g}$  is the span over  $\mathbb{R}$  of the matrices:

$$L = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}, \quad M_x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad M_y = \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}.$$

Let  $e_n(z) = z^n$ . Then the set  $\{e_n \mid n \in \mathbb{N}\}$  forms a basis for  $L^2(\mathbb{T})$ .

**Lemma 4.2.1.** *The operators  $d\pi_\xi(L)$ ,  $d\pi_\xi(M_x)$ , and  $d\pi_\xi(M_y)$  act on the basis vectors  $e_n$  as follows:*

$$\begin{aligned} d\pi_\xi(L)e_n &= -ine_n, \\ d\pi_\xi(M_x)e_n &= \frac{i\xi}{2}(e_{n+1} + e_{n-1}), \\ d\pi_\xi(M_y)e_n &= \frac{\xi}{2}(e_{n+1}(z) - e_{n-1}(z)), \quad z \in \mathbb{C}. \end{aligned}$$

*Proof.* We compute

$$\begin{aligned} (d\pi_\xi(L)e_n)(z) &= \left. \frac{d}{dt} \right|_{t=0} e_n(e^{-it}z) \\ &= -ine_n(z), \\ (d\pi_\xi(M_x)e_n)(z) &= \left. \frac{d}{dt} \right|_{t=0} e^{i\xi \operatorname{Re}(t\bar{z})} e_n(z) \\ &= i\xi \operatorname{Re}(\bar{z})e_n(z) \\ &= i\xi \operatorname{Re}(z)e_n(z) \\ &= \frac{i\xi}{2}(e_{n+1}(z) + e_{n-1}(z)), \quad \text{and} \end{aligned}$$

$$\begin{aligned}
(d\pi_\xi(M_y)e_n)(z) &= \left. \frac{d}{dt} \right|_{t=0} (e^{i\xi \operatorname{Re}(it\bar{z})} e_n)(z) \\
&= i\xi \operatorname{Re}(i\bar{z}) e_n(z) \\
&= i\xi \operatorname{Im}(z) e_n(z) \\
&= \frac{\xi}{2} (e_{n+1}(z) - e_{n-1}(z)). \quad \square
\end{aligned}$$

Extend  $d\pi$  to a  $\mathbb{C}$ -linear map on  $\mathfrak{g} \otimes \mathbb{C}$ . Let  $M = M_y + iM_x$  and  $\bar{M} = M_y - iM_x$ .

**Corollary 4.2.2.** *The operators  $d\pi_\xi(M)$  and  $d\pi_\xi(\bar{M})$  act on the basis vectors  $e_n$  as follows:*

$$d\pi_\xi(M)e_n = -\xi e_{n-1}, \quad \text{and} \quad d\pi_\xi(\bar{M})e_n = \xi e_{n+1}.$$

Let

$$K = \left\{ \left[ \begin{array}{cc} w & 0 \\ 0 & 1 \end{array} \right] \mid |w| = 1 \right\}.$$

Then  $K$  is a compact subgroup of  $G$ . Let  $dk$  denote the Haar probability measure on  $K$ . For  $m \in \mathbb{Z}$ , define  $\chi_m : K \rightarrow \mathbb{C}$  by

$$\chi_m \left( \left[ \begin{array}{cc} w & 0 \\ 0 & 1 \end{array} \right] \right) = w^m.$$

Let  $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$  be a  $\pi_\xi$ -inductive algebra. For each  $g \in G$  define a map  $\kappa(g) : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$  by  $T \mapsto \pi_\xi(g)T\pi_\xi(g)^{-1}$ . Let  $\mathcal{A}_m = \{T \in \mathcal{A} \mid \kappa(g)T = \chi_m(g)T, g \in K\}$ . For  $T \in \mathcal{B}(\mathcal{H})$ , define  $\Pi_m(T) = \int_K \pi_\xi(k)T\pi_\xi(k)^{-1} \overline{\chi_m(k)} dk$ , then  $\mathcal{A}_m = \Pi_m(\mathcal{A})$ . By the Peter-Weyl theorem  $\bigoplus_{m \in \mathbb{Z}} \mathcal{A}_m$  is sequentially dense in  $\mathcal{A}$  (see [11, Lemma 3.2]). Note that if  $T \in \mathcal{A}_m$  then  $Te_n = c_{m+n}e_{m+n}$  for some  $c_{m+n} \in \mathbb{C}$ . Let  $\mathcal{A}^\infty$  be the collection of all smooth vectors in  $\mathcal{A}$ . We use the following facts from Section 3 of [11]:

- $\mathcal{A}_m \cap \mathcal{A}^\infty$  is sequentially dense in  $\mathcal{A}_m$ .
- If  $T \in \mathcal{A}^\infty$  and  $F \in \mathcal{H}^\infty$  then  $TF \in \mathcal{H}^\infty$ .
- $\mathcal{A}^\infty$  is invariant under  $d\kappa(\mathfrak{g})$  as well as  $\kappa(G)$ .

For  $X \in \mathfrak{g}$ , let  $T_X = [T, d\pi_\xi(X)]$ . Because  $\mathcal{A}$  is inductive, it follows that  $T_X \in \mathcal{A}$  and since  $\mathcal{A}$  is abelian, it follows that

$$[T, T_X] = 0.$$

If  $T \in \mathcal{A}_m \cap \mathcal{A}^\infty$ , then by Lemma 4.2.1,

$$\begin{aligned}
T_L(e_n) &= [T, d\pi_\xi(L)](e_n) \\
&= T(d\pi_\xi(L))(e_n) - d\pi_\xi(L)(Te_n) \\
&= T(-ine_n) - c_{m+n}d\pi_\xi(L)(e_{n+m}) \\
&= -inc_{m+n}e_{m+n} + c_{m+n}i(m+n)e_{m+n} \\
&= imc_{m+n}e_{m+n}, \\
T_M(e_n) &= [T, d\pi_\xi(M)](e_n) \\
&= \xi(c_{m+n} - c_{m+n-1})e_{m+n-1}, \quad \text{and} \\
T_{\overline{M}}(e_n) &= [T, d\pi_\xi(\overline{M})](e_n) \\
&= \xi(c_{n+m+1} - c_{n+m})e_{n+m+1}.
\end{aligned}$$

**Lemma 4.2.3.**  $\mathcal{A}_0 = \mathbb{C}I$ , where  $I$  is the identity operator.

*Proof.* It is clear that  $\mathbb{C}I \subset \mathcal{A}_0$ . Let  $T \in \mathcal{A}_0$ . Then there exists  $c_n \in \mathbb{C}$  such that  $Te_n = c_n e_n$  for  $n \in \mathbb{Z}$ . Therefore

$$\begin{aligned} 0 &= [T, T_{\overline{M}}](e_n) \\ &= \xi(c_{n+1} - c_n)^2 e_{n+1}. \end{aligned}$$

It follows that  $c_{n+1} = c_n$  for all  $n \in \mathbb{Z}$ . □

Observe that

$$S(e_n) = e_{n+1}, \quad n \in \mathbb{Z}.$$

**Lemma 4.2.4.**  $\mathcal{A}_m = \mathbb{C}S^m$  for all  $m \in \mathbb{Z}$ .

*Proof.* The case  $m = 0$  is Lemma 4.2.3. Let  $T \in \mathcal{A}_1$ . Then there exists  $c_n \in \mathbb{C}$  such that  $Te_n = c_{n+1}e_{n+1}$  for  $n \in \mathbb{Z}$ . Since  $T_M \in \mathcal{A}_0$ , there exists  $k \in \mathbb{C}$  such that  $T_M = kI$ . Therefore  $T_M e_n = k e_n$  for all  $n \in \mathbb{Z}$ . Therefore

$$\begin{aligned} \xi(c_{n+1} - c_n)e_n &= k e_n, \\ \therefore \xi(c_{n+1} - c_n) &= k, \\ \therefore c_{n+1} &= c_n + \frac{k}{\xi}, \quad \forall n \in \mathbb{Z}. \end{aligned}$$

Therefore  $\{c_n\}$  is an arithmetic progression, i.e. there exists  $a \in \mathbb{C}$  such that  $c_n = a + \frac{k}{\xi}n$  for all  $n \in \mathbb{Z}$ . Since  $T$  is a bounded operator, it must be the case that  $k = 0$ . Thus there exists  $a \in \mathbb{C}$  such that  $c_n = a$  for all  $n \in \mathbb{Z}$ . Therefore  $T = aS$ . This proves the statement for  $m = 1$ . Now assume that the statement is true for  $m = N$ . Let  $T \in \mathcal{A}_{N+1}$ . Then there exists  $c_n \in \mathbb{C}$  such that  $Te_n = c_{N+n+1}e_{N+n+1}$  for  $n \in \mathbb{Z}$ .

Since  $T_M \in \mathcal{A}_N$ , there exists  $k \in \mathbb{C}$  such that  $T_M e_n = k e_{n+N}$  for all  $n \in \mathbb{Z}$ . Therefore

$$\begin{aligned}\xi(c_{N+n+1} - c_{N+n})e_{N+n} &= k e_{N+n}, \\ \therefore \xi(c_{N+n+1} - c_{N+n}) &= k, \\ \therefore c_{N+n+1} &= \frac{k}{\xi} + c_{N+n}, \quad \forall n \in \mathbb{Z}.\end{aligned}$$

Therefore  $\{c_n\}$  is an arithmetic progression, i.e. there exists  $a \in \mathbb{C}$  such that  $c_n = a + \frac{k}{\xi}n$  for all  $n \in \mathbb{Z}$ . Since  $T$  is a bounded operator, it must be the case that  $k = 0$ . Therefore there exists  $a \in \mathbb{C}$  such that  $c_n = a$  for all  $n \in \mathbb{Z}$ . Therefore  $T = aS^{N+1}$ . Therefore  $\mathcal{A}_{N+1} = \mathbb{C}S^{N+1}$ . By induction, it follows that  $\mathcal{A}_m = \mathbb{C}S^m$  for  $m > 0$ . The case  $m < 0$  is similar. Therefore the algebra  $\mathcal{A}$  is generated by the operator  $S$  and  $S^{-1}$ .

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