Chapter 6

Consensus problems in multi-agent systems

6.1 Introduction

In this chapter, we consider different types of consensus problems namely consensus of third-order dynamics multi-agent systems with acceleration and input constraints, a consensus of multi-agent systems with heterogeneous nodes under communication imperfections, synchronization of multi-agent systems with connected and disconnected switching topologies, synchronization of agents in networked systems (like Hopfield networks) with time-varying couplings, and the synchronization of multi-agent systems with underactuated agent dynamics. The objective of all these problems is to drive the agents at a common location in space (where their velocities and accelerations are zero in case of third-order dynamics).

We exploit the extended results on contraction theory using the framework of vector distances to synchronize the motion of agents in multi-agent systems. The proposed results reduce the complexity of proving the negative definiteness property of the largest eigenvalue or measure of the Jacobian matrix of the complex multi-agent systems. Specifically, it provides convergence analysis through some *comparison system*. As a result, the convergence analysis problem of the complicated MAS system simplifies into the convergence analysis of a much simpler *comparison system* and also the controller design for consensus is done using the convergence conditions of the *comparison system*. Hence, the procedure of designing consensus controls for multi-agent systems remains simple despite nonlinearities as compared to the existing literature.

The further part of the chapter follows with the exploitation of vector-based contraction theory to solve different types of consensus problems in multi-agent systems in Section 6.2. Examples with simulation results are illustrated in Section 6.3. Finally, a brief conclusion ends the chapter.

6.2 Vector based contraction theory for consensus problems in multi-agent systems

Consider a multi-agent system consisting of M smooth nonlinear differential systems coupled by some distributed control $u_i(t)$

$$\dot{x}_i = f(t, x_i) + g(t, x_i)u_i(t) \tag{6.1}$$

where $x_i \in \mathbb{R}^n$ represents the vector of states of agent i, $u_i \in \mathbb{R}$ termed as distributed control is the scalar input to the agent i and $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ are general smooth nonlinear functions. The distributed control u_i can be represented as

$$u_i = \psi(x_i, X_i) \tag{6.2}$$

where ψ is a general nonlinear function of x_i and X_i, X_i is the set of the states of the adjacent nodes of the i-th node represented as $X_i = \{x_j\}_{j \in N_i}$, where N_i indicates the set of indices of adjacent agents of agent i. For instance, this u_i can be written in the well known form of linear consensus control as $u_i = \sum_{j \in N_i} b_{ij}(x_j - x_i)$, where b_{ij} are the entries of weighted adjacency matrix, x_i is the agent i and $x_j, j \in N_i$ is the neighboring agent to the agent i, and further as a nonlinear consensus control: $u_i = h_{ij}(x_j - x_i)$, where h_{ij} is the coupling function that represents the nonlinear interactions between agents. The first part for the analysis of all the consensus problems is described by Lemma 6.1.

Lemma 6.1 *Assume the network of agents (6.1) as a connected network, that is, the network has at least one spanning tree. If the corresponding nodes of the network contract with each of the distributed control given in (6.2), then any pair of solution trajectories of the network nodes converge relative to one another.*

Proof: According to [54] (see Theorem 6), the contracting nodes of the connected network synchronize spontaneously. By our assumption that the network is connected and the distributed controls ensure that each node of the network contracts, then any pair of network nodes trajectories converge relative to one another. □

The rest part for the analysis is stated through the following result.

Theorem 6.2 *The corresponding nodes of the connected network (6.1) contract with each of the distributed control (6.2) if there exists the comparison system for the system (6.1) which is quasi-monotone non-decreasing and contracting. Furthermore, they remain contracting if the network is connected for all time.*

Proof: Consider the system (6.1) with u_i of the form (6.2). Assume that there exists a function ϕ which is quasi-monotone non-decreasing and a vector-valued norm as defined by (4.1). Suppose there exist u_i such that the derivative of squared vector-valued norm along the virtual dynamics of system (6.1) for the i-th node follows

$$\frac{d}{dt}\|\delta x_i\|_v^2 \;<<\; \phi(t, P\mathrm{diag}(\delta x_i(t))^2\mathbb{1}, x_i) \tag{6.3}$$

and the comparison system obtained from (6.3) is contracting. Then, the corresponding nodes of the network contract. Now, by the assumption that the network is connected and by Lemma 6.1, it is clear that the trajectories of the nodes of the network converge with respect to one another if every u_i makes corresponding nodes contracting. Hence, the nodes of the multi-agent system (6.1) converge with respect to each other with the control u_i of the form (6.2). □

6.2.1 Consensus problems in third-order dynamics multi-agent systems with acceleration and input constraints

Consider the third-order dynamics of the agents

$$\begin{aligned} \dot{r}_i &= s_i \\ \dot{s}_i &= q_i \\ \dot{q}_i &= u_i, \qquad\qquad i = 1, \cdots, n \end{aligned} \tag{6.4}$$

where $r_i = (x_i, y_i)$ denotes the position vector of the agent i in x and y direction, $s_i = (s_{ix}, s_{iy})$ is the velocity vector of the same agent i, $q_i = (q_{ix}, q_{iy})$ is the acceleration vector of the agent i and $u_i = (u_{ix}, u_{iy})$ represents the control vector for the agent i respectively. We assume that each agent (position) communicates with the agent (position) in its neighbor for all time.

Theorem 6.3 *Consider the multi-agent system (6.4) and assume that it is connected. Then, the agents achieve consensus with the following distributed control* u_i

$$\begin{aligned} u_{ix}(x_i, s_{ix}, q_{ix}, X_i) =& -\gamma_1\gamma_2 \tanh(x_i, X_i) - (\gamma_1 + \gamma_2(1+\gamma_1)) \tanh(s_{ix}) - (1+\gamma_1+\gamma_2) q_{ix} \\ u_{iy}(y_i, s_{iy}, q_{iy}, Y_i) =& -\gamma_1\gamma_2 \tanh(y_i, Y_i) - (\gamma_1 + \gamma_2(1+\gamma_1)) \tanh(s_{iy}) - (1+\gamma_1+\gamma_2) q_{iy}, \end{aligned} \tag{6.5}$$

and

$$\begin{aligned} \|q_i(t)\|_\infty &\leq \frac{(\gamma_1\gamma_2 + (\gamma_1+\gamma_2(1+\gamma_1)))}{1+\gamma_1+\gamma_2}, \\ \|u_i(t)\|_\infty &\leq 2(\gamma_1\gamma_2 + (\gamma_1+\gamma_2(1+\gamma_1))), \end{aligned}$$

holds if, $\|q_i(t_0)\|_\infty \leq \frac{(\gamma_1\gamma_2+(\gamma_1+\gamma_2(1+\gamma_1)))}{1+\gamma_1+\gamma_2}$, *where* $\gamma_1 > 0, \gamma_2 > 0$, (u_{ix}, u_{iy}) *is the control vector for the agent* i *in* x *and* y *direction and* X_i, Y_i *represent the sets of the states of the adjacent nodes of the* i*-th node denoted as* $X_i = \{x_j\}_{j\in N_i}$ *and* $Y_i = \{y_j\}_{j \in N_i}$ *with* N_i *as the set of indices of adjacent agents of agent* i *in* x *and* y *direction respectively.*

Proof: We have to first show that the control given above (6.5) makes each agent dynamics contracting. And further, Lemma 6.1 implies that the agents converge with respect to each other.

The analysis for each x and y direction dynamics can be done separately as the dynamics are decoupled in each direction. Now, observe the x direction dynamics with proposed control u_{ix}:

$$\begin{aligned} \dot{x}_i &= s_{ix} \\ \dot{s}_{ix} &= q_{ix} \\ \dot{q}_{ix} &= -\gamma_1\gamma_2 \tanh(x_i, X_i) - (\gamma_1 + \gamma_2(1+\gamma_1)) \tanh(s_{ix}) - (1+\gamma_1+\gamma_2) q_{ix}. \end{aligned} \tag{6.6}$$

We use vector-based contraction approach and aggregation procedure (discussed in Section 3.3) to show that the dynamics of each agent is contracting. Let us use the transformation $z_i = \mathcal{T}\zeta_i$, with

$$z_i \in \mathbb{R},\ \mathcal{T} = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix},\ \zeta_i = [x_i, s_{ix}, q_{ix}]^\top,$$

to obtain the following dynamics

$$\begin{aligned} \dot{z}_i = 2s_{ix} + q_{ix} + 3(-\gamma_1\gamma_2 \sum_{j\in N_i} \tanh(x_j - x_i) - (\gamma_1 + \gamma_2(1+\gamma_1)) \tanh(s_{ix}) \\ - (1+\gamma_1+\gamma_2) q_{ix}). \end{aligned}$$

On applying pseudoinverse of matrix $\mathcal{T}$, we obtain

$$\begin{aligned}\dot{z}_i =& -(0.286 + 0.643(\gamma_1+\gamma_2))z_i - 3\gamma_1\gamma_2 \sum_{j\in N_i} \tanh(0.143 z_i \\ & - 0.143 z_j) - 3(\gamma_1 + \gamma_2(1+\gamma_1))\tanh(0.071 z_i).\end{aligned}$$

Taking the virtual dynamics,

$$\begin{aligned}\delta\dot{z}_i =& -(0.286 + 0.643(\gamma_1+\gamma_2))\delta z_i - 0.429\gamma_1\gamma_2 \sum_{j\in N_i} \mathrm{sech}^2 \\ & (0.143 z_i - 0.143 z_j)\delta z_i - 0.213(\gamma_1+\gamma_2(1+\gamma_1))\mathrm{sech}^2(0.071 z_i)\delta z_i.\end{aligned} \tag{6.7}$$

Now, the derivative of the squared virtual distance along system (6.7) is given by

$$\begin{aligned}\frac{d}{dt}(\delta z_i)^2 \leq & -2((0.286 + 0.643(\gamma_1+\gamma_2)) + 0.429\gamma_1\gamma_2 \sum_{j\in N_i} \mathrm{sech}^2(0.143 z_i - 0.143 z_j) \\ & + 0.213(\gamma_1+\gamma_2(1+\gamma_1))\mathrm{sech}^2(0.071 z_i))\delta z_i^2.\end{aligned} \tag{6.8}$$

There is no method which directly finds solutions of this inequality, so in order to estimate its solution, we have to apply the comparison result as described in Theorem 4.2. For this to happen, some comparison system from this inequality is constructed which follows the property of quasi-monotone non-decreasing (positive system). This property ensures that the solutions of the comparison system are non-negative for non-negative initial conditions in order to make the comparison feasible. To further prove that the nodes of the original system converge with respect to each other (contracting), comparison system must also be contracting since if the condition $P\ (\mathrm{diag}(\delta x_0))^2 \mathbb{1} << u_0$, satisfies, then $P\ (\mathrm{diag}(\delta x(t)))^2 \mathbb{1} << R(t)$ holds for all $t \geq t_0$ (as described in Theorem 4.2). Now, the comparison system obtained from the above inequality (6.8) is quasi-monotone non-decreasing and contracting if $\gamma_1 > 0$ and $\gamma_2 > 0$. Hence, from Theorem 4.2, the original agent dynamics (6.6) is contracting. In a similar way, we can prove the contraction of the agents for the y direction dynamics. Further applying the results of Lemma 6.1, it is proved that the nodes of system (6.4) converge with respect to each other. Hence, the agents achieve consensus as they indeed converge to an equilibrium point $x = \bar{x}, y = \bar{y}, s_x = 0, s_y = 0, q_x = 0, q_y = 0$, where $(\bar{x}, \bar{y})$ is a constant point in the space.

Now, we prove that acceleration and control input are constrained. Firstly, consider the x-direction dynamics (6.6). Let $u'_{ix} = -\gamma_1\gamma_2 \tanh(x_i, X_i) - (\gamma_1 + \gamma_2(1+\gamma_1))\tanh(s_{ix})$, then, one can say that $\|u_{ix}(t)\|_\infty \leq (\gamma_1\gamma_2 + (\gamma_1 + \gamma_2(1+\gamma_1)))$. From (6.4) and (6.5), $\dot{q}_{ixk} = -(1+\gamma_1+\gamma_2)q_{ixk} + u'_{ixk}$, for each dimension k of q_{ix}. It follows that

$$q_{ixk}(t) \leq \frac{(\gamma_1\gamma_2 + (\gamma_1 + \gamma_2(1+\gamma_1)))}{(1+\gamma_1+\gamma_2)}.$$

Suppose that it is not true. We prove it by contradiction. Let there exist $t_0 \leq t' < t''$ such that $q_{ixk}(t') = \frac{(\gamma_1\gamma_2+(\gamma_1+\gamma_2(1+\gamma_1)))}{(1+\gamma_1+\gamma_2)}$ and $q_{ixk}(t'') > \frac{(\gamma_1\gamma_2+(\gamma_1+\gamma_2(1+\gamma_1)))}{(1+\gamma_1+\gamma_2)}$ on $t \in (t', t'']$. From the mean value theorem, let there exists $t' < \bar{t} < t''$ such that $\dot{q}_{ixk}(\bar{t}) = \frac{(q_{ixk}(t'')-q_{ixk}(t'))}{(t''-t')} > 0$.

However, $\dot{q}_{ixk}(\bar{t}) = -(1+\gamma_1+\gamma_2)q_{ixk}(\bar{t}) + u'_{ixk} < 0$, since, $q_{ixk}(\bar{t}) > \frac{(\gamma_1\gamma_2+(\gamma_1+\gamma_2(1+\gamma_1)))}{(1+\gamma_1+\gamma_2)}$, which is a contradiction. Therefore, $q_{ixk}(t) \leq \frac{(\gamma_1\gamma_2+(\gamma_1+\gamma_2(1+\gamma_1)))}{(1+\gamma_1+\gamma_2)}$ holds for all $t \geq t_0$. In a similar way, it can be proved that $-\frac{(\gamma_1\gamma_2+(\gamma_1+\gamma_2(1+\gamma_1)))}{(1+\gamma_1+\gamma_2)} \leq q_{ixk}(t)$. Hence, $\|q_{ix}(t)\|_\infty \leq \frac{(\gamma_1\gamma_2+(\gamma_1+\gamma_2(1+\gamma_1)))}{(1+\gamma_1+\gamma_2)}$. Also, it follows that $\|u_{ix}(t)\|_\infty \leq 2((\gamma_1\gamma_2 + (\gamma_1+\gamma_2(1+\gamma_1))))$, since,

$$\|u_{ix}(t)\|_\infty \leq (1+\gamma_1+\gamma_2)\|q_{ix}(t)\|_\infty + (\gamma_1\gamma_2 + (\gamma_1+\gamma_2(1+\gamma_1))).$$

Similar result follows for y-direction dynamics. This completes the proof. □

6.2.2 Consensus of multi-agent systems with heterogeneous nodes under communication imperfections

The loss of communication is a very frequent problem in multi-agent systems. In the following part, we analyze a multi-agent system with heterogeneous nodes, for instance, the network comprises of some nodes with third-order and others with second-order dynamics under communication imperfections.

Consider a heterogeneous multi-agent system with n number of agents where m number of agents are represented by third-order dynamics

$$\begin{aligned} \dot{r}_i(t) &= s_i(t) \\ \dot{s}_i(t) &= q_i(t) \\ \dot{q}_i(t) &= u_i(t), \qquad i \in I_m \end{aligned} \tag{6.9}$$

where $r_i \in \mathbb{R}$, $s_i \in \mathbb{R}$, $q_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the position, velocity, acceleration and control respectively for the agent i and $I_m = \{i = 1, \ldots, m\}$. Suppose $r_i(0) = r_{i0}$, $s_i(0) = s_{i0}$ and $q_i(0) = q_{i0}$ as the initial conditions. And $\{m+1, \ldots, n\}$ nodes are represented by second-order dynamics

$$\begin{aligned} \dot{r}_p(t) &= s_p(t) \\ \dot{s}_p(t) &= u'_p(t), \qquad p \in \{m+1, \ldots, n\} \end{aligned} \tag{6.10}$$

where $r_p \in \mathbb{R}$, $s_p \in \mathbb{R}$ and $u'_p \in \mathbb{R}$ are the position, velocity and control for the agent p respectively. We discuss the below consensus control formulation for the heterogeneous system (6.9)-(6.10) under intermittent (imperfect) communication. We assume that each agent (position) communicates with the agent (position) in its neighbor for all time. By communication imperfections, we mean that there is loss of communication between agents at some time instants. The controls u_i and u'_p are fully distributed controls since local information between agents is needed to implement these controls.

Theorem 6.4 *Consider the system (6.9)-(6.10). Assume that this system is represented by a graph G which is connected. Then, the multi-agent system (6.9)-(6.10) achieve consensus with the distributed controls u_i and u'_p*

$$\begin{aligned} u_i(r_i, s_i, q_i, R_i) &= -\gamma_1\gamma_2\alpha_1(r_i, R_i) - (\gamma_1 + \gamma_2(1+\gamma_1))s_i - (1+\gamma_1+\gamma_2)q_i \\ u'_p(r_p, s_p, R_p) &= -\gamma'_1\alpha_2(r_p, R_p) - (\gamma'_1 + 1)s_p \end{aligned}$$

under communication imperfections, where α_1 and α_2 are general nonlinear functions, $\gamma_1 > \frac{1}{2}$, $\gamma_2 > 0$, $\gamma'_1 > 0$ and R_i denotes the set of the states of the adjacent nodes of the i-th node denoted as $R_i = \{r_j\}_{j\in N_i}$ with N_i as the set of indices of adjacent agents of agent i. Similarly, R_p denotes the set of the states of the adjacent nodes of the p-th node denoted as $R_p = \{r_j\}_{j\in N_p}$ with N_p as the set of indices of adjacent agents of agent p.

Proof: Consider the m number of agents modeled by third-order dynamics (6.9). We use the backstepping approach with vector-based contraction framework to design the control u_i for the system (6.9). To start with the first subsystem, we assume the virtual control, $s_i = \psi(r_i) = -r_i$ to obtain the subsystem $\dot{r}_i = -r_i$ to be globally asymptotic stable. Next, we assume the deviation variable as $z_{1i} = s_i + r_i$, to obtain the structure of the form

$$\begin{aligned} \dot{r}_i &= -r_i + z_{1i} \\ \dot{z}_{1i} &= -r_i + z_{1i} + q_i \end{aligned} \tag{6.11}$$

Now we design virtual control, $q_i = r_i - (\gamma_1+1)z_{1i}, \gamma_1 > 0$ for the second subsystem (6.11) to obtain $\dot{z}_{1i} = -\gamma_1 z_{1i}$ to be globally asymptotic stable. We assume second deviation variable as $z_{2i} = q_i - r_i + (\gamma_1+1)z_{1i}$, to obtain the following structure

$$\begin{aligned} \dot{r}_i &= -r_i + z_{1i} \\ \dot{z}_{1i} &= -\gamma_1 z_{1i} + z_{2i} \\ \dot{z}_{2i} &= r_i - (1+\gamma_1+\gamma_1^2)z_{1i} + (\gamma_1+1)z_{2i} + u_i \end{aligned} \tag{6.12}$$

Now, consider the $n-m$ number of agents modeled by second-order dynamics. We apply the same backstepping procedure to this system, then, we obtain

$$\begin{aligned} \dot{r}_p &= -r_p + z_{1p} \\ \dot{z}_{1p} &= -r_p + z_{1p} + u'_p \end{aligned} \tag{6.13}$$

Let the squared vector-valued norm, defined by (4.1), assuming the matrix P as diag($\mathbb{1}$) for the system (6.12)-(6.13) be: $\|\delta x_i\|_v^2 = [\delta r_i^2, \delta z_{1i}^2, \delta z_{2i}^2, \delta r_p^2, \delta z_{1p}^2]^\top$. Then, the rate of this squared vector-valued norm along the virtual dynamics of the system (6.12)-(6.13) is given by

$$\begin{aligned} \frac{d}{dt}(\delta r_i)^2 &\leq -\delta r_i^2 + \delta z_{1i}^2 \\ \frac{d}{dt}(\delta z_{1i})^2 &\leq (-2\gamma_1 + 1)\delta z_{1i}^2 + \delta z_{2i}^2 \\ \frac{d}{dt}(\delta z_{2i})^2 &= 2\delta z_{1i}(\delta r_i - (1+\gamma_1+\gamma_1^2)\delta z_{1i} + (\gamma_1+1)\delta z_{2i} + \delta u_i) \\ \frac{d}{dt}(\delta r_p)^2 &\leq -\delta r_p^2 + \delta z_{1p}^2 \\ \frac{d}{dt}(\delta z_{1p})^2 &= 2\delta z_{1p}(-\delta r_p + \delta z_{1p} + \delta u'_p) \end{aligned}$$

We select $\delta u_i = -\delta r_i + (1+\gamma_1+\gamma_1^2)\delta z_{1i} - (1+\gamma_1+\gamma_2)\delta z_{2i}, \gamma_1 > 0, \gamma_2 > 0$ and $\delta u'_p = \delta r_p - (\gamma'_1 + 1)\delta z_{1p}$, $\gamma'_1 > 0$ to obtain the following comparison system

$$\begin{aligned} \dot{w}_1 &= -w_1 + w_2 \\ \dot{w}_2 &= (-2\gamma_1 + 1)w_2 + w_3 \\ \dot{w}_3 &= -2\gamma_2 w_3 \\ \dot{w}_4 &= -w_4 + w_5 \\ \dot{w}_5 &= -2\gamma'_1 w_5 \end{aligned}$$

which is quasi-monotone non-decreasing (off-diagonal entries non-negative) and contracting when $\gamma_1 > \frac{1}{2}$, as a result, the system (6.9)-(6.10) is contracting. We obtain control u_i by substituting $\alpha_1(r_i, R_i)$ in place of δr_i and similarly u'_p, since we assume that the agent communicates its position state with its neighbor. Now, assume that there is communication loss between agents at some time instants for some durations, lets say $t_k \leq t \leq t_{k'}$, k, k' are positive constants and each agent is contracting with the convergence rate λ when there exists communication (interaction) between agents. Since the convergence rate is zero for the durations $t_k \leq t \leq t_{k'}$ and the overall convergence rate is given by $\max\{0, \lambda\}$, hence, it is concluded that the agent dynamics converge with the convergence rate λ for all time $t \geq t_0$. Further, applying the results of Lemma 6.1, it is proved that the agents of the system (6.9)-(6.10) converge with respect to each other. Hence, the agents achieve consensus as they indeed converge to an equilibrium point $r = \bar{r}, s = 0, q = 0$, where $\bar{r}$ is a constant point on a line. $\square$

In a similar manner, one can easily extend this result to the consensus of higher-order dynamics heterogeneous multi-agent systems.

6.2.3 Synchronization of multi-agent systems with connected and disconnected switching topologies

Consider a network of agents as a graph G with switching topologies

$$\dot{x}_i(t) = f(x_i) + \sum_{j=1}^{M} L_{ij}^{s(t)} x_j, \qquad i = 1, 2, \ldots, M \tag{6.14}$$

where $x_i \in \mathbb{R}^n$ indicates the vector of states of agent i, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear smooth vector field, $s(t) : [0, \infty] \to \bar{Q} = \{1, 2, \ldots, q\}$ represents the piecewise continuous switching signal which tells about the topology switching between the intervals with $\bar{Q}$ finite. Laplacian matrix $L^{s(t)} = L_{ij}^{s(t)}$ is the $M \times M$ matrix related to the switching function $s(t)$.

Theorem 6.5 *Consider a system (6.14) as a graph G which is $\sigma-$jointly connected. Suppose there exist $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ as a quasi-monotone non-decreasing function and a squared vector-valued norm $\|\delta x_i\|_v^2 : \mathbb{R}^n \to \mathbb{R}^n$, defined by (4.1), such that the derivative of squared vector-valued norm along the virtual dynamics of (6.14) follows*

$$\frac{d}{dt}(\|\delta x_i\|_v^2) << \phi(P diag(\delta x_i(t))^2 \mathbb{1}, x_i)$$

and the switching topologies are connected or disconnected for all time. Further, assume that the maximal solution $R(t)$ of the following comparison system

$$\dot{w}_i = \phi(w_i, z_i), \qquad w_i(t_0) = w_{0i} \geq\geq 0 \tag{6.15}$$

converges with the rate λ_k for $k \in s(t)$. Then, any solution $\delta x_i(t)$ which holds

$$P \ (diag(\delta x_i(t_0)))^2 \mathbb{1} << w_{0i}$$

follows that the infinitesimal distance $\|\delta x_i\|$ between trajectories of the system (6.14) converges fast with a minimum rate of

$$p^* = \min_{k\in s(t)} \lambda_k$$

for any switching signal $s(t)$ taken arbitrarily. In other way, $\|\delta x_i(t)\| \leq \|\delta x_i(t_0)\| e^{-p^* t}$.

Proof: Consider the system (6.14) and taking the virtual dynamics of the system

$$\delta \dot{x}_i(t) = \frac{\partial f(x_i)}{\partial x_i}\delta x_i + \sum_{j=1}^{M} L_{ij}^{s(t)} \delta x_j. \tag{6.16}$$

From the definition of the differential equations, it is clear that $x(t)$ is continuous for all t. Let the squared vector-valued norm, defined by (4.1), assuming the matrix P as diag($\mathbb{1}$). Thus, its derivative along the dynamics (6.16) is given by

$$\frac{d}{dt}(\|\delta x_i\|_v^2) << \Big(2\frac{\partial f(z_i)}{\partial x_i} + \sum_{j=1}^{M} |L_{ij}^{s(t)}|\Big)\delta x_i^2 + \sum_{j=1}^{M} |L_{ij}^{s(t)}|\delta x_j^2.$$

Since, by the assumption that the graph G is $\sigma-$jointly connected, then in every time interval $[t, t+\tau]$ of duration τ, there exists a graph G_k (current graph) such that the above equation follows

$$\frac{d}{dt}\|\delta x_i\|_v^2 << \phi(\text{diag}(\delta x_i(t))^2 \mathbb{1}, x_i), \tag{6.17}$$

where ϕ is a quasi-monotone non-decreasing function. Moreover, the comparison system (6.15) converges with the rate λ_k, so if $(\text{diag}(\delta x_i(t_0)))^2 \mathbb{1} << w_i(t_0)$, then from Theorem 4.2,

$$\|\delta x_i(t)\|^2 \;<<\; w_i(t_0)\exp(-\lambda_k t).$$

As the switching signal is piecewise continuous in t, then we can write $p^* = \min_{k\in s(t)} \frac{\lambda_k}{2}$. As a result

$$\|\delta x_i(t)\| \leq \|\delta x_i(t_0)\| \exp(-p^* t).$$

Hence, it is proved that the solution trajectories of the system (6.14) converge with respect to each other for the switching signal taken arbitrarily. □

6.2.4 Synchronization in networked systems with time-varying couplings

Consider the networked model of Hopfield Network (HN) with [115]

$$\dot{x}_i = -\frac{x_i}{R} + \sum_{j \in N_i} (g_{ij}(t, x_j) - g_{ij}(t, x_i)) + u_i(t) \tag{6.18}$$

where $i = 1, \ldots, 6$, $x_i \in \mathbb{R}$ denotes the neural voltages, $g : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ is the coupling function, R is the resistance, N_i represents the set of indices of the neighbors of each node i, and $u_i(t) \in \mathbb{R}$ is the periodic input to each of the nodes.

Now, the virtual dynamics of the system (6.18),

$$\delta\dot{x}_i = -\frac{1}{R}\delta x_i - \frac{\partial g_{ij}}{\partial x_i}\delta x_i. \tag{6.19}$$

Thus, the rate of squared virtual distance along the dynamics (6.19) is given by,

$$\frac{d}{dt}(\delta x_i^2) = -\frac{2}{R}\delta x_i^2 - 2\frac{\partial g_{ij}}{\partial x_i}\delta x_i^2. \tag{6.20}$$

The coupling functions g_{ij} are designed in order to obtain the following comparison system (6.21)

$$\dot{w}_i = \phi(t, w_i, x_i) \tag{6.21}$$

contracting, where ϕ is a quasi-monotone non-decreasing function in w_i. Thus, the agents of the original network system (6.18) converge with respect to one another.

6.2.5 Synchronization control design for multi-agent systems with underactuated agent dynamics

Consider the multi-agent system consisting n number of agents, where each agent has the following underactuated dynamics

$$\begin{aligned}
\dot{x}_i &= 0.8s_{1i} - z_i s_{2i} - cx_i \\
\dot{y}_i &= z_i s_{1i} + 0.8s_{2i} - cy_i \\
\dot{z}_i &= s_{3i} - cz_i \\
\dot{s}_{1i} &= 0.513u_{1i} + 1.23s_{2i}s_{3i} - 1.54s_{1i} \\
\dot{s}_{2i} &= -0.8125s_{1i}s_{3i} - 5.42s_{2i} \\
\dot{s}_{3i} &= 16.67u_{2i} - 0.67s_{1i}s_{2i} - 0.833s_{3i}
\end{aligned} \tag{6.22}$$

for $i = \{1, \ldots, n\}$, x_i, y_i, z_i are the positions of the agent i, c is any positive constant, s_{1i}, s_{2i}, s_{3i} are the velocities of the same agent i and u_{1i} and u_{2i} are two control inputs. Let us apply the transformation $\zeta_i = \mathcal{T}\eta_i$, where

$$\mathcal{T} = \begin{bmatrix} 1 & 0 & 0 & 2 & 4 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{bmatrix}, \ \eta_i = [x_i, y_i, z_i, s_{1i}, s_{2i}s_{3i}]^\top$$

and $\zeta_i = [\zeta_{1i}, \zeta_{2i}]^\top$ to transform the system (6.22) into

$$\begin{aligned} \dot{\zeta}_{1i} =& -(0.217 + 0.0476c)\zeta_{1i} - 0.0076\zeta_{1i}\zeta_{2i} + 1.026u_{1i} \\ \dot{\zeta}_{2i} =& 0.041\zeta_{1i}\zeta_{2i} + 3.42\zeta_{1i} + (0.155 - 0.9358c)\zeta_{2i} + 16.67u_{2i} \end{aligned} \tag{6.23}$$

Let the squared vector-valued norm, defined by (4.1), assuming the matrix P as diag($\mathbb{1}$) for the system (6.23) be: $\|\delta\zeta\|_v^2 = [\delta\zeta_{1i}^2, \delta\zeta_{2i}^2]^\top$. Thus, the rate of squared vector-valued norm along the virtual dynamics of the system (6.23)

$$\begin{aligned} \frac{d}{dt}(\delta\zeta_{1i}^2) &= 2\delta\zeta_{i1}(-(0.217 + 0.0476c)\delta\zeta_{1i} - 0.0076(\delta\zeta_{1i}\zeta_{2i} + \zeta_{1i}\delta\zeta_{2i}) + 1.026\delta u_{1i}) \\ \frac{d}{dt}(\delta\zeta_{2i}^2) &= 2\delta\zeta_{2i}(0.041(\delta\zeta_{1i}\zeta_{2i} + \zeta_{1i}\delta\zeta_{2i}) + 3.42\delta\zeta_{1i} + (0.155 - 0.9358c)\delta\zeta_{2i} + 16.67\delta u_{2i}) \end{aligned}$$

On substituting δu_{i1} and δu_{2i} as

$$\begin{aligned} \delta u_{1i} &= \frac{1}{1.026}(0.0076(\delta\zeta_{1i}\zeta_{2i} + \zeta_{1i}\delta\zeta_{2i})) \\ \delta u_{2i} &= \frac{1}{16.67}(-0.041(\delta\zeta_{1i}\zeta_{2i} + \zeta_{1i}\delta\zeta_{2i}) - 3.42\delta\zeta_{1i} - 20\delta\zeta_{2i}). \end{aligned} \tag{6.24}$$

The comparison system obtained is

$$\begin{aligned} \dot{w}_{1i} &= -(0.217 + 0.0476c)w_{1i} \\ \dot{w}_{2i} &= -(0.9358c + 19.845)w_{2i} \end{aligned}$$

quasi-monotone non-decreasing and contracting if $c > 0$. Hence, from Theorem 4.2, each agent dynamics (6.22) is contracting with the controls obtained after integrating (6.24). Further, applying the results of Lemma 6.1, it is proved that all the nodes of the network synchronize with respect to each other.

6.3 Examples

Example 6.6 *Consider the system (6.4) with the initial position of the agents given in Fig. 6.1.*

The simulation results with the following nonlinear distributed controls are shown in Fig. 6.2 and Fig. 6.3 with $\gamma_1 = 2$ and $\gamma_2 = 2$.

$$
\begin{aligned}
u_{ix} = & -\gamma_1\gamma_2 \sum_{j\in N_i} \tanh(x_i - x_j) - (\gamma_1 + \gamma_2(1+\gamma_1))\tanh(s_{ix}) - (1+\gamma_1+\gamma_2)q_{ix} \\
u_{iy} = & -\gamma_1\gamma_2 \sum_{j\in N_i} \tanh(y_i - y_j) - (\gamma_1 + \gamma_2(1+\gamma_1))\tanh(s_{iy}) - (1+\gamma_1+\gamma_2)q_{iy}
\end{aligned}
\tag{6.25}
$$

where N_i is the set of indices of adjacent agents of agent i.

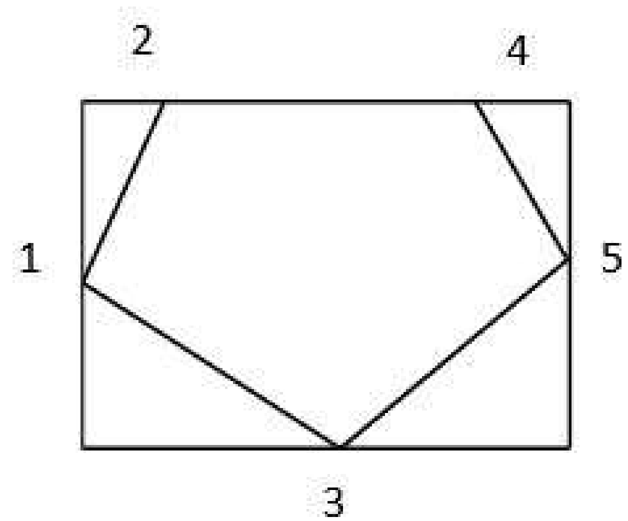

Figure 6.1: Initial positions of the agents of system (6.4)

The agents converge to the common point where their velocities and accelerations are zero, and further acceleration and control inputs are also constrained. This approach is different from the work in [82] in the way that it considers third-order dynamics of the agents with constrained acceleration to limit its high values, since high values may damage the structure of the aircraft.

Example 6.7 *Consider the graph in Fig. 6.4 of five nodes which is undirected and connected. Nodes $1, 3, 4$ are governed by third-order dynamics (6.9) and nodes $2, 5$ are governed by second-order dynamics (6.10). We assume $\gamma_1 = 2$, $\gamma_2 = 3$, $\gamma_1' = 4$, $r(0) = [3, -6, 4, 3, 5]$, $s(0) = [5, 4, -5, 3, -5]$ and $q(0) = [7, -6, -3]$. Further, we assume that the communication is lost at two time instants $t = 1$ sec and $t = 3$ sec for 0.3 sec and 0.5 sec durations respectively. The simulation results of the multi-agent system with heterogeneous nodes (6.9)-(6.10) controlled by the following nonlinear consensus controls are*

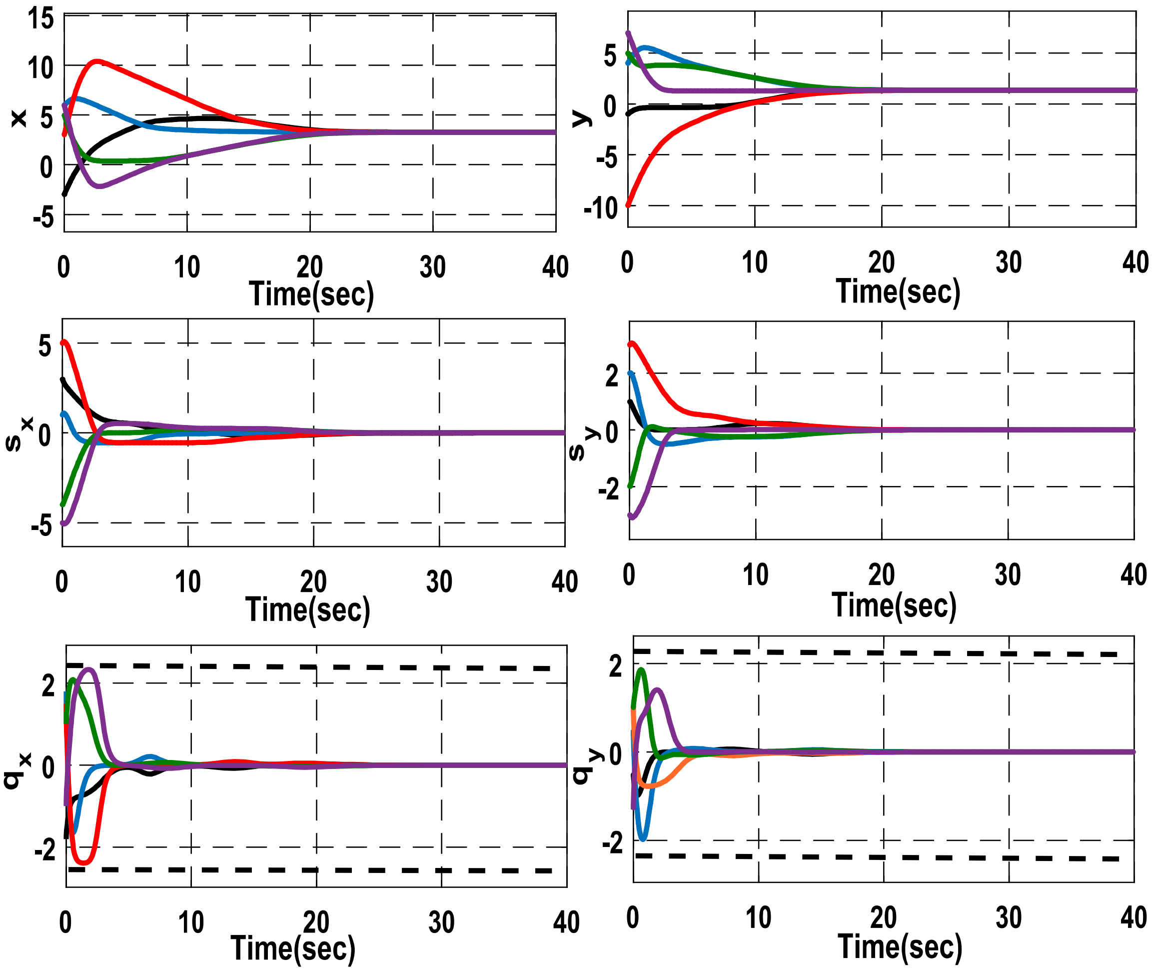

Figure 6.2: Solution trajectories of the states of the multi-agent system (6.4) controlled by (6.25)

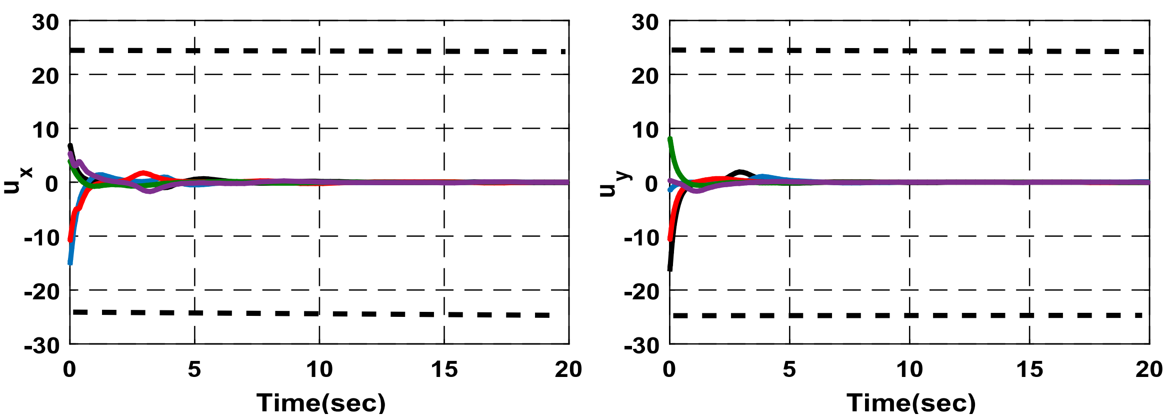

Figure 6.3: Control (6.25) for the multi-agent system (6.4)

shown in Fig. 6.5.

$$
\begin{aligned}
u_i = & -\gamma_1\gamma_2 \sum_{j\in N_i} exp(r_i - r_j) - (\gamma_1 + \gamma_2(1+\gamma_1))s_i - (1+\gamma_1+\gamma_2)q_i \\
u_i' = & -\gamma_1' \sum_{j\in N_i} \arctan(r_i - r_j) - (\gamma_1' + 1)s_i
\end{aligned} \tag{6.26}
$$

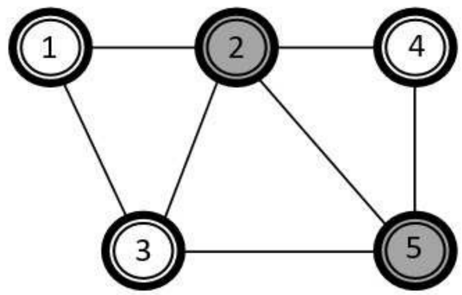

Figure 6.4: Graph with heterogeneous nodes for Example 6.7

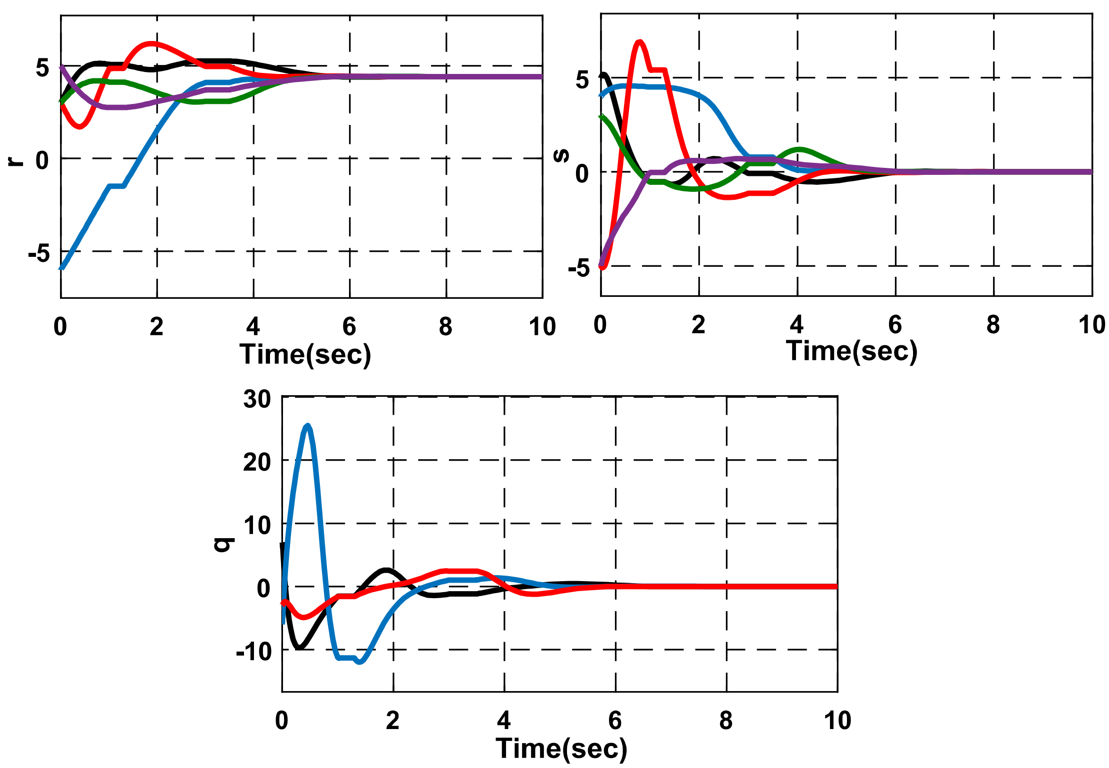

Figure 6.5: Solution trajectories of the states of the multi-agent system (6.9)-(6.10) controlled by (6.26)

Example 6.8 *Consider a nonlinear multi-agent system of five agents with each agent dynamics as*

$$\dot{x}_i = -x_i^5 - \arctan(x_i) + \sum_{j \in N_i} L_{ij}^{s(t)} x_j, \quad i = 1, \ldots, 5 \tag{6.27}$$

where $x_i \in \mathbb{R}$ is the position of agent i, N_i represents all the nodes that are neighbors to agent i and $s(t)$ is the aperiodic switching signal as shown in Fig. 6.6. The Laplacian matrix L_{ij} is selected according to the three possible switching topologies shown in Fig.

6.7. The simulation results of the system (6.27) are shown in Fig. 6.8 with $x_i(0) = \{-5, 8, 5, 6, -9\}$ *and* $\sigma = 5$*. The proposed framework offers a significant advantage over classical contraction in the sense that the calculation of the Jacobian matrix for the linear comparison system (obtained in this case) is much easier than the original multi-agent system (6.27).*

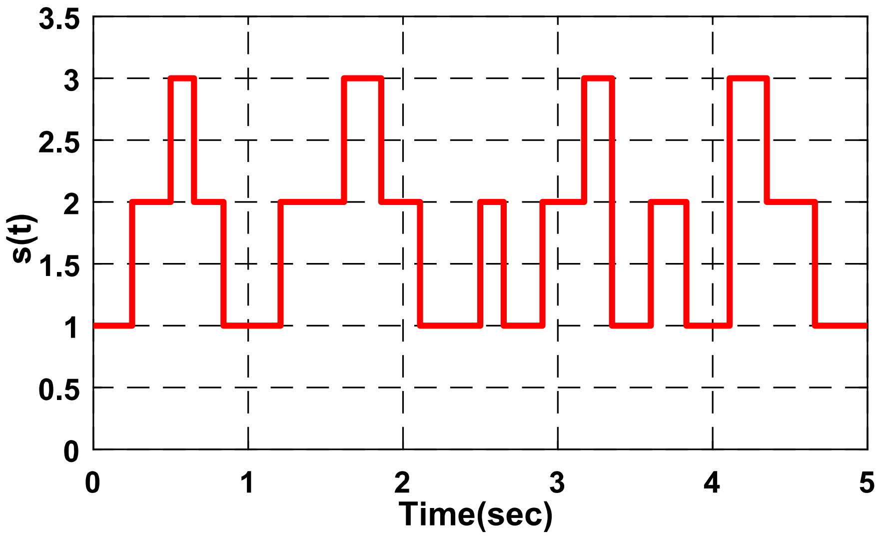

Figure 6.6: Piecewise continuous switching signal for Example 6.8

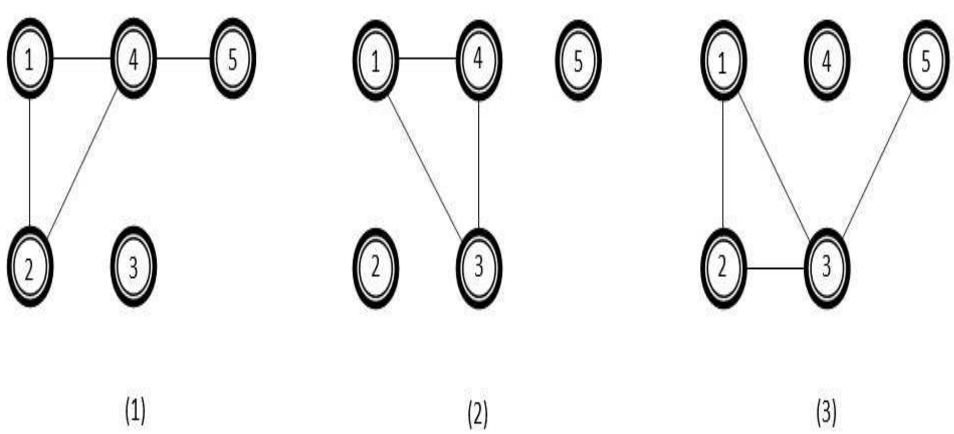

Figure 6.7: Three switching topologies for Example 6.8

Example 6.9 *Consider a network used for simulation of the model (6.18) in Fig. 6.9. The coupling functions* g_{ij} *given in Table 6.1 makes the original system (6.18) contracting with* $H = 0.5, R = 1$ *and* $K = 4$ *respectively, since the largest eigenvalue of the symmetric*

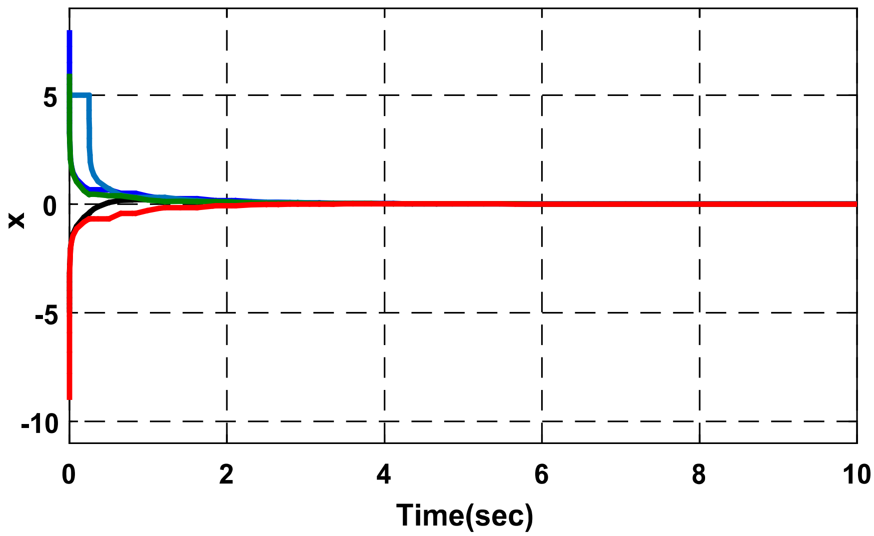

Figure 6.8: State evolutions of switching system (6.27)

part of the Jacobian matrix of the obtained linear comparison system is

$$
\begin{aligned}
\dot{w}_1 &= -9.5w_1 + 4w_2 + 4w_3 + 0.5w_6 \\
\dot{w}_2 &= 4w_1 - 5.5w_2 + 0.5w_4 \\
\dot{w}_3 &= 4w_1 - 9.5w_3 + w_4 + 4w_6 \\
\dot{w}_4 &= 0.5w_2 + 4w_3 - 4.5w_4 + w_5 \\
\dot{w}_5 &= w_4 - 5w_5 + 4w_6 \\
\dot{w}_6 &= 0.5w_1 + w_3 + 4w_5 - 4.5w_6
\end{aligned}
$$

negative definite. The simulation results are shown in Fig. 6.10 with $R = 1, H = 0.5, K = 4$ *and* $u(t) = 2\sin 3t$. *By applying the proposed procedure, we are able to get more arbitrary values of constants* H *and* K *and also remove the difficulty of proving the measure of the matrix for large networks to be negative definite as compared to the work in [86].*

Table 6.1: Coupling functions for network

Edge	g_{ij}
$1 \to 2, \quad 1 \to 3, \quad 3 \to 4, \quad 5 \to 6$	Kx
$1 \to 6, \quad 2 \to 4$	$H\cos^4(t)\arctan(x)$
$3 \to 6, \quad 4 \to 5$	$0.5\sin^2(t)\frac{1-e^{-x}}{1+e^{-x}}$

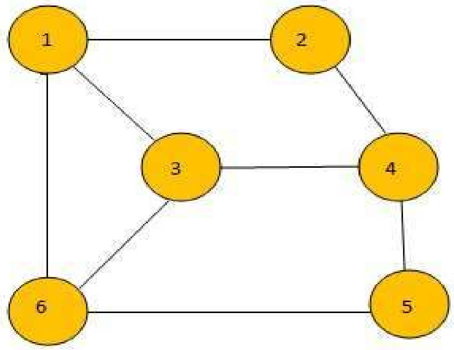

Figure 6.9: Network used for simulation of system (6.18)

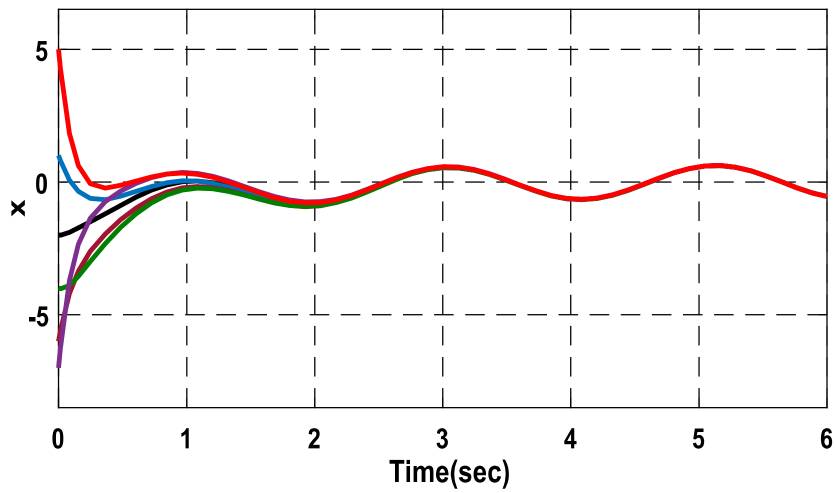

Figure 6.10: Evolution of all the states of the network (6.18)

Example 6.10 *Consider a graph of four agents which is undirected and connected in Fig. 6.11. Each of the agents is governed by the underactuated dynamics (6.22). We assume that each of the agents (position states) communicates with its neighbor (position states) for all time. The simulation outcomes of the multi-agent system with underactuated agent dynamics (6.22) with the control designed in Subsection 6.2.5 considering* $c = 2$*, are shown in Fig. 6.12 and Fig. 6.13.*

6.4 Conclusion

The general results of vector distance-based contraction theory are exploited to analyze the coordination of motion in multi-agent systems. The proposed procedure has the

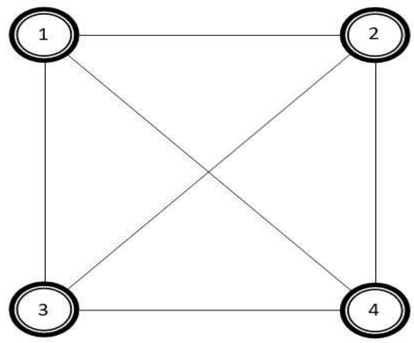

Figure 6.11: Undirected graph for the Example 6.10

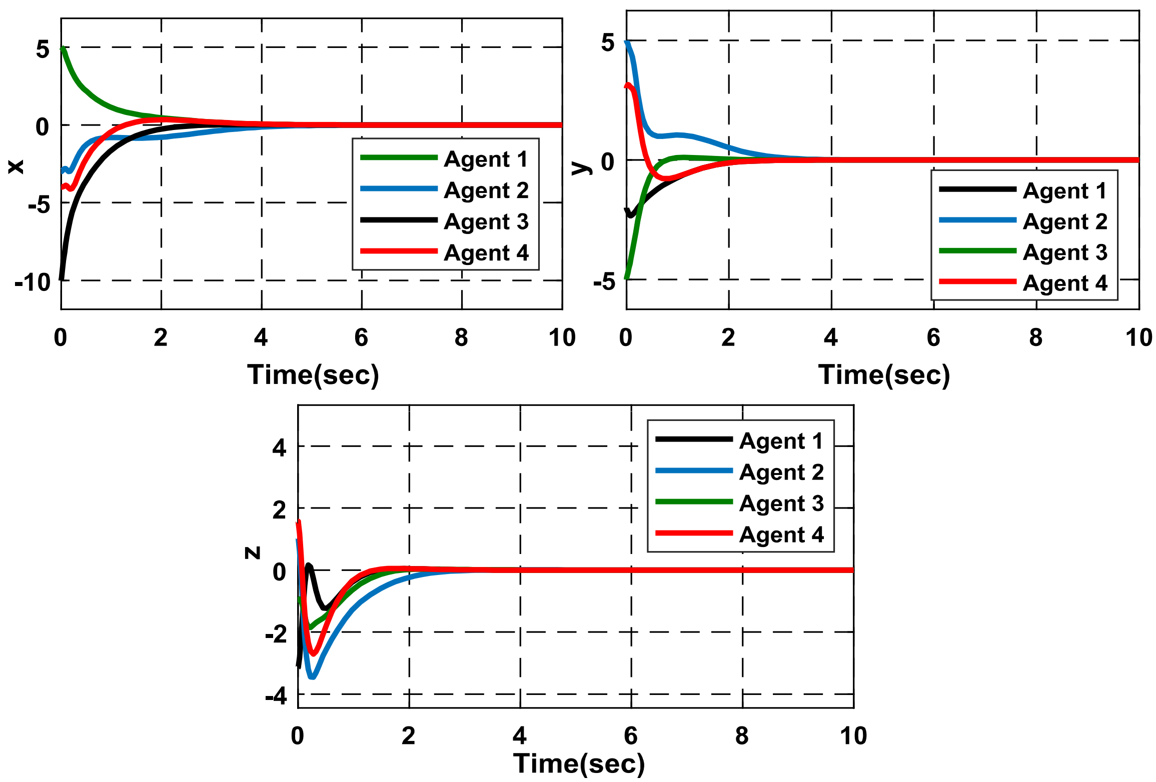

Figure 6.12: Evolution of position states with time

advantage that proving the contraction for the comparison system is much easier than for the original complex multi-agent system in the sense of the Jacobian matrix which is quite complex for complex (large-scale and nonlinear) multi-agent systems. The results are demonstrated on consensus problems: third-order consensus subjected to acceleration and input constraints, heterogeneous systems under communication imperfections, systems with connected and disconnected switching topologies, networked systems with time-varying couplings, multi-agent systems with underactuated dynamics.

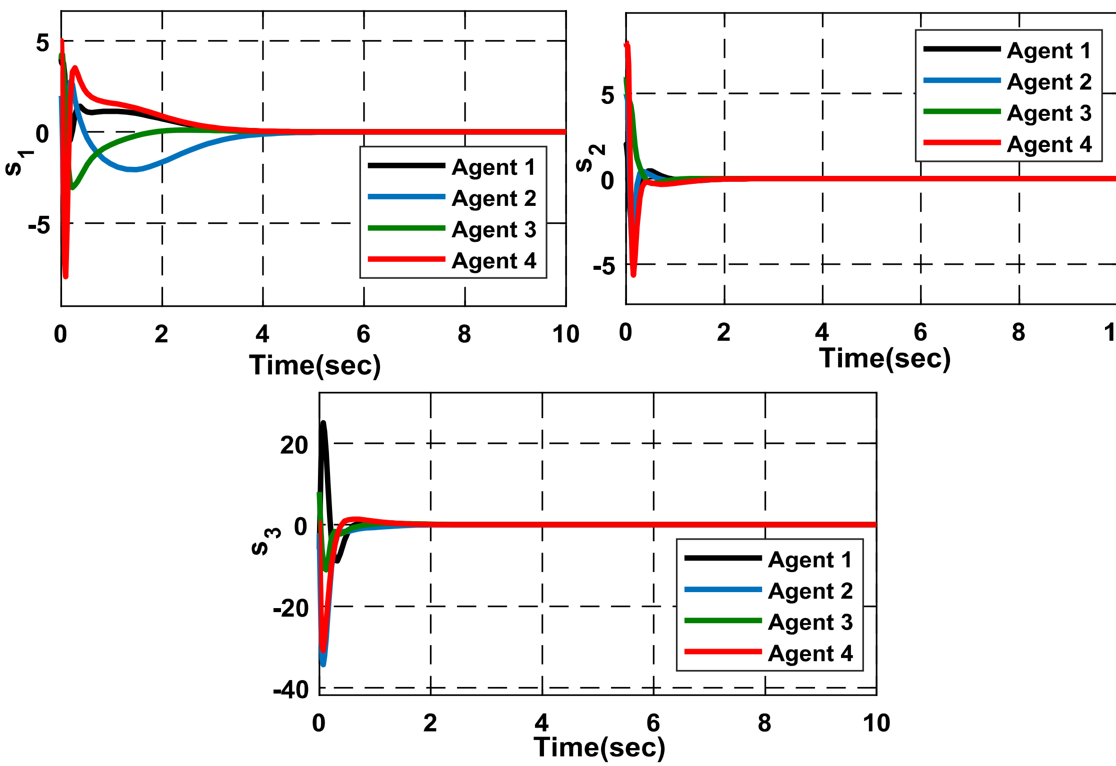

Figure 6.13: Evolution of agent velocities with time