Chapter 2

Coexistence of locally multistable equilibrium points for n-neuron delayed quaternion-valued neural networks with continuous piecewise nonlinear activation functions

2.1 Introduction

Quaternion valued neural network (QVNN) is neural network model whose state variables are quaternion numbers, and is an extension of real valued neural network (RVNN). Since all quaternions are four-dimensional, hence QVNNs are able to store four times the amount of information stored by RVNNs. Thus, due to their high storage capacity, QVNNs have gained immense popularity in the field of neural networks and their applications. QVNNs are extensively applied in classify images coming from radar [39], image processing [40], robot manipulator control[41], adaptive filtering [42], etc. Some applications such as associative memory and pattern recognition require systems with high storage capacity and the number of stable EPs of the systems plays an important role in improving the storage capacity. In this chapter, we analyze the coexistence of local multistable EPs for n-neuron delayed quaternion-valued neural networks (DQVNNs) with continuous piecewise nonlinear activation functions. Before giving the model description, some notations are given those are used throughout this chapter.

Notations: In the present chapter, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ denote, respectively, the set of real numbers, the set of complex numbers, and the set of quaternion numbers. $\mathbb{R}^{n\times m}$, $\mathbb{C}^{n\times m}$, and $\mathbb{H}^{n\times m}$ denote $n\times m$ matrices, whose entries belong to $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, respectively. Let $x=(x_1,x_2,...,x_n)^T\in\mathbb{R}^n$ denotes n-dimensional vector. $\mathcal{C}([-\tau,0],\mathbb{R}^n)$ represents the Banach space of continuous vector-valued functions which map the interval $[-\tau,0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. Let $\|x\|_\xi$ denotes the ξ- norm of x, with $\|x\|_\xi=\max_p\{\xi_p|x_p|\}$, where $\xi=(\xi_1,\xi_2,...,\xi_n)^T$ and $\xi_p>0$ for $p=1,2,...,n$. D^- represents the upper left Dini derivative operator.

2.2 Model Description

Let us consider the DQVNNs described by the following system of nonlinear delayed differential equations as

$$\dot{h}_p(t) = -a_p h_p(t) + \sum_{q=1}^{n} b_{pq} g_q(h_q(t)) + \sum_{q=1}^{n} c_{pq} g_q(h_q(t-\tau_q)) + k_q,$$
$$\text{for } p = 1, 2, ..., n \text{ and } t \geq 0, \tag{2.1}$$

where n is the number of neurons in the network; $h(t) = (h_1(t), h_2(t), ..., h_n(t))^T \in \mathbb{H}^n$, $h_p(t)$ represents the state variable of p-th neuron at a time t; $a_p > 0$ is the self-feedback connection weight; $b_{pq} \in \mathbb{H}$, $c_{pq} \in \mathbb{H}$ corresponds to the connection weights from q-th to p-th neuron at time t and $t - \tau_q$, respectively; $k_q \in \mathbb{H}$ is the external input; $\tau_q > 0$ denotes the time delay with $\tau = \max_q\{\tau_q | q = 1, 2, ..., n\}$; $g_q(\cdot) \in \mathbb{H}$ denotes the neuron activation. To check the dynamics of DQVNNs given in (2.1), let us make the following assumption.

Assumption 2.1. Let $h_q = h_q^{(0)} + h_q^{(1)} i + h_q^{(2)} j + h_q^{(3)} k \in \mathbb{H}$ be any arbitrary. For $q = 1, 2, ..., n$, $g_q(h_q)$ can be described by

$$g_q(h_q) = g_q^{(0)}(h_q^{(0)}) + g_q^{(1)}(h_q^{(1)})i + g_q^{(2)}(h_j^{(2)})j + g_q^{(3)}(h_q^{(3)})k,$$

where $g_q^{(b)}(\cdot)$ are continuous nonlinear functions on $\mathbb{R}$ which are defined as

$$g_q^{(b)}(r) = \begin{cases} -1, & -\infty < r < -\frac{\pi}{2} \\ -sin(3r), & -\frac{\pi}{2} \leq r \leq \frac{\pi}{2} \\ 1, & \frac{\pi}{2} < r < +\infty \end{cases}, \tag{2.2}$$

with $b = 0, 1, 2, 3$.

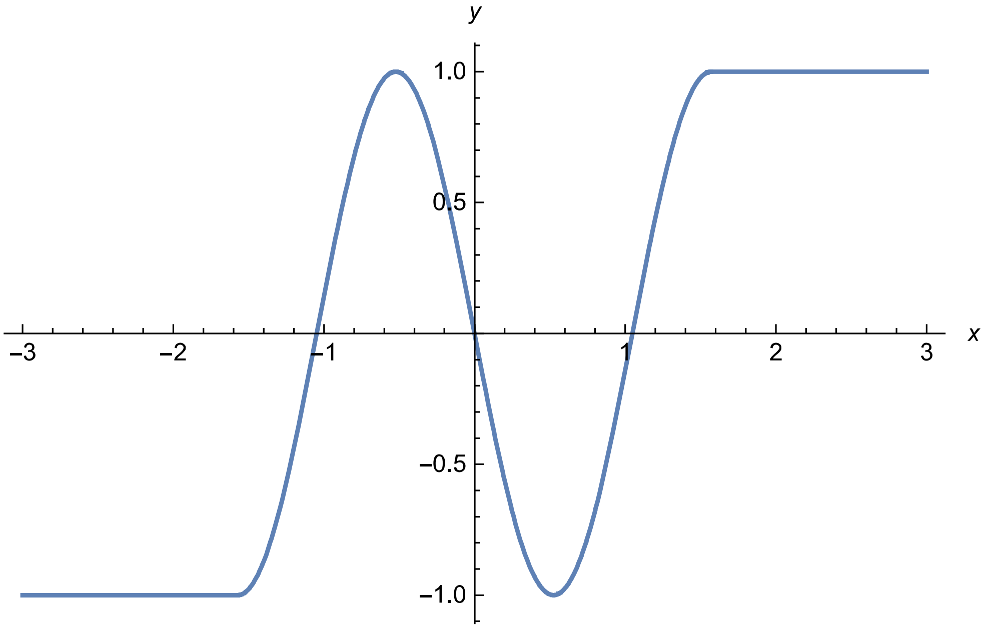

FIGURE 2.1: Configuration of piecewise nonlinear activation function $g_q^{(b)}(r)$ defined in equation (2.2)

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The typical configuration of this activation function (2.2) is depicted in Figure 2.1.

Considering Assumption 2.1 and by the multiplication of quaternions, the system (2.1) can be rewritten as the following four real-valued systems as

$$\begin{aligned}\dot{h}_p^{(0)}(t) =& -a_p h_p^{(0)}(t) + \sum_{q=1}^{n} \Big[b_{pq}^{(0)} g_q^{(0)}(h_q^{(0)}(t)) - b_{pq}^{(1)} g_q^{(1)}(h_q^{(1)}(t)) \\ &\qquad - b_{pq}^{(2)} g_q^{(2)}(h_q^{(2)}(t)) - b_{pq}^{(3)} g_q^{(3)}(h_q^{(3)}(t)) \Big] \\ &+ \sum_{q=1}^{n} \Big[c_{pq}^{(0)} g_q^{(0)}(h_q^{(0)}(t-\tau_q)) - c_{pq}^{(1)} g_q^{(1)}(h_q^{(1)}(t-\tau_q)) \\ &\qquad - c_{pq}^{(2)} g_q^{(2)}(h_q^{(2)}(t-\tau_q)) - c_{pq}^{(3)} g_q^{(3)}(h_q^{(3)}(t-\tau_q)) \Big] + k_p^{(0)}, \qquad (2.3)\end{aligned}$$

$$\begin{aligned}\dot{h}_p^{(1)}(t) = &- a_p h_p^{(1)}(t) + \sum_{q=1}^{n} \Big[b_{pq}^{(0)} g_q^{(1)}(h_q^{(1)}(t)) + b_{pq}^{(1)} g_q^{(0)}(h_q^{(0)}(t)) \\ &\qquad\qquad + b_{pq}^{(2)} g_q^{(3)}(h_q^{(3)}(t)) - b_{pq}^{(3)} g_q^{(2)}(h_q^{(2)}(t)) \Big] \\ &+ \sum_{q=1}^{n} \Big[c_{pq}^{(0)} g_q^{(1)}(h_q^{(1)}(t-\tau_q)) + c_{pq}^{(1)} g_q^{(0)}(h_q^{(0)}(t-\tau_q)) \\ &\qquad + c_{pq}^{(2)} g_q^{(3)}(h_q^{(3)}(t-\tau_q)) - c_{pq}^{(3)} g_q^{(2)}(h_q^{(2)}(t-\tau_q)) \Big] + k_p^{(1)}, \end{aligned} \tag{2.4}$$

$$\begin{aligned}\dot{h}_p^{(2)}(t) = &- a_p h_p^{(2)}(t) + \sum_{q=1}^{n} \Big[b_{pq}^{(0)} g_q^{(2)}(h_q^{(2)}(t)) + b_{pq}^{(2)} g_q^{(0)}(h_q^{(0)}(t)) \\ &\qquad\qquad - b_{pq}^{(1)} g_q^{(3)}(h_q^{(3)}(t)) + b_{pq}^{(3)} g_q^{(1)}(h_q^{(1)}(t)) \Big] \\ &+ \sum_{q=1}^{n} \Big[c_{pq}^{(0)} g_q^{(2)}(h_q^{(2)}(t-\tau_q)) + c_{pq}^{(2)} g_q^{(0)}(h_q^{(0)}(t-\tau_q)) \\ &\qquad - c_{pq}^{(1)} g_q^{(3)}(h_q^{(3)}(t-\tau_q)) + c_{pq}^{(3)} g_q^{(1)}(h_q^{(1)}(t-\tau_q)) \Big] + k_p^{(2)}, \end{aligned} \tag{2.5}$$

$$\begin{aligned}\dot{h}_p^{(3)}(t) = &- a_p h_p^{(3)}(t) + \sum_{q=1}^{n} \Big[b_{pq}^{(0)} g_q^{(3)}(h_q^{(3)}(t)) + b_{pq}^{(3)} g_q^{(0)}(h_q^{(0)}(t)) \\ &\qquad\qquad + b_{pq}^{(1)} g_q^{(2)}(h_q^{(2)}(t)) - b_{pq}^{(2)} g_q^{(1)}(h_q^{(1)}(t)) \Big] \\ &+ \sum_{q=1}^{n} \Big[c_{pq}^{(0)} g_q^{(3)}(h_q^{(3)}(t-\tau_q)) + c_{pq}^{(3)} g_q^{(0)}(h_q^{(0)}(t-\tau_q)) \\ &\qquad + c_{pq}^{(1)} g_q^{(2)}(h_q^{(2)}(t-\tau_q)) - c_{pq}^{(2)} g_q^{(1)}(h_q^{(1)}(t-\tau_q)) \Big] + k_p^{(3)}. \end{aligned} \tag{2.6}$$

The initial conditions associated with the DQVNNs (2.1) are given by

$$h_p^{(b)}(\theta) = \phi_p^{(b)}(\theta), \qquad \theta \in [-\tau, 0],$$

where $\phi_p^{(b)}(\theta) \in \mathbf{C}([-\tau,0],\mathbb{R})$, $b = 0,1,2,3$, and $p = 1,2,...,n$.

Thus, for $p = 1, 2, ..., n$, we can define the continuous functions on $(-\infty, +\infty)$ as

$$\hat{G}_p^{(b)}(r) = -\, a_p r + (b_{pp}^{(0)} + c_{pp}^{(0)}) g_p^{(b)}(r) + \hat{\eta}_p^{(b)}, \tag{2.7}$$

$$\check{G}_p^{(b)}(r) = -\, a_p r + (b_{pp}^{(0)} + c_{pp}^{(0)}) g_p^{(b)}(r) + \check{\eta}_p^{(b)}, \tag{2.8}$$

where $r \in \mathbb{R}$, $b = 0, 1, 2, 3$, and the constants $\hat{\eta}_p^{(b)}, \check{\eta}_p^{(b)} \in \mathbb{R}$ are defined as

$$\hat{\eta}_p^{(b)} = \sum_{q=1, q\neq p}^{n} |(b_{pq}^{(0)} + c_{pq}^{(0)})| - \sum_{q=1}^{n} \Bigg[-|(b_{pq}^{(1)} + c_{pq}^{(1)})| - |b_{pq}^{(2)} + c_{pq}^{(2)})| - |b_{pq}^{(3)} + c_{pq}^{(3)})| \Bigg] + k_p^{(b)},$$

$$\check{\eta}_p^{(b)} = -\sum_{q=1, q\neq p}^{n} |(b_{pq}^{(0)} + c_{pq}^{(0)})| - \sum_{q=1}^{n} \Bigg[|(b_{pq}^{(1)} + c_{pq}^{(1)})| + |b_{pq}^{(2)} + c_{pq}^{(2)})| + |b_{pq}^{(3)} + c_{pq}^{(3)})| \Bigg] + k_p^{(b)},$$

Remark 2.1. It follows from the definitions of $\hat{\eta}_p^{(b)}$ and $\check{\eta}_p^{(b)}$ that $\hat{G}_p^{(b)}(r) \geq \check{G}_p^{(b)}(r)$ for all $b = 0, 1, 2, 3$ and $p = 1, 2, ..., n$. In terms of Assumption 2.1 and $a_p > 0$, we see that

$$\lim_{r \to -\infty} \check{G}_p^{(b)}(r) = +\infty, \quad \lim_{r \to +\infty} \hat{G}_p^{(b)}(r) = -\infty.$$

Thus, there exist $\check{r}_p^{(b)} < 0$ and $\hat{r}_p^{(b)} > 0$, both are suffiently large such that $\check{r}_p^{(b)} < -\frac{\pi}{2}$, $\frac{\pi}{2} < \hat{r}_p^{(b)}$ and

$$\hat{G}_p^{(b)}(\check{r}_p^{(b)}) \geq \check{G}_p^{(b)}(\check{r}_p^{(b)}) > 0, \quad \check{G}_p^{(b)}(\hat{r}_p^{(b)}) \leq \hat{G}_p^{(b)}(\hat{r}_p^{(b)}) < 0. \tag{2.9}$$

Next, for $p = 1, 2, ..., n$ and $b = 0, 1, 2, 3$, let us define functions $\hat{\tilde{G}}_p^{(b)}(r)$ and $\check{\tilde{G}}_p^{(b)}(r)$ on $(-\infty, +\infty)$ as

$$\hat{\tilde{G}}_p^{(b)}(r) = -a_p r + b_{pp}^{(0)} g_p^{(b)}(r) + \hat{\tilde{\eta}}_p^{(b)}, \tag{2.10}$$

$$\check{\tilde{G}}_p^{(b)}(r) = -a_p r + b_{pp}^{(0)} g_p^{(b)}(r) + \check{\tilde{\eta}}_p^{(b)}, \tag{2.11}$$

where

$$\begin{aligned}\hat{\tilde{\eta}}_p^{(b)} = & \sum_{q=1, q\neq p}^{n} |b_{pq}^{(0)}| + \sum_{q=1}^{n} |c_{pq}^{(0)}| + \sum_{q=1}^{n} \Big[|b_{pq}^{(1)}| + |b_{pq}^{(2)}| + |b_{pq}^{(3)}|\Big] + \sum_{q=1}^{n} \Big[|c_{pq}^{(1)}| + |c_{pq}^{(2)}| + |c_{pq}^{(3)}|\Big] \\ & + k_p^{(b)},\end{aligned}$$

$$\begin{aligned}\check{\tilde{\eta}}_p^{(b)} = & - \sum_{q=1, q\neq p}^{n} |b_{pq}^{(0)}| - \sum_{q=1}^{n} |c_{pq}^{(0)}| - \sum_{q=1}^{n} \Big[|b_{pq}^{(1)}| + |b_{pq}^{(2)}| + |b_{pq}^{(3)}|\Big] - \sum_{q=1}^{n} \Big[|c_{pq}^{(1)}| + |c_{pq}^{(2)}| + |c_{pq}^{(3)}|\Big] \\ & + k_p^{(b)}.\end{aligned}$$

To facilitate the discussion further, let us introduce some more functions. For any given $h = (h_1, h_2, ..., h_n)^T \in \mathbb{H}^n$, let us introduce the continuous functions $G_p^{(b)}(\cdot) : \mathbb{R} \to \mathbb{R}$ as

$$G_p^{(b)}(r) = -a_p r + (b_{pp}^{(0)} + c_{pp}^{(0)}) g_p^{(b)}(r) + \eta_p^{(b)}, \tag{2.12}$$

where $b = 0, 1, 2, 3,\ p = 1, 2, ...n$, and

$$\begin{aligned}\eta_p^{(0)} = & \sum_{q=1, q\neq p}^{n} (b_{pp}^{(0)} + c_{pp}^{(0)}) g_q^{(0)}(h_q^{(0)}) \\ & - \sum_{q=1}^{n} \Big[(b_{pq}^{(1)} + c_{pq}^{(1)}) g_q^{(1)}(h_q^{(1)}) + (b_{pq}^{(2)} + c_{pq}^{(2)}) g_q^{(2)}(h_q^{(2)}) + (b_{pq}^{(3)} + c_{pq}^{(3)}) g_q^{(3)}(h_q^{(3)})\Big] + k_p^{(0)},\end{aligned}$$

$$\eta_i^{(1)} = \sum_{j=1,j\neq p}^{n} (b_{ii}^{(0)} + c_{ii}^{(0)})g_j^{(1)}(h_j^{(1)}) \\ + \sum_{j=1}^{n} \Big[(b_{ij}^{(1)} + c_{ij}^{(1)})g_j^{(0)}(h_j^{(0)}) + (b_{ij}^{(2)} + c_{ij}^{(2)})g_j^{(3)}(h_j^{(3)}) - (b_{ij}^{(3)} + c_{ij}^{(3)})g_j^{(2)}(h_j^{(2)}) \Big] + k_i^{(1)},$$

$$\eta_p^{(2)} = \sum_{q=1,q\neq p}^{n} (b_{pp}^{(0)} + c_{pp}^{(0)})g_q^{(2)}(h_q^{(2)}) \\ + \sum_{q=1}^{n} \Big[(b_{pq}^{(2)} + c_{pq}^{(2)})g_q^{(0)}(h_q^{(0)}) - (b_{pq}^{(1)} + c_{pq}^{(1)})g_q^{(3)}(h_q^{(3)}) + (b_{pq}^{(3)} + c_{pq}^{(3)})g_q^{(1)}(h_q^{(1)}) \Big] + k_p^{(2)},$$

$$\eta_p^{(3)} = \sum_{q=1,q\neq p}^{n} (b_{pp}^{(0)} + c_{qq}^{(0)})g_q^{(3)}(h_q^{(3)}) \\ + \sum_{q=1}^{n} \Big[(b_{pq}^{(3)} + c_{pq}^{(3)})g_q^{(0)}(h_q^{(0)}) + (b_{pq}^{(1)} + c_{pq}^{(1)})g_q^{(2)}(h_q^{(2)}) - (b_{pq}^{(2)} + c_{pq}^{(2)})g_q^{(1)}(h_q^{(1)}) \Big] + k_p^{(3)}.$$

From (2.7)-(2.8) and (2.10)-(2.12), we can conclude that

$$\check{\check{G}}_p^{(b)}(r) \leq \check{G}_p^{(b)}(r) \leq G_p^{(b)}(r) \leq \hat{G}_p^{(b)}(r) \leq \hat{\hat{G}}_p^{(b)}(r), \quad \forall r \in \mathbb{R}. \tag{2.13}$$

2.3 Main Results

In this section, the dynamics of the DQVNNs (2.1) are studied. Here, several sufficient conditions are introduced to ensure that the system (2.1) can have 5^{4n} EPs, out of which 3^{4n} are locally stable.

2.3.1 Existence of multiple stable EPs and positively invariant sets for the DQVNNs (2.1)

For the coexistence of multiple EPs, let us denote the following intervals as

$\tilde{I}_p^{(b)\mathrm{I}} = (-\infty, -\frac{\pi}{2})$, $\tilde{I}_p^{(b)\mathrm{II}} = [-\frac{\pi}{2}, -\frac{\pi}{6}]$, $\tilde{I}_p^{(b)\mathrm{III}} = (-\frac{\pi}{6}, \frac{\pi}{6})$, $\tilde{I}_p^{(b)\mathrm{IV}} = [\frac{\pi}{6}, \frac{\pi}{2}]$, $\tilde{I}_p^{(b)\mathrm{V}} = (\frac{\pi}{2}, \infty)$, where $b = 0, 1, 2, 3$.

The, superscripts “I”, “II”, “III”, “IV” and “V”, respectively represent “First”, “Second”, “Third”, “Fourth” and “Fifth”. Then for each index $\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2, ..., \tilde{\alpha}_{4n})$, where $\tilde{\alpha}_p$ is “I”, “II”, “III”, “IV” or “V”, we denote

$$\begin{aligned}\tilde{\Omega}^{\tilde{\alpha}} = \Big\{ &(x_1, x_2, ..., x_n)^T \in \mathbb{H}^n | x_p^{(0)} \in \tilde{I}_p^{(0)\tilde{\alpha}_p}, x_p^{(1)} \in \tilde{I}_p^{(1)\tilde{\alpha}_{n+p}}, x_i^{(2)} \in \tilde{I}_p^{(2)\tilde{\alpha}_{2n+p}}, \\ &x_p^{(3)} \in \tilde{I}_p^{(3)\tilde{\alpha}_{3n+p}}, \text{for } p = 1, 2, .., n \Big\},\end{aligned}$$

and

$\tilde{\Omega} = \Big\{ \tilde{\Omega}^{\tilde{\alpha}} | \tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2, ..., \tilde{\alpha}_{4n})$ with $\tilde{\alpha}_p$ is “I”, “II”, “III”, “IV”, or “V”$\Big\}$. It should be noticed that, there are 5^{4n} elements $\tilde{\Omega}^{\tilde{\alpha}}$ in $\tilde{\Omega}$, which are disjoint regions in $\mathbb{H}^n$.

Now, we choose an infinitesimal number $\epsilon > 0$ which is very very small. Further, denote the intervals as

$I_p^{(b)\mathrm{I}} = [\check{r}_p^{(b)} + \epsilon, -\frac{\pi}{2} - \epsilon]$, $I_p^{(b)\mathrm{II}} = [-\frac{\pi}{2} + \epsilon, -\frac{\pi}{6} - \epsilon]$, $I_p^{(b)\mathrm{III}} = [-\frac{\pi}{6} + \epsilon, \frac{\pi}{6} - \epsilon]$, $I_p^{(b)\mathrm{IV}} = [\frac{\pi}{6} + \epsilon, \frac{\pi}{2} - \epsilon]$, $I_p^{(b)\mathrm{V}} = [\frac{\pi}{2} + \epsilon, \hat{r}_p^{(b)} - \epsilon]$, where $b = 0, 1, 2, 3$.

Then for each index $\alpha = (\alpha_1, \alpha_2, ..., \alpha_{4n})$, where α_p is “I”, “II”, “III”, “IV” or

"V", let us denote

$$\Omega^{\alpha} = \Big\{(x_1, x_2, ..., x_n)^T \in \mathbb{H}^n | x_p^{(0)} \in I_p^{(0)\alpha_p}, x_p^{(1)} \in I_p^{(1)\alpha_{n+p}}, x_i^{(2)} \in I_p^{(2)\alpha_{2n+p}}, \\ x_p^{(3)} \in I_p^{(3)\alpha_{3n+p}}, \text{for } p = 1, 2, .., n\Big\},$$

and

$\Omega = \Big\{\Omega^{\alpha} | \alpha = (\alpha_1, \alpha_2, ..., \alpha_{4n})$ with α_p is "I", "II", "III", "IV", or "V"$\Big\}$.

Clearly, Ω consists of 5^{4n} disjoint regions of the type Ω^{α}. Moreover, each region $\Omega^{\alpha} \in \Omega$ is a closed and bounded set.

Theorem 2.1. Suppose that Assumption 2.1 holds, then DQVNNs (2.1) have at least 5^{4n} EPs such that each is located in one of the regions $\Omega^{\alpha} \in \Omega$ if

$$\check{G}_p^{(b)}(\frac{\pi}{2}) > 0, \hat{G}_p^{(b)}(\frac{-\pi}{2}) < 0, \tag{2.14}$$

where $b = 0, 1, 2, 3$ and $p = 1, 2, ..., n$.

Before giving the proof of the Theorem 2.1, let us present a proposition to identify the relationship between the parameters of the system (2.1) to make the proof of Theorem 2.1 more understandable.

Proposition 2.1. Conditions in (2.14) imply that $(b_{pp}^{(0)} + c_{pp}^{(0)}) > 0, -a_p + 3(b_{pp}^{(0)} + c_{pp}^{(0)}) > 0$ and $-a_p - 3(b_{pp}^{(0)} + c_{pp}^{(0)}) < 0$, where $p = 1, 2, ..., n$.

Proof. From (2.14), we have $\check{G}_p^{(b)}(\frac{\pi}{2}) > 0$ i.e.,

$$-\frac{\pi}{2}a_p + (b_{pp}^{(0)} + c_{pp}^{(0)}) + \check{\eta}_p^{(b)} > 0.$$

Since $a_p > 0$, then we must have

$$\frac{\pi}{6}a_p + (b_{pp}^{(0)} + c_{pp}^{(0)}) + \check{\eta}_p^{(b)} > 0. \tag{2.15}$$

Owing to $\check{G}_p^{(b)}(\frac{-\pi}{2}) < 0$ and condition (2.15), we have

$$\frac{\pi}{6}a_p + (b_{pp}^{(0)} + c_{pp}^{(0)}) + k_p^{(b)} \geq \frac{\pi}{6}a_p + (b_{pp}^{(0)} + c_{pp}^{(0)}) + \check{\eta}_p^{(b)} > 0,$$

and

$$\frac{\pi}{6}a_p + (b_{pp}^{(0)} + c_{pp}^{(0)}) + k_p^{(b)} > \frac{\pi}{2}a_p - (b_{pp}^{(0)} + c_{pp}^{(0)}) + \hat{\eta}_p^{(b)} \geq \frac{\pi}{2}a_p - (b_{pp}^{(0)} + c_{pp}^{(0)}) + k_p^{(b)}.$$

$$\text{Therefore,} \quad (b_{pp}^{(0)} + c_{pp}^{(0)}) > \frac{\pi}{6}a_p > 0. \tag{2.16}$$

Equation (2.16) implies that $(b_{pp}^{(0)} + c_{pp}^{(0)}) > 0$ and $-a_p - 3(b_{pp}^{(0)} + c_{pp}^{(0)}) < 0$. Again from equation (2.16), we have $-a_p + \frac{6}{\pi}(b_{pp}^{(0)} + c_{pp}^{(0)}) > 0$. As $\frac{6}{\pi} < 3$ and $(b_{pp}^{(0)} + c_{pp}^{(0)}) > 0$, we have $-a_p + 3(b_{pp}^{(0)} + c_{pp}^{(0)}) > 0$. This completes the proof of the proposition. ☐

Now, we wish to give the proof of the Theorem 2.1 as follows.

Proof. For any $h = (h_1, h_2, ..., h_n) \in \mathbb{H}^n$, let us define the real valued continuous functions $G_p^{(b)}(\cdot)$ on $\mathbb{R}$ in the form of (2.12). For better understanding, the following discussions are presented in the point-wise form.

(a) Since $\check{G}_p^{(b)}(\check{r}_p^{(b)}) > 0$ and from (2.13), we have $G_p^{(b)}(\check{r}_p^{(b)}) > 0$. Again, from $\hat{G}_p^{(b)}(\frac{-\pi}{2}) < 0$ and from (2.13), we get $G_p^{(b)}(\frac{-\pi}{2}) < 0$. Hence, because of continuity of $G_p^{(b)}(\cdot)$, there is a point $\tilde{r}_p^{(b)\mathrm{I}} \in [\check{r}_p^{(b)} + \epsilon, \frac{-\pi}{2} - \epsilon]$ such that $G_p^{(b)}(\tilde{r}_p^{(b)\mathrm{I}}) = 0$.

(b) From the above point (a), note that $G_p^{(b)}(\frac{-\pi}{2}) < 0$. Now, $\check{G}_p^{(b)}(\frac{-\pi}{6}) = -a_p(\frac{-\pi}{6}) + (b_{pp}^{(0)} + c_{pp}^{(0)})g_p^{(b)}(\frac{-\pi}{6}) + \check{\eta}_p^{(b)} = a_p(\frac{\pi}{6}) + (b_{pp}^{(0)} + c_{pp}^{(0)}) + \check{\eta}_p^{(b)}$ and by using (2.15), we get $\check{G}_p^{(b)}(\frac{-\pi}{6}) > 0$. Owing to (2.13) and $\check{G}_p^{(b)}(\frac{-\pi}{6}) > 0$, we obtain $G_p^{(b)}(\frac{-\pi}{6}) > 0$. Hence, because of continuity of $G_p^{(b)}(\cdot)$, there is a point $\tilde{r}_p^{(b)\mathrm{II}} \in [\frac{-\pi}{2} + \epsilon, \frac{-\pi}{6} - \epsilon]$ such that $G_p^{(b)}(\tilde{r}_p^{(b)\mathrm{II}}) = 0$.

(c) From the above point (b), note that $G_p^{(b)}(\frac{-\pi}{6}) > 0$. Since, $\hat{G}_p^{(b)}(\frac{\pi}{6}) = -a_p(\frac{\pi}{6}) + (b_{pp}^{(0)} + c_{pp}^{(0)})g_p^{(b)}(\frac{\pi}{6}) + \hat{\eta}_p^{(b)} = -a_p(\frac{\pi}{6}) - (b_{pp}^{(0)} + c_{pp}^{(0)}) + \hat{\eta}_p^{(b)}$. Next, from (2.14) we must have $\hat{G}_p^{(b)}(\frac{-\pi}{2}) = -a_p(\frac{-\pi}{2}) + (b_{pp}^{(0)} + c_{pp}^{(0)})g_p^{(b)}(\frac{-\pi}{2}) + \hat{\eta}_p^{(b)} = a_p(\frac{\pi}{2}) - (b_{pp}^{(0)} + c_{pp}^{(0)}) + \hat{\eta}_p^{(b)}$¡0. Therefore, it can be conclude that $\hat{G}_p^{(b)}(\frac{\pi}{6}) < 0$, and because of (2.13), we get $G_p^{(b)}(\frac{\pi}{6}) < 0$. Hence, there is a point $\tilde{r}_p^{(b)\mathrm{III}} \in [\frac{-\pi}{6} + \epsilon, \frac{\pi}{6} - \epsilon]$ such that $G_p^{(b)}(\tilde{r}_p^{(b)\mathrm{III}}) = 0$ due to continuity of $G_p^{(b)}(\cdot)$.

(d) From the above point (c) and equation (2.15), we have $G_p^{(b)}(\frac{\pi}{6}) < 0$ and $\check{G}_p^{(b)}(\frac{\pi}{2}) > 0$, respectively. Owing to (2.13) and $\check{G}_p^{(b)}(\frac{\pi}{2}) > 0$, we obtain $G_p^{(b)}(\frac{\pi}{2}) > 0$. Hence, because of continuity of $G_p^{(b)}(\cdot)$, there is a point $\tilde{r}_p^{(b)\mathrm{IV}} \in [\frac{\pi}{6} + \epsilon, \frac{\pi}{2} - \epsilon]$ such that $G_p^{(b)}(\tilde{r}_p^{(b)\mathrm{IV}}) = 0$.

(e) Since $\hat{G}_p^{(b)}(\hat{r}_p^{(b)}) < 0$ and from (2.13), we have $G_p^{(b)}(\hat{r}_p^{(b)}) < 0$. Again, from $\check{G}_p^{(b)}(\frac{\pi}{2}) > 0$ and from (2.13), we get $G_p^{(b)}(\frac{\pi}{2}) > 0$. Hence, because of continuity of $G_p^{(b)}(\cdot)$, there is a point $\tilde{r}_p^{(b)\mathrm{V}} \in [\frac{\pi}{2} + \epsilon, \hat{r}_p^{(b)} - \epsilon]$ such that $G_p^{(b)}(\tilde{r}_p^{(b)\mathrm{V}}) = 0$.

From the above discussion, we can see that there exist exactly five points $\tilde{r}_p^{(b)\mathrm{I}} \in I_p^{(b)\mathrm{I}}, \tilde{r}_p^{(b)\mathrm{II}} \in I_p^{(b)\mathrm{II}}, \tilde{r}_p^{(b)\mathrm{III}} \in I_p^{(b)\mathrm{III}}, \tilde{r}_p^{(b)\mathrm{IV}} \in I_p^{(b)\mathrm{IV}}$ and $\tilde{r}_p^{(b)\mathrm{V}} \in I_p^{(b)\mathrm{V}}$ such that

$$G_p^{(b)}(\tilde{r}_p^{(b)\mathrm{I}}) = G_p^{(b)}(\tilde{r}_p^{(b)\mathrm{II}}) = G_p^{(b)}(\tilde{r}_p^{(b)\mathrm{III}}) = G_p^{(b)}(\tilde{r}_p^{(b)\mathrm{IV}}) = G_p^{(b)}(\tilde{r}_p^{(b)\mathrm{V}}) = 0.$$

For any given region Ω^{α}, let us assume an index $\alpha = (\alpha_1, \alpha_2, ..., \alpha_{4n})$, where α_p is "I", "II", "III", "IV" or "V". By choosing $h = (h_1, h_2, ..., h_n) \in \Omega^{\alpha}$ be any arbitrary and putting it into (2.12), we obtain the functions of the $G_p^{(b)}(r)$ and exactly one $\tilde{r} = (\tilde{r}_1, \tilde{r}_2, ..., \tilde{r}_n \in \Omega^{\alpha}$, where $\tilde{r}_p = \tilde{r}_p^{(0)\alpha_p} + \tilde{r}_p^{(1)\alpha_{n+p}} \imath + \tilde{r}_p^{(2)\alpha_{2n+p}} \jmath + \tilde{r}_p^{(3)\beta_{3n+p}} k$, such that $G_p^{(b)}(\tilde{r}_p^{(b)\alpha_{bn+p}}) = 0$ with $b = 0, 1, 2, 3$ and $p = 1, 2, ..., n$. Then a mapping is defined as $F : \Omega^{\alpha} \to \Omega^{\alpha}$ such that $F(h) = \tilde{r}$. Obviously, the mapping F is continuous. Hence, it can be concluded from Brouwer's fixed point theorem, the mapping F has at least one fixed point $\tilde{r} \in \Omega^{\alpha}$, which is also an EP of the system (2.1) in Ω^{α}. Hence, there exist 5^{4n} EPs of the system (2.1) and each of those lies in one of the 5^{4n} regions $\Omega^{\alpha} \in \Omega$. This completes the proof of the theorem. $\square$

Again, for the index $\beta = (\beta_1, \beta_2, ..., \beta_n)$ where β_p is I, III or V, let us denote

$$\Pi^{\beta} = \Big\{ (x_1, x_2, ..., x_n)^T \in \mathbb{H}^n | x_p^{(0)} \in I_p^{(0)\alpha_p}, x_p^{(1)} \in I_p^{(1)\alpha_{n+p}}, x_i^{(2)} \in I_p^{(2)\alpha_{2n+p}}, \\ x_p^{(3)} \in I_p^{(3)\alpha_{3n+p}}, \text{for } p = 1, 2, .., n \Big\},$$

and

$$\Pi = \Big\{ \Pi^{\beta} | \beta = (\beta_1, \beta_2, ..., \beta_{4n}) \text{ with } \beta_p \text{ is "I", "III", or "V"} \Big\},$$

where $I_p^{(b)\mathrm{I}} = [\check{r}_p^{(b)} + \epsilon, -\frac{\pi}{2} - \epsilon]$, $I_p^{(b)\mathrm{III}} = [-\frac{\pi}{6} + \epsilon, \frac{\pi}{6} - \epsilon]$, $I_p^{(b)\mathrm{V}} = [\frac{\pi}{2} + \epsilon, \hat{r}_p^{(b)} - \epsilon]$, with $b = 0, 1, 2, 3,$ and $p = 1, 2, ..., n$.

Clearly, there are 3^{4n} disjoint regions $\Pi^{\beta} \in \Pi$ and each Π^{β} is a closed and bounded set. Moreover, it is easy to see that $\Pi \subset \Omega$.

Theorem 2.2. Under the Assumption 2.1, if DQVNNs (2.1) satisfy the following conditions for $p = 1, 2, ..., n$:

$$\hat{\tilde{G}}_p^{(b)}(\frac{-\pi}{2}) < 0, \check{\tilde{G}}_p^{(b)}(\frac{\pi}{2}) > 0, \qquad b = 0, 1, 2, 3. \tag{2.17}$$

Then each region $\Pi^\beta \in \Pi$ is positively invariant.

Proof. Using the inequality (2.13), we can conclude that, the conditions given in (2.17) are stronger than the conditions in (2.14). It means that if DQVNNs (2.1) satisfy the conditions in (2.17) then it also satisfies the conditions given in (2.14). Moreover, with the same method as used in Preposition 2.1, from the conditions in (2.17), we can obtain that $b_{pp}^{(0)} > 0$, $-a_p + 3b_{pp}^{(0)} > 0$ and $-a_p - 3b_{pp}^{(0)} < 0$, where $b = 0, 1, 2, 3$ and $p = 1, 2, ..., n$.

We may write inequalities in equation (2.17) as

$$\hat{\tilde{G}}_p^{(b)}(\frac{-\pi}{2}) = -a_p(\frac{-\pi}{2}) + b_{pp}^{(0)} g_p^{(b)}(\frac{-\pi}{2}) + \hat{\tilde{\eta}}_p^{(b)} < 0, \tag{2.18}$$

$$\check{\tilde{G}}_p^{(b)}(\frac{\pi}{2}) = -a_p(\frac{\pi}{2}) + b_{pp}^{(0)} g_p^{(b)}(\frac{\pi}{2}) + \check{\tilde{\eta}}_p^{(b)} > 0. \tag{2.19}$$

From (2.18) and (2.19), we can obtain the following conditions as

$$-a_p(\frac{\pi}{6}) + b_{pp}^{(0)} g_p^{(b)}(\frac{-\pi}{2}) + \hat{\tilde{\eta}}_p^{(b)} < 0, \tag{2.20}$$

$$a_p(\frac{\pi}{6}) + b_{pp}^{(0)} g_p^{(b)}(\frac{\pi}{2}) + \check{\tilde{\eta}}_p^{(b)} > 0. \tag{2.21}$$

Since the conditions (2.14) hold, according to Theorem 2.1 the DQVNNs (2.1) can have 5^{4n} EPs and each of them lies in one of the 5^{4n} regions $\Omega^{\alpha} \in \Omega$. Next, we want to show that each 3^{4n} region $\Pi^{\beta} \in \Pi$ is positively invariant. That is, for each $\Pi^{\beta} \in \Pi$, if any state trajectory $h(t)$ of the system (2.1) with the initial condition is chosen from Π^{β}, then $h(t) \in \Pi^{\beta}$ for all $t \geq 0$. It means, $h(t)$ will stay in Π^{β} for all $t \geq 0$.

Without loss of generality, suppose the initial condition for the QVNNs (2.1) is

$$\begin{aligned} h_p(\theta) &= h_p^{(0)}(\theta) + h_p^{(1)}(\theta)i + h_p^{(2)}(\theta)j + h_p^{(3)}(\theta)k \\ &= \phi_p^{(0)}(\theta) + \phi_p^{(1)}(\theta)i + \phi_p^{(2)}(\theta)j + \phi_p^{(3)}(\theta)k, \quad \theta \in [-\tau, 0], \end{aligned} \tag{2.22}$$

where $p = 1, 2, ..., n$; $\phi_p^{(0)}(\theta) \in [\check{r}_p^{(0)} + \epsilon, -\frac{\pi}{2} - \epsilon]$; $\phi_p^{(1)}(\theta) \in [-\frac{\pi}{6} + \epsilon, \frac{\pi}{6} - \epsilon]$; $\phi_p^{(2)}(\theta) \in [\frac{\pi}{2} + \epsilon, \hat{r}_p^{(2)} - \epsilon]$; $\phi_p^{(3)}(\theta) \in [\frac{\pi}{2} + \epsilon, \hat{r}_p^{(3)} - \epsilon]$. Now, we claim that for all $t \geq 0$, the state trajectory $h_p(t)$ of the system (2.1) satisfies

$$\begin{aligned} &\check{r}_p^{(0)} + \epsilon \leq h_p^{(0)}(t) \leq -\frac{\pi}{2} - \epsilon, \quad -\frac{\pi}{6} + \epsilon \leq h_p^{(1)}(t) \leq \frac{\pi}{6} - \epsilon, \\ &\frac{\pi}{2} + \epsilon \leq h_p^{(2)}(t) \leq \hat{r}_p^{(2)} - \epsilon, \quad \frac{\pi}{2} + \epsilon \leq h_p^{(3)}(t) \leq \hat{r}_p^{(3)} - \epsilon; \quad p = 1, 2, ..., n. \end{aligned} \tag{2.23}$$

The proof of this claim will be done by using the contradiction method. If possible, let us suppose that (2.23) is not true, then there exist an index $p_1 \in \{1, 2, ..., n\}$ and a time time $t_0 > 0$, such that for all $p = 1, 2, ..., n$:

$$\begin{aligned} &\check{r}_p^{(0)} + \epsilon \leq h_p^{(0)}(t) \leq -\frac{\pi}{2} - \epsilon, \quad -\frac{\pi}{6} + \epsilon \leq h_p^{(1)}(t) \leq \frac{\pi}{6} - \epsilon, \\ &\frac{\pi}{2} + \epsilon \leq h_p^{(2)}(t) \leq \hat{r}_p^{(2)} - \epsilon, \quad \frac{\pi}{2} + \epsilon \leq h_p^{(3)}(t) \leq \hat{r}_p^{(3)} - \epsilon; \quad t \in [0, t_0) \end{aligned} \tag{2.24}$$

and one of the following cases holds:

$$h_{p_1}^{(0)}(t_0) = \check{r}_p^{(0)} + \epsilon \quad \text{and} \quad \dot{h}_{p_1}^{(0)}(t_0) < 0, \tag{2.25}$$

$$h_{p_1}^{(0)}(t_0) = -\frac{\pi}{2} - \epsilon \quad \text{and} \quad \dot{h}_{p_1}^{(0)}(t_0) > 0, \tag{2.26}$$

$$h_{p_1}^{(1)}(t_0) = -\frac{\pi}{6} + \epsilon \quad \text{and} \quad \dot{h}_{p_1}^{(1)}(t_0) < 0, \tag{2.27}$$

$$h_{p_1}^{(1)}(t_0) = \frac{\pi}{6} - \epsilon \quad \text{and} \quad \dot{h}_{p_1}^{(1)}(t_0) > 0, \tag{2.28}$$

$$h_{p_1}^{(2)}(t_0) = \frac{\pi}{2} + \epsilon \quad \text{and} \quad \dot{h}_{p_1}^{(2)}(t_0) < 0, \tag{2.29}$$

$$h_{p_1}^{(2)}(t_0) = \hat{r}_p^{(2)} - \epsilon \quad \text{and} \quad \dot{h}_{p_1}^{(2)}(t_0) > 0, \tag{2.30}$$

$$h_{p_1}^{(3)}(t_0) = \frac{\pi}{2} + \epsilon \quad \text{and} \quad \dot{h}_{p_1}^{(3)}(t_0) < 0, \tag{2.31}$$

$$h_{p_1}^{(3)}(t_0) = \hat{r}_p^{(3)} - \epsilon \quad \text{and} \quad \dot{h}_{p_1}^{(3)}(t_0) > 0. \tag{2.32}$$

For the case (2.25), noting that $g^{(0)}_{p_1}(h^{(0)}_{p_1}(t_0)) = g^{(0)}_{p_1}(h^{(0)}_{p_1}(t_0 - \tau_{p_1})) = g^{(0)}_{p_1}(\check{r}^{(0)}_{p_1} + \epsilon) = g^{(0)}_{p_1}(\check{r}^{(0)}_{p_1}) = -1$. Therefore, from (2.3) and (2.9), we can compute

$$
\begin{aligned}
\dot{h}^{(0)}_{p_1}(t_0) = -a_{p_1}h^{(0)}_{p_1}(t_0) + \sum_{q=1}^{n}\Big[b^{(0)}_{p_1q}g^{(0)}_{q}(h^{(0)}_{q}(t_0)) - b^{(1)}_{p_1q}g^{(1)}_{q}(h^{(1)}_{q}(t_0)) - b^{(2)}_{p_1q}g^{(2)}_{q}(h^{(2)}_{q}(t_0)) - \\
b^{(3)}_{p_1q}g^{(3)}_{q}(h^{(3)}_{q}(t_0))\Big] \quad + \sum_{q=1}^{n}\Big[c^{(0)}_{p_1q}g^{(0)}_{q}(h^{(0)}_{q}(t_0 - \tau_q)) - c^{(1)}_{p_1q}g^{(1)}_{q}(h^{(1)}_{q}(t_0 - \tau_q)) - \\
c^{(2)}_{p_1q}g^{(2)}_{q}(h^{(2)}_{q}(t_0 - \tau_q)) - c^{(3)}_{p_1q}g^{(3)}_{q}(h^{(3)}_{q}(t_0 - \tau_q))\Big] + k^{(0)}_{p_1} \\
\geq -a_{p_1}(\check{r}^{(0)}_{p_1} + \epsilon) + (b^{(0)}_{p_1p_1} + c^{(0)}_{p_1p_1})g^{(0)}_{p_1}(\check{r}^{(0)}_{p_1} + \epsilon) - \sum_{q=1,q\neq p}^{n}|(b^{(0)}_{pq} + c^{(0)}_{pq})| - \\
\sum_{q=1}^{n}\left[|(b^{(1)}_{pq} + c^{(1)}_{pq})| + |b^{(2)}_{pq} + c^{(2)}_{pq})| + |b^{(3)}_{pq} + c^{(3)}_{pq})|\right] + k^{(0)}_{p} \\
= -a_{p_1}\check{r}^{(0)}_{p_1} - a_{p_1}\epsilon + (b^{(0)}_{p_1p_1} + c^{(0)}_{p_1p_1})g^{(0)}_{p_1}(\check{r}^{(0)}_{p_1}) - \sum_{q=1,q\neq p}^{n}|(b^{(0)}_{p_1q} + c^{(0)}_{p_1q})| - \\
\sum_{q=1}^{n}\left[|(b^{(1)}_{p_1q} + c^{(1)}_{p_1q})| + |b^{(2)}_{p_1q} + c^{(2)}_{p_1q})| + |b^{(3)}_{p_1q} + c^{(3)}_{p_1q})|\right] + k^{(0)}_{p_1} \\
= \check{G}^{(0)}_{p_1}(\check{r}^{(0)}_{p_1}) - a_{p_1}\epsilon \geq 0,
\end{aligned}
\tag{2.33}
$$

which contradicts (2.25).

For case (2.26), noting that $g^{(0)}_{p_1}(\frac{-\pi}{2} - \epsilon) = g^{(0)}_{p_1}(\frac{-\pi}{2}) = -1$. Therefore, from (2.3) and

(2.17), we can compute

$$\begin{aligned}
\dot{h}_{p_1}^{(0)}(t_0) = -a_{p_1}h_{p_1}^{(0)}(t_0) + \sum_{q=1}^{n}\Big[b_{p_1q}^{(0)}g_q^{(0)}(h_q^{(0)}(t_0)) - b_{p_1q}^{(1)}g_q^{(1)}(h_q^{(1)}(t_0)) - b_{p_1q}^{(2)}g_q^{(2)}(h_q^{(2)}(t_0)) -& \\
b_{p_1q}^{(3)}g_q^{(3)}(h_q^{(3)}(t_0))\Big] + \sum_{q=1}^{n}\Big[c_{p_1q}^{(0)}g_q^{(0)}(h_q^{(0)}(t_0-\tau_q)) - c_{p_1q}^{(1)}g_q^{(1)}(h_q^{(1)}(t_0-\tau_q)) -& \\
c_{p_1q}^{(2)}g_q^{(2)}(h_q^{(2)}(t_0-\tau_q)) - c_{p_1q}^{(3)}g_q^{(3)}(h_q^{(3)}(t_0-\tau_q))\Big] + k_{p_1}^{(0)}& \\
\leq -a_{p_1}(\frac{-\pi}{2} - \epsilon) + b_{p_1p_1}^{(0)}g_{p_1}^{(0)}(\frac{-\pi}{2} - \epsilon) + \sum_{q=1, q\neq p_1}^{n}|b_{p_1q}^{(0)}| + \sum_{q=1}^{n}|c_{p_1q}^{(0)}| +& \\
\sum_{q=1}^{n}\Big[|b_{p_1q}^{(1)}| + |b_{p_1q}^{(2)}| + |b_{p_1q}^{(3)}|\Big] + \sum_{q=1}^{n}\Big[|c_{p_1q}^{(1)}| + |c_{p_1q}^{(2)}| + |c_{p_1q}^{(3)}|\Big] + k_{p_1}^{(0)}& \\
= -a_{p_1}(\frac{-\pi}{2}) + a_{p_1}\epsilon + b_{p_1p_1}^{(0)}g_{p_1}^{(0)}(\frac{-\pi}{2}) + \sum_{q=1, q\neq p_1}^{n}|b_{p_1q}^{(0)}| + \sum_{q=1}^{n}|c_{p_1q}^{(0)}| +& \\
\sum_{q=1}^{n}\Big[|b_{p_1q}^{(1)}| + |b_{p_1q}^{(2)}| + |b_{p_1q}^{(3)}|\Big] + \sum_{q=1}^{n}\Big[|c_{p_1q}^{(1)}| + |c_{p_1q}^{(2)}| + |c_{p_1q}^{(3)}|\Big] + k_{p_1}^{(0)}& \\
= \hat{\tilde{G}}_p^{(b)}(\frac{-\pi}{2}) + a_{p_1}\epsilon \leq 0,& \qquad (2.34)
\end{aligned}$$

which contradicts (2.26).

For the case (2.27), noting that $g_p^{(1)}(\frac{\pi}{2}) = 1 = g_{p_1}^{(1)}(-\frac{\pi}{6})$. Since ϵ is very very small, hence we can conclude, $g_{p_1}^{(1)}(-\frac{\pi}{6} + \epsilon) + \epsilon = g_{p_1}^{(1)}(-\frac{\pi}{6})$. Then from (2.4) and (2.21), we may compute

$$
\begin{aligned}
\dot{h}_{p_1}^{(1)}(t_0) = &-a_{p_1}h_{p_1}^{(1)}(t_0) + \sum_{q=1}^{n}\Big[b_{p_1q}^{(0)}g_q^{(1)}(h_q^{(1)}(t_0)) + b_{p_1q}^{(1)}g_q^{(0)}(h_q^{(0)}(t_0)) + b_{p_1q}^{(2)}g_q^{(3)}(h_q^{(3)}(t_0)) - \\
&b_{p_1q}^{(3)}g_q^{(2)}(h_q^{(2)}(t_0))\Big] + \sum_{q=1}^{n}\Big[c_{p_1q}^{(0)}g_q^{(1)}(h_q^{(1)}(t_0-\tau_q)) + c_{p_1q}^{(1)}g_q^{(0)}(h_q^{(0)}(t_0-\tau_q)) + \\
&c_{p_1q}^{(2)}g_q^{(3)}(h_q^{(3)}(t_0-\tau_q)) - c_{p_1q}^{(3)}g_q^{(2)}(h_q^{(2)}(t_0-\tau_q))\Big] + k_{p_1}^{(1)} \\
\geq &-a_{p_1}(-\frac{\pi}{6}+\epsilon) + b_{p_1p_1}^{(0)}g_{p_1}^{(1)}(-\frac{\pi}{6}+\epsilon) - \sum_{q=1,q\neq p_1}^{n}|b_{p_1q}^{(0)}| - \sum_{q=1}^{n}|c_{p_1q}^{(0)}| - \\
&\sum_{q=1}^{n}\Big[|b_{p_1q}^{(1)}| + |b_{p_1q}^{(2)}| + |b_{p_1q}^{(3)}|\Big] - \sum_{q=1}^{n}\Big[|c_{p_1q}^{(1)}| + |c_{p_1q}^{(2)}| + |c_{p_1q}^{(3)}|\Big] + k_{p_1}^{(1)} \\
= &-a_{p_1}(-\frac{\pi}{6}) - a_{p_1}\epsilon + b_{p_1p_1}^{(0)}g_{p_1}^{(1)}(-\frac{\pi}{6}+\epsilon) - \sum_{q=1,q\neq p_1}^{n}|b_{p_1q}^{(0)}| - \sum_{q=1}^{n}|c_{p_1q}^{(0)}| - \\
&\sum_{q=1}^{n}\Big[|b_{p_1q}^{(1)}| + |b_{p_1q}^{(2)}| + |b_{p_1q}^{(3)}|\Big] - \sum_{q=1}^{n}\Big[|c_{p_1q}^{(1)}| + |c_{p_1q}^{(2)}| + |c_{p_1q}^{(3)}|\Big] + k_{p_1}^{(1)} \\
= &\, a_{p_1}(\frac{\pi}{6}) + b_{p_1p_1}^{(0)}g_p^{(1)}(-\frac{\pi}{6}+\epsilon) + \check{\tilde{\eta}}_{p_1}^{(1)} - a_{p_1}\epsilon \\
= &\, a_{p_1}(\frac{\pi}{6}) + b_{p_1p_1}^{(0)}g_p^{(1)}(-\frac{\pi}{6}) + \check{\tilde{\eta}}_{p_1}^{(1)} - \epsilon(a_{p_1}+1) \geq 0, \qquad (2.35)
\end{aligned}
$$

which contradicts (2.27). For the case (2.28), noting that $g_p^{(1)}(\frac{-\pi}{2}) = -1 = g_{p_1}^{(1)}(\frac{\pi}{6})$. Since ϵ is very very small, hence we can conclude $g_{p_1}^{(1)}(\frac{\pi}{6}-\epsilon) - \epsilon = g_{p_1}^{(1)}(\frac{\pi}{6})$. Then, from (2.4) and (2.20), we may compute

$$\begin{aligned}
\dot{h}_{p_1}^{(1)}(t_0) = -a_{p_1}h_{p_1}^{(1)}(t_0) + \sum_{q=1}^{n}\Big[& b_{p_1q}^{(0)}g_q^{(1)}(h_q^{(1)}(t_0)) + b_{p_1q}^{(1)}g_q^{(0)}(h_q^{(0)}(t_0)) + b_{p_1q}^{(2)}g_q^{(3)}(h_q^{(3)}(t_0)) - \\
& b_{p_1q}^{(3)}g_q^{(2)}(h_q^{(2)}(t_0))\Big] + \sum_{q=1}^{n}\Big[c_{p_1q}^{(0)}g_q^{(1)}(h_q^{(1)}(t_0-\tau_q)) + c_{p_1q}^{(1)}g_q^{(0)}(h_q^{(0)}(t_0-\tau_q)) + \\
& c_{p_1q}^{(2)}g_q^{(3)}(h_q^{(3)}(t_0-\tau_q)) - c_{p_1q}^{(3)}g_q^{(2)}(h_q^{(2)}(t_0-\tau_q))\Big] + k_{p_1}^{(1)} \\
\le -a_{p_1}(\frac{\pi}{6}-\epsilon) + b_{p_1p_1}^{(0)}g_{p_1}^{(1)}(\frac{\pi}{6}-\epsilon) + \sum_{q=1,q\neq p_1}^{n}& |b_{p_1q}^{(0)}| + \sum_{q=1}^{n}|c_{p_1q}^{(0)}| + \\
& \sum_{q=1}^{n}\Big[|b_{p_1q}^{(1)}| + |b_{p_1q}^{(2)}| + |b_{p_1q}^{(3)}|\Big] + \sum_{q=1}^{n}\Big[|c_{p_1q}^{(1)}| + |c_{p_1q}^{(2)}| + |c_{p_1q}^{(3)}|\Big] + k_{p_1}^{(1)} \\
= -a_{p_1}(\frac{\pi}{6}) + a_{p_1}\epsilon + b_{p_1p_1}^{(0)}g_{p_1}^{(1)}(\frac{\pi}{6}-\epsilon) + & \sum_{q=1,q\neq p_1}^{n}|b_{p_1q}^{(0)}| + \sum_{q=1}^{n}|c_{p_1q}^{(0)}| + \\
& \sum_{q=1}^{n}\Big[|b_{p_1q}^{(1)}| + |b_{p_1q}^{(2)}| + |b_{p_1q}^{(3)}|\Big] + \sum_{q=1}^{n}\Big[|c_{p_1q}^{(1)}| + |c_{p_1q}^{(2)}| + |c_{p_1q}^{(3)}|\Big] + k_{p_1}^{(1)} \\
= -a_{p_1}(\frac{\pi}{6}) + b_{p_1p_1}^{(0)}g_p^{(1)}(\frac{\pi}{6}-\epsilon) + \hat{\tilde{\eta}}_{p_1}^{(1)} + a_{p_1}\epsilon \le 0 & \\
= -a_{p_1}(\frac{\pi}{6}) + b_{p_1p_1}^{(0)}g_p^{(1)}(\frac{\pi}{6}) + \hat{\tilde{\eta}}_{p_1}^{(1)} + (a_{p_1}+1)\epsilon \le 0, & \qquad (2.36)
\end{aligned}$$

which contradicts (2.28).

For the case (2.29), noting that $g^{(2)}_{p_1}(\frac{\pi}{2}) = 1 = g^{(2)}_{p_1}(\frac{\pi}{2}+\epsilon)$. Therefore from (2.5) and (2.19), we can compute

$$
\begin{aligned}
\dot{h}^{(2)}_{p_1}(t_0) = {} & -a_{p_1}h^{(2)}_{p_1}(t_0) + \sum_{q=1}^{n}\Big[b^{(0)}_{p_1q}g^{(2)}_q(h^{(2)}_q(t_0)) + b^{(2)}_{p_1q}g^{(0)}_q(b^{(0)}_q(t_0)) - b^{(1)}_{p_1q}g^{(3)}_q(h^{(3)}_q(t_0)) + \\
& b^{(3)}_{p_1q}g^{(1)}_q(h^{(1)}_q(t_0))\Big] + \sum_{q=1}^{n}\Big[c^{(0)}_{p_1q}g^{(2)}_q(h^{(2)}_q(t_0-\tau_q)) + c^{(2)}_{p_1q}g^{(0)}_q(h^{(0)}_q(t_0-\tau_q)) - \\
& c^{(1)}_{p_1q}g^{(3)}_q(h^{(3)}_q(t_0-\tau_q)) + c^{(3)}_{p_1q}g^{(1)}_q(h^{(1)}_q(t_0-\tau_q))\Big] + k^{(2)}_{p_1} \\
\geq {} & -a_{p_1}(\frac{\pi}{2}+\epsilon) + b^{(0)}_{p_1p_1}g^{(2)}_{p_1}(\frac{\pi}{2}+\epsilon) - \sum_{q=1, q\neq p_1}^{n}|b^{(0)}_{p_1q}| - \sum_{q=1}^{n}|c^{(0)}_{p_1q}| - \\
& \sum_{q=1}^{n}\Big[|b^{(1)}_{p_1q}| + |b^{(2)}_{p_1q}| + |b^{(3)}_{p_1q}|\Big] - \sum_{q=1}^{n}\Big[|c^{(1)}_{p_1q}| + |c^{(2)}_{p_1q}| + |c^{(3)}_{p_1q}|\Big] + k^{(2)}_{p_1} \\
= {} & -a_{p_1}(\frac{\pi}{2}) - a_{p_1}\epsilon + b^{(0)}_{p_1p_1}g^{(2)}_{p_1}(\frac{\pi}{2}+\epsilon) - \sum_{q=1, q\neq p_1}^{n}|b^{(0)}_{p_1q}| - \sum_{q=1}^{n}|c^{(0)}_{p_1q}| - \\
& \sum_{q=1}^{n}\Big[|b^{(1)}_{p_1q}| + |b^{(2)}_{p_1q}| + |b^{(3)}_{p_1q}|\Big] - \sum_{q=1}^{n}\Big[|c^{(1)}_{p_1q}| + |c^{(2)}_{p_1q}| + |c^{(3)}_{p_1q}|\Big] + k^{(1)}_{p_1} \\
= {} & -a_{p_1}(\frac{\pi}{2}) + b^{(0)}_{p_1p_1}g^{(2)}_{p}(\frac{\pi}{2}+\epsilon) + \check{\tilde{\eta}}^{(2)}_{p_1} - a_{p_1}\epsilon \geq 0,
\end{aligned}
\tag{2.37}
$$

which contradicts (2.29).

For the case (2.30), noting that $g^{(2)}_{p_1}(h^{(2)}_{p_1}(t_0)) = g^{(2)}_{p_1}(h^{(2)}_{p_1}(t_0 - \tau_{p_1})) = g^{(2)}_{p_1}(\hat{r}^{(2)}_{p_1} - \epsilon) = g^{(2)}_{p_1}(\hat{r}^{(2)}_{p_1}) = 1$. Therefore, from (2.5) and (2.9), we can compute

$$
\begin{aligned}
\dot{h}^{(2)}_{p_1}(t_0) = {} & -a_{p_1} h^{(2)}_{p_1}(t_0) + \sum_{q=1}^{n} \Big[b^{(0)}_{p_1q} g^{(2)}_q(h^{(2)}_q(t_0)) + b^{(2)}_{p_1q} g^{(0)}_q(b^{(0)}_q(t_0)) - b^{(1)}_{p_1q} g^{(3)}_q(h^{(3)}_q(t_0)) + \\
& b^{(3)}_{p_1q} g^{(1)}_q(h^{(1)}_q(t_0))\Big] + \sum_{q=1}^{n} \Big[c^{(0)}_{p_1q} g^{(2)}_q(h^{(2)}_q(t_0 - \tau_q)) + c^{(2)}_{p_1q} g^{(0)}_q(h^{(0)}_q(t_0 - \tau_q)) - \\
& c^{(1)}_{p_1q} g^{(3)}_q(h^{(3)}_q(t_0 - \tau_q)) + c^{(3)}_{p_1q} g^{(1)}_q(h^{(1)}_q(t_0 - \tau_q))\Big] + k^{(2)}_{p_1} \\
\le {} & -a_{p_1}(\hat{r}^{(2)}_{p_1} - \epsilon) + (b^{(0)}_{p_1p_1} + c^{(0)}_{p_1p_1}) g^{(2)}_{p_1}(\hat{r}^{(2)}_{p_1} - \epsilon) + \\
& \sum_{q=1,q\neq p}^{n} |(b^{(0)}_{pq} + c^{(0)}_{pq})| + \sum_{q=1}^{n} \Big[|(b^{(1)}_{pq} + c^{(1)}_{pq})| + |b^{(2)}_{pq} + c^{(2)}_{pq})| + |b^{(3)}_{pq} + c^{(3)}_{pq})| \Big] + k^{(2)}_{p_1} \\
= {} & -a_{p_1}\hat{r}^{(2)}_{p_1} + a_{p_1}\epsilon + (b^{(0)}_{p_1p_1} + c^{(0)}_{p_1p_1}) g^{(2)}_{p_1}(\hat{r}^{(2)}_{p_1}) + \sum_{q=1,q\neq p}^{n} |(b^{(0)}_{p_1q} + c^{(0)}_{p_1q})| + \\
& \sum_{q=1}^{n} \Big[|(b^{(1)}_{p_1q} + c^{(1)}_{p_1q})| + |b^{(2)}_{p_1q} + c^{(2)}_{p_1q})| + |b^{(3)}_{p_1q} + c^{(3)}_{p_1q})| \Big] + k^{(2)}_{p_1} \\
= {} & \hat{G}^{(2)}_{p_1}(\hat{r}^{(2)}_{p_1}) + a_{p_1}\epsilon \le 0,
\end{aligned}
\tag{2.38}
$$

which contradicts (2.30). Similarly, we can infer that cases (2.31) and (2.32) do not hold.

To sum up, we can obtain that each $\Pi^{\beta} \in \Pi$ is positively invariant of DQVNNs (2.1). This completes the proof. □

Theorem 2.3. Under the Assumption 2.1, suppose the conditions (2.18)-(2.19) are hold. If there also exist positive real constants $\xi^{(b)}_1, \xi^{(b)}_2, ..., \xi^{(b)}_n$ for $b = 0, 1, 2, 3$ satisfying the following conditions:

$$
\begin{aligned}
& -a_p(\xi^{(0)}_p)^{-1} + 3\Big[\sum_{q=1,q\neq p}^{n} |b^{(0)}_{pq}|(\xi^{(0)}_q)^{-1} + \sum_{q=1}^{n} |b^{(1)}_{pq}|(\xi^{(1)}_q)^{-1} + \sum_{q=1}^{n} |b^{(2)}_{pq}|(\xi^{(2)}_q)^{-1} + \sum_{q=1}^{n} |b^{(3)}_{pq}| \times \\
& (\xi^{(3)}_q)^{-1} + \sum_{q=1}^{n} |c^{(0)}_{pq}|(\xi^{(0)}_q)^{-1} + \sum_{q=1}^{n} |c^{(1)}_{pq}|(\xi^{(1)}_q)^{-1} + \sum_{q=1}^{n} |c^{(2)}_{pq}|(\xi^{(2)}_q)^{-1} + \sum_{q=1}^{n} |c^{(3)}_{pq}|(\xi^{(3)}_q)^{-1} \Big] < 0,
\end{aligned}
\tag{2.39a}
$$

$$
\begin{aligned}
&- a_p(\xi_p^{(1)})^{-1} + 3\Big[\sum_{q=1,q\neq p}^{n} |b_{pq}^{(0)}|(\xi_q^{(1)})^{-1} + \sum_{q=1}^{n}|b_{pq}^{(1)}|(\xi_q^{(0)})^{-1} + \sum_{q=1}^{n}|b_{pq}^{(2)}|(\xi_q^{(3)})^{-1} + \sum_{q=1}^{n}|b_{pq}^{(3)}|\times \\
&(\xi_q^{(2)})^{-1} + \sum_{q=1}^{n}|c_{pq}^{(0)}|(\xi_q^{(1)})^{-1} + \sum_{q=1}^{n}|c_{pq}^{(1)}|(\xi_q^{(0)})^{-1} + \sum_{q=1}^{n}|c_{pq}^{(2)}|(\xi_q^{(3)})^{-1} + \sum_{q=1}^{n}|c_{pq}^{(3)}|(\xi_q^{(2)})^{-1}\Big] < 0,
\end{aligned}
\tag{2.39b}
$$

$$
\begin{aligned}
&- a_p(\xi_p^{(2)})^{-1} + 3\Big[\sum_{q=1,q\neq p}^{n} |b_{pq}^{(0)}|(\xi_q^{(2)})^{-1} + \sum_{q=1}^{n}|b_{pq}^{(2)}|(\xi_q^{(0)})^{-1} + \sum_{q=1}^{n}|b_{pq}^{(1)}(\xi_q^{(3)})^{-1} + \sum_{q=1}^{n}|b_{pq}^{(3)}|\times \\
&(\xi_q^{(1)})^{-1} + \sum_{q=1}^{n}|c_{pq}^{(0)}|(\xi_q^{(2)})^{-1} + \sum_{q=1}^{n}|c_{pq}^{(2)}|(\xi_q^{(0)})^{-1} + \sum_{q=1}^{n}|c_{pq}^{(1)}|(\xi_q^{(3)})^{-1} + \sum_{q=1}^{n}|c_{pq}^{(3)}|(\xi_q^{(1)})^{-1}\Big] < 0,
\end{aligned}
\tag{2.39c}
$$

$$
\begin{aligned}
&- a_p(\xi_p^{(3)})^{-1} + 3\Big[\sum_{q=1,q\neq p}^{n} |b_{pq}^{(0)}|(\xi_q^{(3)})^{-1} + \sum_{q=1}^{n}|b_{pq}^{(3)}|(\xi_q^{(0)})^{-1} + \sum_{q=1}^{n}|b_{pq}^{(1)}|(\xi_q^{(2)})^{-1} + \sum_{q=1}^{n}|b_{pq}^{(2)}||\times \\
&(\xi_q^{(1)})^{-1} + \sum_{q=1}^{n}|c_{pq}^{(0)}|(\xi_q^{(3)})^{-1} + \sum_{q=1}^{n}|c_{pq}^{(3)}|(\xi_q^{(0)})^{-1} + \sum_{q=1}^{n}|c_{pq}^{(1)}|(\xi_q^{(2)})^{-1} + \sum_{q=1}^{n}|c_{pq}^{(2)}|(\xi_q^{(1)})^{-1}\Big] < 0,
\end{aligned}
\tag{2.39d}
$$

then DQVNNs (2.1) have 5^{4n} EPs, among which 3^{4n} are locally stable.

Proof. Let $h(t;\phi)$ be the solution of the DQVNNs (2.1) with the random initial condition $\phi(\theta) \in \Pi^{\beta}$, where $\theta \in [-\tau, 0]$. By Theorem 2.1, DQVNNs (2.1) have exactly one EP $\tilde{h} = (\tilde{h}_1\tilde{h}_2, ..., \tilde{h}_n)$ in the region Π^{β}. By setting $z(t) = h(t) - \tilde{h}$, the systems (2.3)-(2.6) can be transformed into

$$\begin{aligned}\dot{z}_p^{(0)}(t) =& -a_p z_p^{(0)}(t) + \sum_{q=1}^{n}\Big[b_{pq}^{(0)}\tilde{g}_q^{(0)}(z_q^{(0)}(t)) - b_{pq}^{(1)}\tilde{g}_q^{(1)}(z_q^{(1)}(t)) - \\ & \qquad b_{pq}^{(2)}\tilde{g}_q^{(2)}(z_q^{(2)}(t)) - b_{pq}^{(3)}\tilde{g}_q^{(3)}(z_q^{(3)}(t))\Big] \\ & + \sum_{q=1}^{n}\Big[c_{pq}^{(0)}\tilde{g}_q^{(0)}(z_q^{(0)}(t-\tau_q)) - c_{pq}^{(1)}\tilde{g}_q^{(1)}(z_q^{(1)}(t-\tau_q)) \\ & \qquad - c_{pq}^{(2)}\tilde{g}_q^{(2)}(z_q^{(2)}(t-\tau_q)) - c_{pq}^{(3)}\tilde{g}_q^{(3)}(z_q^{(3)}(t-\tau_q))\Big],\end{aligned} \tag{2.40a}$$

$$\begin{aligned}\dot{z}_p^{(1)}(t) =& -a_p z_p^{(1)}(t) + \sum_{q=1}^{n}\Big[b_{pq}^{(0)}\tilde{g}_q^{(1)}(z_q^{(1)}(t)) + b_{pq}^{(1)}\tilde{g}_q^{(0)}(z_q^{(0)}(t)) + \\ & \qquad b_{pq}^{(2)}\tilde{g}_q^{(3)}(z_q^{(3)}(t)) - b_{pq}^{(3)}\tilde{g}_q^{(2)}(z_q^{(2)}(t))\Big] \\ & + \sum_{q=1}^{n}\Big[c_{pq}^{(0)}\tilde{g}_q^{(1)}(z_q^{(1)}(t-\tau_q)) + c_{pq}^{(1)}\tilde{g}_q^{(0)}(z_q^{(0)}(t-\tau_q)) \\ & \qquad + c_{pq}^{(2)}\tilde{g}_q^{(3)}(z_q^{(3)}(t-\tau_q)) - c_{pq}^{(3)}\tilde{g}_q^{(2)}(z_q^{(2)}(t-\tau_q))\Big],\end{aligned} \tag{2.40b}$$

$$\begin{aligned}\dot{z}_p^{(2)}(t) =& -a_p z_p^{(2)}(t) + \sum_{q=1}^{n}\Big[b_{pq}^{(0)}\tilde{g}_q^{(2)}(z_q^{(2)}(t)) + b_{pq}^{(2)}\tilde{g}_q^{(0)}(z_q^{(0)}(t)) - \\ & \qquad b_{pq}^{(1)}\tilde{g}_q^{(3)}(z_q^{(3)}(t)) + b_{pq}^{(3)}\tilde{g}_q^{(1)}(z_q^{(1)}(t))\Big] \\ & + \sum_{q=1}^{n}\Big[c_{pq}^{(0)}\tilde{g}_q^{(2)}(z_q^{(2)}(t-\tau_q)) + c_{pq}^{(2)}\tilde{g}_q^{(0)}(z_q^{(0)}(t-\tau_q)) \\ & \qquad - c_{pq}^{(1)}\tilde{g}_q^{(3)}(z_q^{(3)}(t-\tau_q)) + c_{pq}^{(3)}\tilde{g}_q^{(1)}(z_q^{(1)}(t-\tau_q))\Big],\end{aligned} \tag{2.40c}$$

$$\begin{aligned}\dot{z}_p^{(3)}(t) =& - a_p z_p^{(3)}(t) + \sum_{q=1}^{n} \Big[b_{pq}^{(0)} \tilde{g}_q^{(3)}(z_q^{(3)}(t)) + b_{pq}^{(3)} \tilde{g}_q^{(0)}(z_q^{(0)}(t)) + \\ &\qquad b_{pq}^{(1)} \tilde{g}_q^{(2)}(z_q^{(2)}(t)) - b_{pq}^{(2)} \tilde{g}_q^{(1)}(z_q^{(1)}(t)) \Big] \\ &+ \sum_{q=1}^{n} \Big[c_{pq}^{(0)} \tilde{g}_q^{(3)}(z_q^{(3)}(t-\tau_q)) + c_{pq}^{(3)} \tilde{g}_q^{(0)}(z_q^{(0)}(t-\tau_q)) \\ &\qquad + c_{pq}^{(1)} \tilde{g}_q^{(2)}(z_q^{(2)}(t-\tau_q)) - c_{pq}^{(2)} \tilde{g}_q^{(1)}(z_q^{(1)}(t-\tau_q)) \Big], \end{aligned} \tag{2.40d}$$

where $p = 1, 2, ...n$, $\tilde{g}_q^{(b)}(z_q(t)) = g_q^{(b)}(h_q^{(b)}(t)) - g_q^{(b)}(\tilde{h}_q^{(b)})$, $\tilde{g}_q^{(b)}(z_q(t-\tau_q)) = g_q^{(b)}(h_q^{(b)}(t-\tau_q)) - g_q^{(b)}(\tilde{h}_q^{(b)})$ with $p = 1, 2, ..., n$ and $b = 0, 1, 2, 3$.

Let,

$$N_1^{(b)} = \big\{ q \mid h_q^{(b)}(t) \in I_q^{(b)\mathrm{I}},\ q = 1, 2, ..., n \big\},$$

$$N_3^{(b)} = \big\{ q \mid h_q^{(b)}(t) \in I_q^{(b)\mathrm{III}},\ q = 1, 2, ..., n \big\},$$

$$N_5^{(b)} = \big\{ q \mid h_q^{(b)}(t) \in I_q^{(b)\mathrm{V}},\ q = 1, 2, ..., n \big\},$$

for each $b = 0, 1, 2, 3$; $N_1^{(b)}, N_3^{(b)}, N_5^{(b)}$ are subsets of $\{1, 2, ..., n\}$ and $N_1^{(b)} \cup N_3^{(b)} \cup N_5^{(b)} = \{1, 2, ..., n\}$, $N_l^{(b)} \cap N_m^{(b)} = \varphi$, where $l \neq m; l, m = 1, 3, 5$. Since Π^β is positive invariant, we can obtain that $h(t; \phi) \in \Pi^\beta\ \forall\ t > 0$. When $h(t; \phi) \in \Pi^\beta$, then $\sum_{q \in N_1^{(b)}} g_q^{(b)}(h_q^{(b)})$ and $\sum_{q \in N_5^{(b)}} g_q^{(b)}(h_q^{(b)})$ are constants. That is $\sum_{q \in N_1^{(b)}} g_q^{(b)}(h_q^{(b)}(t)) = \sum_{q \in N_1^{(b)}} g_q^{(b)}(\tilde{h}_q^{(b)})$ and $\sum_{q \in N_3^{(b)}} g_q^{(b)}(h_q^{(b)}(t)) = \sum_{q \in N_3^{(b)}} g_q^{(b)}(\tilde{h}_q^{(b)})$. By virtue of Lagrange's mean value theorem, there exist $\gamma_q^{(b)}$ between $h_q^{(b)}(t)$ and $\tilde{h}_q^{(b)}$, and also $\kappa_q^{(b)}$ between $h_q^{(b)}(t-\tau_q)$ and $\tilde{h}_q^{(b)}$ such that $\tilde{g}_q^{(b)}(z_q^{(b)}(t)) = g_q^{(b)'}(\gamma_q^{(b)}) z_q^{(b)}(t)$ and $\tilde{g}_q^{(b)}(z_q^{(b)}(t-\tau_q)) = g_q^{(b)'}(\kappa_q^{(b)}) z_q^{(b)}(t-\tau_q)$ for $q \in N_2^{(b)}$. Clearly, for $b = 0, 1, 2, 3$ and $q \in N_2^{(b)}$, we have $-3 < g_q^{(b)'}(\gamma_q^{(b)}) < 0$ and $-3 < g_q^{(b)'}(\kappa_q^{(b)}) < 0$. According to (2.39a)-(2.39d), we can get a very small number ϵ such that

$$\begin{aligned}
&(-a_p+\epsilon)(\xi_p^{(0)})^{-1}+\sum_{q=1,q\neq p}^{n}3|b_{pq}^{(0)}|(\xi_q^{(0)})^{-1}+\sum_{q=1}^{n}3|b_{pq}^{(1)}|(\xi_q^{(1)})^{-1}+\sum_{q=1}^{n}3|b_{pq}^{(2)}|(\xi_q^{(2)})^{-1}+\\
&\sum_{q=1}^{n}3|b_{pq}^{(3)}|(\xi_q^{(3)})^{-1}+e^{\epsilon\tau}\Bigg\{\sum_{q=1}^{n}3|c_{pq}^{(0)}|(\xi_q^{(0)})^{-1}+\sum_{q=1}^{n}3|c_{pq}^{(1)}|(\xi_q^{(1)})^{-1}+\\
&\qquad\qquad\sum_{q=1}^{n}3|c_{pq}^{(2)}|(\xi_q^{(2)})^{-1}+\sum_{q=1}^{n}3|c_{pq}^{(3)}|(\xi_q^{(3)})^{-1}\Bigg\}<0,
\end{aligned}\tag{2.41a}$$

$$\begin{aligned}
&(-a_p+\epsilon)(\xi_p^{(1)})^{-1}+\sum_{q=1q\neq p}^{n}3|b_{pq}^{(0)}|(\xi_q^{(1)})^{-1}+\sum_{q=1}^{n}3|b_{pq}^{(1)}|(\xi_q^{(0)})^{-1}+\sum_{q=1}^{n}3|b_{pq}^{(2)}|(\xi_q^{(3)})^{-1}+\\
&\sum_{q=1}^{n}3|b_{pq}^{(3)}|(\xi_q^{(2)})^{-1}+e^{\epsilon\tau}\Bigg\{\sum_{q=1}^{n}3|c_{pq}^{(0)}|(\xi_q^{(1)})^{-1}+\sum_{q=1}^{n}3|c_{pq}^{(1)}|(\xi_q^{(0)})^{-1}+\\
&\qquad\qquad\sum_{q=1}^{n}3|c_{pq}^{(2)}|(\xi_q^{(3)})^{-1}+\sum_{q=1}^{n}3|c_{pq}^{(3)}|(\xi_q^{(2)})^{-1}\Bigg\}<0,
\end{aligned}\tag{2.41b}$$

$$\begin{aligned}
&(-a_p+\epsilon)(\xi_p^{(2)})^{-1}+\sum_{q=1,q\neq p}^{n}3|b_{pq}^{(0)}|(\xi_q^{(2)})^{-1}+\sum_{q=1}^{n}3|b_{pq}^{(2)}|(\xi_q^{(0)})^{-1}+\sum_{q=1}^{n}3|b_{pq}^{(1)}|(\xi_q^{(3)})^{-1}+\\
&\sum_{q=1}^{n}3|b_{pq}^{(3)}|(\xi_q^{(1)})^{-1}+e^{\epsilon\tau}\Bigg\{\sum_{q=1}^{n}3|c_{pq}^{(0)}|(\xi_q^{(2)})^{-1}+\sum_{q=1}^{n}3|c_{pq}^{(2)}|(\xi_q^{(0)})^{-1}+\\
&\qquad\qquad\sum_{q=1}^{n}3|c_{pq}^{(1)}|(\xi_q^{(3)})^{-1}+\sum_{q=1}^{n}3|c_{pq}^{(3)}|(\xi_q^{(1)})^{-1}\Bigg\}<0,
\end{aligned}\tag{2.41c}$$

$$\begin{aligned}
&(-a_p+\epsilon)(\xi_p^{(3)})^{-1}+\sum_{q=1,q\neq p}^{n}3|b_{pq}^{(0)}|(\xi_q^{(3)})^{-1}+\sum_{q=1}^{n}3|b_{pq}^{(3)}|(\xi_q^{(0)})^{-1}+\sum_{q=1}^{n}3|b_{pq}^{(1)}|(\xi_q^{(2)})^{-1}+\\
&\sum_{q=1}^{n}3|b_{pq}^{(2)}|(\xi_q^{(1)})^{-1}+e^{\epsilon\tau}\Bigg\{\sum_{q=1}^{n}3|c_{pq}^{(0)}|(\xi_q^{(3)})^{-1}+\sum_{q=1}^{n}3|c_{pq}^{(3)}|(\xi_q^{(0)})^{-1}+\\
&\qquad\qquad\sum_{q=1}^{n}3|c_{pq}^{(1)}|(\xi_q^{(2)})^{-1}+\sum_{q=1}^{n}3|c_{pq}^{(2)}|(\xi_q^{(1)})^{-1}\Bigg\}<0.
\end{aligned}\tag{2.41d}$$

Let $H_p^{(b)}(t) = e^{\epsilon t} z_p^{(b)}(t)$; $H(t) = (H^{(0)}(t); H^{(1)}(t); H^{(2)}(t); H^{(3)}(t))$ with $H^{(b)}(t) = (H_1^{(b)}(t), H_2^{(b)}(t), ..., H_n^{(b)}(t))^T$, where $b = 0, 1, 2, 3$.

Let us define

$$M(t) = \sup_{u \leq t} \|H(u)\|_\xi, \quad t \geq 0, \tag{2.42}$$

where $\|H(u)\|_\xi = \max\left\{\|H^{(0)}(u)\|_{\xi^{(0)}}, \|H^{(1)}(u)\|_{\xi^{(1)}}, \|H^{(2)}(u)\|_{\xi^{(2)}}, \|H^{(3)}(u)\|_{\xi^{(3)}}\right\}$ with $\|H^{(b)}(u)\|_{\xi^{(b)}} = \max_{p \in N_2^{(b)}} \left\{\xi_p^{(b)} |H_p^{(b)}(u)|\right\}$, $b = 0, 1, 2, 3$.

We claim that $M(t)$ is bounded, consequently $\|H(u)\|_\xi$ is bounded too. According to the definition of $M(t)$, we have $\|H(u)\|_\xi \leq M(t)$ for all $t \geq 0$. If at a special time point $t_0 \geq 0$ such that $\|H(t_0)\|_\xi = M(t_0)$, we have the following four possibilities:

Case 2.1. An index $p^{(0)} = p^{(0)}(t_0) \in N_2^{(0)}$ depending on t_0 exists such that

$$\|H(t_0)\|_\xi = \|H^{(0)}(t_0)\|_{\xi^{(0)}} = \xi_{P^{(0)}}^{(0)} |H_{p^{(0)}}^{(0)}(t_0)|. \tag{2.43}$$

Case 2.2. An index $p^{(1)} = p^{(1)}(t_0) \in N_2^{(1)}$ depending on t_0 exists such that

$$\|H(t_0)\|_\xi = \|H^{(1)}(t_0)\|_{\xi^{(1)}} = \xi_{p^{(1)}}^{(1)} |H_{p^{(1)}}^{(1)}(t_0)|. \tag{2.44}$$

Case 2.3. An index $p^{(2)} = p^{(2)}(t_0) \in N_2^{(2)}$ depending on t_0 exists such that

$$\|H(t_0)\|_\xi = \|H^{(2)}(t_0)\|_{\xi^{(2)}} = \xi_{p^{(2)}}^{(2)} |H_{p^{(2)}}^{(2)}(t_0)|. \tag{2.45}$$

Case 2.4. An index $p^{(3)} = p^{(3)}(t) \in N_2^{(3)}$ depending on t_0 exists such that

$$\|H(t_0)\|_\xi = \|H^{(3)}(t_0)\|_{\xi^{(3)}} = \xi^{(3)}_{p^{(3)}} |H^{(3)}_{p^{(3)}}(t_0)|. \tag{2.46}$$

Here, only Case 2.1 will be discussed. The rest of the cases can be proved in a similar way.

For Case 2.1, we have $(\xi^{(0)}_{p^{(0)}})^{-1}\|H(t_0)\|_\xi = |H^{(0)}_{p^{(0)}}(t_0)|$. Due to (2.40a) and (2.41a), we can get

$$\begin{aligned}
D^-|H^{(0)}_{p^{(0)}}(t_0)| &= sign\{H^{(0)}_{p^{(0)}}(t_0)\}\dot{H}^{(0)}_{p^{(0)}}(t_0)\\
&= sign\{z^{(0)}_{p^{(0)}}(t_0)\}\epsilon e^{\epsilon t_0} z^{(0)}_{p^{(0)}}(t_0) + sign\{z^{(0)}_{p^{(0)}}(t_0)\}e^{\epsilon t_0}\dot{z}^{(0)}_{p^{(0)}}(t_0)\\
&= sign\{z^{(0)}_{p^{(0)}}(t_0)\}\epsilon e^{\epsilon t_0} z^{(0)}_{p^{(0)}}(t_0) + sign\{z^{(0)}_{p^{(0)}}(t_0)\}e^{\epsilon t_0}\Bigg\{ - a_{p_0} z^{(0)}_{p^{(0)}}(t_0) +\\
&\quad \sum_{q=1}^{n}\Big[b^{(0)}_{p^{(0)}q}\tilde{g}^{(0)}_q(z^{(0)}_q(t_0)) - b^{(1)}_{p^{(0)}q}\tilde{g}^{(1)}_q(z^{(1)}_q(t_0)) - b^{(2)}_{p^{(0)}q}\tilde{g}^{(2)}_q(z^{(2)}_q(t_0)) -\\
&\quad b^{(3)}_{p^{(0)}q}\tilde{g}^{(3)}_q(z^{(3)}_q(t_0))\Big] + \sum_{q=1}^{n}\Big[c^{(0)}_{p^{(0)}q}\tilde{g}^{(0)}_q(z^{(0)}_q(t_0-\tau_q)) - c^{(1)}_{p^{(0)}q}\tilde{g}^{(1)}_q(z^{(1)}_q(t_0-\tau_q))\\
&\quad - c^{(2)}_{p^{(0)}q}\tilde{g}^{(2)}_q(z^{(2)}_q(t_0-\tau_q)) - c^{(3)}_{p^{(0)}q}\tilde{g}^{(3)}_q(z^{(3)}_q(t_0-\tau_q))\Big]\Bigg\}\\
&\le \epsilon e^{\epsilon t_0}|z^{(0)}_{p^{(0)}}(t_0)| + e^{\epsilon t_0}\Bigg\{ - a_{p^{(0)}}|z^{(0)}_{p^{(0)}}(t_0)| + \sum_{q\in N_2^{(0)},\ q\neq p^{(0)}} |b^{(0)}_{p^{(0)}q}||\tilde{g}^{(0)}_q(z^{(0)}_q(t_0))| +\\
&\quad \sum_{q\in N_2^{(1)}} |b^{(1)}_{p^{(0)}q}||\tilde{g}^{(1)}_q(z^{(1)}_q(t_0))| + \sum_{q\in N_2^{(2)}} |b^{(2)}_{p^{(0)}q}||\tilde{g}^{(2)}_q(z^{(2)}_q(t_0))|\\
&\quad + \sum_{q\in N_2^{(3)}} |b^{(3)}_{p^{(0)}q}||\tilde{g}^{(3)}_q(z^{(3)}_q(t_0))| + \sum_{q\in N_2^{(0)}} |c^{(0)}_{p^{(0)}q}||\tilde{g}^{(0)}_q(z^{(0)}_q(t_0-\tau_q))| +\\
&\quad \sum_{q\in N_2^{(1)}} |c^{(1)}_{p^{(0)}q}||\tilde{g}^{(1)}_q(z^{(1)}_q(t_0-\tau_q))| + \sum_{q\in N_2^{(2)}} |c^{(2)}_{p^{(0)}q}||\tilde{g}^{(2)}_q(z^{(2)}_q(t_0-\tau_q))| +\\
&\quad \sum_{q\in N_2^{(3)}} |c^{(3)}_{p^{(0)}q}||\tilde{g}^{(3)}_q(z^{(3)}_q(t_0-\tau_q))|\Bigg\}
\end{aligned}$$

$$\begin{aligned}
&\leq (-a_{p^{(0)}}+\epsilon)e^{\epsilon t_0}|z^{(0)}_{p^{(0)}}(t_0)| + e^{\epsilon t_0}\Bigg\{ \sum_{q\in N_2^{(0)},\ q\neq p^{(0)}} |b^{(0)}_{p^{(0)}q}||g_q^{(0)'}(\gamma_q^{(0)})||z_q^{(0)}(t_0)| \\
&\quad + \sum_{q\in N_2^{(1)}} |b^{(1)}_{p^{(0)}q}||g_q^{(1)'}(\gamma_q^{(1)})||z_q^{(1)}(t_0)| + \sum_{q\in N_2^{(2)}} |b^{(2)}_{p^{(0)}q}||g_q^{(2)'}(\gamma_q^{(2)})||z_q^{(2)}(t_0)| \\
&\quad + \sum_{q\in N_2^{(3)}} |b^{(3)}_{p^{(0)}q}||g_q^{(3)'}(\gamma_q^{(3)})||z_q^{(3)}(t_0)| + \sum_{q\in N_2^{(0)}} |c^{(0)}_{p^{(0)}q}||g_q^{(0)'}(\kappa_q^{(0)})||z_q^{(0)}(t_0-\tau_q)| + \\
&\sum_{q\in N_2^{(1)}} |c^{(1)}_{p^{(0)}q}||g_q^{(1)'}(\kappa_q^{(1)})||z_q^{(1)}(t_0-\tau_q)| + \sum_{q\in N_2^{(2)}} |c^{(2)}_{p^{(0)}q}||g_q^{(2)'}(\kappa_q^{(2)})||z_q^{(2)}(t_0-\tau_q)| + \\
&\sum_{q\in N_2^{(3)}} |c^{(3)}_{p^{(0)}q}||g_q^{(3)'}(\kappa_q^{(3)})||z_q^{(3)}(t_0-\tau_q)|\Bigg\} \\
&\leq (-a_{p^{(0)}}+\epsilon)|H^{(0)}_{p^{(0)}}(t_0)| + \Bigg\{ \sum_{q\in N_2^{(0)},\ q\neq p^{(0)}} |b^{(0)}_{p^{(0)}q}||g_q^{(0)'}(\gamma_q^{(0)})||H_q^{(0)}(t_0)| + \\
&\sum_{q\in N_2^{(1)}} |b^{(1)}_{p^{(0)}q}||g_q^{(1)'}(\gamma_q^{(1)})||H_q^{(1)}(t_0)| + \sum_{q\in N_2^{(2)}} |b^{(2)}_{p^{(0)}q}||g_q^{(2)'}(\gamma_q^{(2)})||H_q^{(2)}(t_0)| + \\
&\sum_{q\in N_2^{(3)}} |b^{(3)}_{p^{(0)}q}||g_q^{(3)'}(\gamma_q^{(3)})||H_q^{(3)}(t_0)| + \sum_{q\in N_2^{(0)}} |c^{(0)}_{p^{(0)}q}||g_q^{(0)'}(\kappa_q^{(0)})||H_q^{(0)}(t_0-\tau_q)|e^{\epsilon\tau_q} + \\
&\sum_{q\in N_2^{(1)}} |c^{(1)}_{p^{(0)}q}||g_q^{(1)'}(\kappa_q^{(1)})||H_q^{(1)}(t_0-\tau_q)|e^{\epsilon\tau_q} + \sum_{q\in N_2^{(2)}} |c^{(2)}_{p^{(0)}q}||g_q^{(2)'}(\kappa_q^{(2)})||H_q^{(2)}(t_0-\tau_q)|e^{\epsilon\tau_q} + \\
&\sum_{q\in N_2^{(3)}} |c^{(3)}_{p^{(0)}q}||g_q^{(3)'}(\kappa_q^{(3)})||H_q^{(3)}(t_0-\tau_q)|e^{\epsilon\tau_q}\Bigg\} \\
&\leq (-a_{p^{(0)}}+\epsilon)(\xi^{(0)}_{P^{(0)}})^{-1}\|H^{(0)}_{p^{(0)}}(t_0)\|_{\xi^{(0)}} + \sum_{q\in N_2^{(0)},\ q\neq p^{(0)}} |b^{(0)}_{p^{(0)}q}||g_q^{(0)'}(\gamma_q^{(0)})|(\xi_q^{(0)})^{-1} \\
&\quad \times \|H_q^{(0)}(t_0)\|_{\xi^{(0)}} + \sum_{q\in N_2^{(1)}} |b^{(1)}_{p^{(0)}q}||g_q^{(1)'}(\gamma_q^{(1)})|(\xi_q^{(1)})^{-1}\|H_q^{(1)}(t_0)\|_{\xi^{(1)}} + \sum_{q\in N_2^{(2)}} |b^{(2)}_{p^{(0)}q}| \\
&\quad \times |g_q^{(2)'}(\gamma_q^{(2)})|(\xi_q^{(2)})^{-1}\|H_q^{(2)}(t_0)\|_{\xi^{(2)}} + \sum_{q\in N_2^{(3)}} |b^{(3)}_{p^{(0)}q}||g_q^{(3)'}(\gamma_q^{(3)})|(\xi_q^{(3)})^{-1}\|H_q^{(3)}(t_0)\|_{\xi^{(3)}} \\
&\quad + \sum_{q\in N_2^{(0)}} |c^{(0)}_{p^{(0)}q}||g_q^{(0)'}(\kappa_q^{(0)})|(\xi_q^{(0)})^{-1}\|H_q^{(0)}(t_0-\tau_q)\|_{\xi^{(0)}}e^{\epsilon\tau_q} + \sum_{q\in N_2^{(1)}} |c^{(1)}_{p^{(0)}q}||g_q^{(1)'}(\kappa_q^{(1)})| \\
&\quad \times (\xi_q^{(1)})^{-1}\|H_q^{(1)}(t_0-\tau_q)\|_{\xi^{(1)}}e^{\epsilon\tau_q} + \sum_{q\in N_2^{(2)}} |c^{(2)}_{p^{(0)}q}||g_q^{(2)'}(\kappa_q^{(2)})|(\xi_q^{(2)})^{-1}\|H_q^{(2)}(t_0-\tau_q)\|_{\xi^{(2)}}e^{\epsilon\tau_q} \\
&\quad + \sum_{q\in N_2^{(3)}} |c^{(3)}_{p^{(0)}q}||g_q^{(3)'}(\kappa_q^{(3)})|(\xi_q^{(3)})^{-1}\|H_q^{(3)}(t_0-\tau_q)\|_{\xi^{(3)}}e^{\epsilon\tau_q}
\end{aligned}$$

$$
\begin{aligned}
&\leq \Bigg\{ (-a_{p^{(0)}}+\epsilon)(\xi^{(0)}_{P^{(0)}})^{-1} + \sum_{q\in N_2^{(0)},\ q\neq p^{(0)}} 3|b^{(0)}_{p^{(0)}q}|(\xi^{(0)}_q)^{-1} + \sum_{q\in N_2^{(1)}} 3|b^{(1)}_{p^{(0)}q}|(\xi^{(1)}_q)^{-1} + \\
&\quad \sum_{q\in N_2^{(2)}} 3|b^{(2)}_{p^{(0)}q}|(\xi^{(2)}_q)^{-1} + \sum_{q\in N_2^{(3)}} 3|b^{(3)}_{p^{(0)}q}|(\xi^{(3)}_q)^{-1} + \sum_{q\in N_2^{(0)}} 3|c^{(0)}_{p^{(0)}q}|(\xi^{(0)}_q)^{-1}e^{\epsilon\tau_q} + \sum_{q\in N_2^{(1)}} 3 \\
&\quad \times |c^{(1)}_{p^{(0)}q}|(\xi^{(1)}_q)^{-1}e^{\epsilon\tau_q} + \sum_{q\in N_2^{(2)}} 3|c^{(2)}_{p^{(0)}q}|(\xi^{(2)}_q)^{-1}e^{\epsilon\tau_q} + \sum_{q\in N_2^{(3)}} 3|c^{(3)}_{p^{(0)}q}|(\xi^{(3)}_q)^{-1}e^{\epsilon\tau_q} \Bigg\} M(t_0) \\
&< 0. \qquad (2.47)
\end{aligned}
$$

From the above inequality, it means that $\|H(t)\|_\xi$ is strictly decreasing when time passes the point t_0. Now from Case 2.1- Case 2.4 and from the definition of $M(t)$, we can conclude that $M(t)$ is non-increasing. Clearly $M(t)$ and $\|H(t)\|_\xi$ are bounded. Consequently, for $b=0,1,2,3$, each $\|H^{(b)}(t)\|_{\xi^{(b)}}$ is bounded. Therefore, for $p=1,2,...,n$ and $b=0,1,2,3$, each $|H_p^{(b)}(t)|$ is bounded, hence there exist positive constants $\lambda_i^{(b)}>0$ such that $|H_p^{(b)}(t)|\leq \lambda_p^{(b)}$, $\forall\ t\geq t_0$. Hence, we must have

$$
|z_p^{(b)}(t)| \leq \lambda_p^{(b)} e^{-\epsilon t}, \qquad \forall t\geq t_0,\ p\in N_2^{(b)}. \qquad (2.48)
$$

Consequently, $\tilde{h}_p^{(b)}$ satisfies the definition of Lyapunov stability when $p\in N_2^{(b)}$. So, for any $\epsilon^{(b)}>0$, we can find $\delta^{(b)}>0$ such that the following inequality

$$
\begin{aligned}
&\sum_{q\in N_2^{(0)}} |b^{(0)}_{pq}||\tilde{g}^{(0)}_q(z^{(0)}_q(t))| + \sum_{q\in N_2^{(1)}} |b^{(1)}_{pq}||\tilde{g}^{(1)}_q(z^{(1)}_q(t))| + \sum_{q\in N_2^{(2)}} |b^{(2)}_{pq}||\tilde{g}^{(2)}_q(z^{(2)}_q(t))| + \sum_{q\in N_2^{(3)}} |b^{(3)}_{pq}| \\
&\times |\tilde{g}^{(3)}_q(z^{(3)}_q(t))| + \sum_{q\in N_2^{(0)}} |c^{(0)}_{pq}||\tilde{g}^{(0)}_q(z^{(0)}_q(t-\tau_q))| + \sum_{q\in N_2^{(1)}} |c^{(1)}_{pq}\tilde{g}^{(1)}_q(z^{(1)}_q(t-\tau_q))| + \\
&\sum_{q\in N_2^{(2)}} |c^{(2)}_{pq}||\tilde{g}^{(2)}_q(z^{(2)}_q(t-\tau_q))| + \sum_{q\in N_2^{(3)}} |c^{(3)}_{pq}||\tilde{g}^{(3)}_q(z^{(3)}_q(t-\tau_q))| < \epsilon^{(0)}. \qquad (2.49)
\end{aligned}
$$

holds $\forall\ p\in N_2^{(0)}$ and $\forall\ t>0$, when $|h_p^{(0)}(\theta)-\tilde{h}_p^{(0)}|<\delta^{(0)}$ for all $p\in N_2^{(0)}$ and $\theta\in[\tau,0]$. According to (2.40a) and (2.48), we can obtain

$$
D^{-}\{|z_p^{(0)}(t)|\} \leq -a_p|z_p^{(0)}(t)| + \epsilon^{(0)}, \qquad t\geq 0,\ p\in N_1^{(0)}\cup N_3^{(0)},
$$

which immediately assures that

$$|z_p^{(0)}(t)| \leq e^{-a_p t}|z_p^{(0)}(0)| + \frac{\epsilon^{(0)}}{a_p}(1 - e^{-a_p t}), \qquad t \geq 0,\ p \in N_1^{(0)} \cup N_3^{(0)}. \tag{2.50a}$$

Similarly, we can obtain

$$|z_p^{(1)}(t)| \leq e^{-a_p t}|z_p^{(1)}(0)| + \frac{\epsilon^{(1)}}{a_p}(1 - e^{-a_p t}), \qquad t \geq 0,\ p \in N_1^{(1)} \cup N_3^{(1)}. \tag{2.50b}$$

$$|z_p^{(2)}(t)| \leq e^{-a_p t}|z_p^{(2)}(0)| + \frac{\epsilon^{(2)}}{a_p}(1 - e^{-a_p t}), \qquad t \geq 0,\ p \in N_1^{(2)} \cup N_3^{(2)}. \tag{2.50c}$$

$$|z_p^{(3)}(t)| \leq e^{-a_p t}|z_p^{(3)}(0)| + \frac{\epsilon^{(3)}}{a_p}(1 - e^{-a_p t}), \qquad t \geq 0,\ p \in N_1^{(3)} \cup N_3^{(3)}. \tag{2.50d}$$

According to (2.50a)-(2.50d), it is seen that for $b = 0, 1, 2, 3$, $\tilde{h}_p^{(b)}$ satisfies the definition of Lyapunov stability when $p \in N_2^{(b)} \cup N_3^{(b)}$. Therefore, we can observe that $\tilde{h}$ is stable in Π^β. Then from (2.48) and (2.50a)-(2.50d), for all $p = 1, 2, ..., n$, we can get $\lim_{t\to\infty}|z_p^{(b)}(t)| = 0$. It follows that the EP $\tilde{h}$ is attractive in Π^β.

Hence, the EP $\tilde{h}$ is locally asymptotically stable in Π^β. Since Π^β is any arbitrary in Π, therefore DQVNNs (2.1) have 3^{4n} locally stable EPs. This completes the proof. □

Remark 2.2. In [7] and [43], the theory of multistability of QVNNs was established, in which the authors showed that QVNNs could have 2^{4n} locally stable EPs. Whereas, the present chapter has increased the number of locally stable EPs of QVNNs by taking continuous piecewise nonlinear activation functions. It should be noted that no one can ignore the role of activation functions. In [7], multistability analysis with piecewise linear activation functions is discussed. And in [43], the authors have studied multistability using two different types of activations, the first one is a bounded sigmoid function and the second one is a non-decreasing function with saturation. So it is clearly seen that the considered DQVNNs (2.1) have more stable EPs as compared to the QVNNs taken in [7] and [43].

2.4 Numerical example

In this section, Two numerical examples are presented to illustrate the effectiveness of the derived results.

Example 2.1. *For $n = 2$, assume the parameters of DQVNNs (2.1) are as follows:*

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}, \quad k_1 = k_2 = 0, \quad \tau_1 = \tau_2 = 1, \tag{2.51}$$

with the activation functions as

$$g_1^{(b)}(r) = g_1^{(b)}(r) = \begin{cases} -1, & -\infty < r < -\frac{\pi}{2} \\ -sin(3r), & -\frac{\pi}{2} \leq r \leq \frac{\pi}{2} \\ 1, & \frac{\pi}{2} < r < +\infty \end{cases}, \tag{2.52}$$

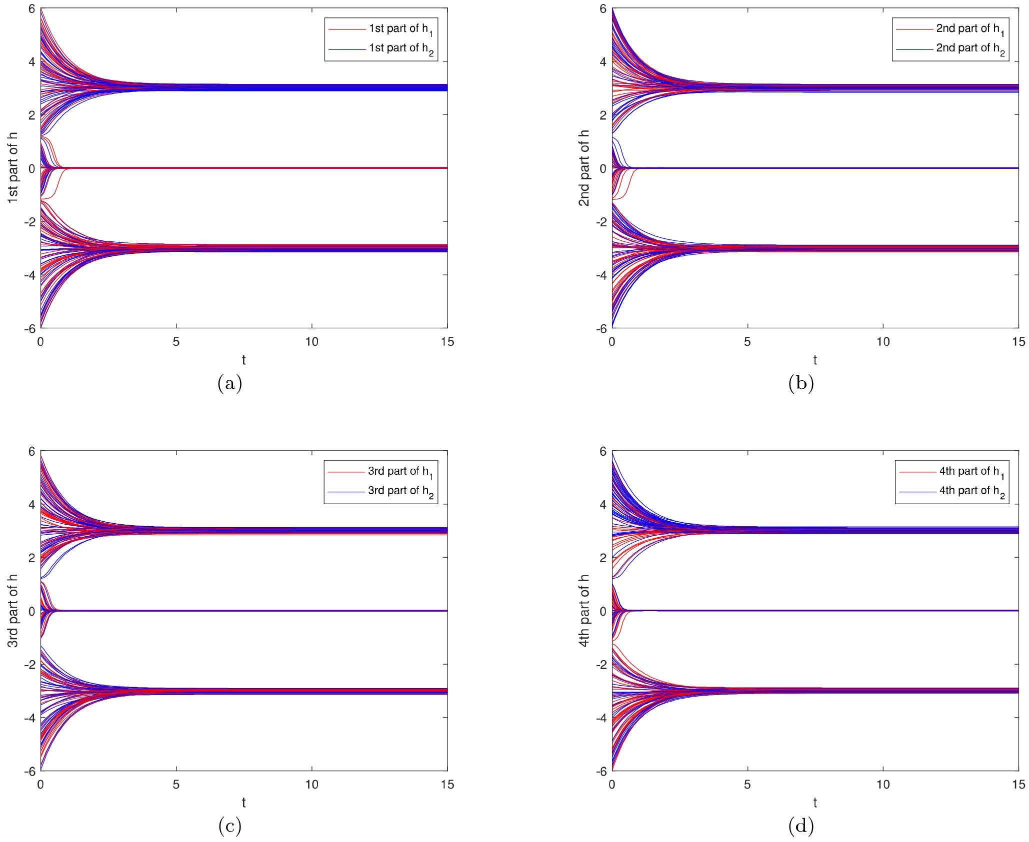
where $b = 0, 1, 2, 3,$ and

$$b_{11} = 3 + 0.0125i + 0.0125j + 0.0125k, \qquad b_{12} = 0.0125 + 0.0125i + 0.0125j + 0.0125k,$$

$$b_{21} = 0.0125 + 0.0125i + 0.0125j + 0.0125k, \qquad b_{22} = 3 + 0.0125i + 0.0125j + 0.0125k,$$

$$c_{11} = 0.0125 + 0.0125i + 0.0125j + 0.0125k, \quad c_{12} = 0.0125 + 0.0125i + 0.0125j + 0.0125k,$$

$$c_{21} = 0.0125 + 0.0125i + 0.0125j + 0.0125k, \quad c_{22} = 0.0125 + 0.0125i + 0.0125j + 0.0125k.$$



(a) (b) (c) (d)

FIGURE 2.2: Figures (a)-(d) demonstrate the state trajectories of the 1st part-4th part of the system (2.1), respectively.

Then, we have

$$\hat{\tilde{G}}_1^{(b)}(r) = -r + 3g_1^{(0)}(r) + \hat{\tilde{\eta}}_1^{(b)}, \qquad \hat{\tilde{G}}_2^{(b)}(r) = -r + 3g_2^{(0)}(r) + \hat{\tilde{\eta}}_2^{(b)},$$

$$\check{\tilde{G}}_1^{(b)}(r) = -r + 3g_1^{(0)}(r) + \check{\tilde{\eta}}_1^{(b)}, \qquad \check{\tilde{G}}_1^{(b)}(r) = -r + 3g_1^{(0)}(r) + \check{\tilde{\eta}}_1^{(b)},$$

where $\hat{\tilde{\eta}}_1^{(b)} = 0.1875,\ \check{\tilde{\eta}}_1^{(b)} = -0.1875,\ \hat{\tilde{\eta}}_2^{(b)} = 0.1875,\ \check{\tilde{\eta}}_2^{(b)} = -0.1875,\ b = 0, 1, 2, 3.$ *Consequently, we can get*

$$\hat{\tilde{G}}_1^{(b)}(-\frac{\pi}{2}) = -1.241 < 0, \qquad \hat{\tilde{G}}_2^{(b)}(-\frac{\pi}{2}) = -1.241 < 0,$$

$$\check{\tilde{G}}_1^{(b)}(\frac{\pi}{2}) = 1.241 > 0, \qquad \check{\tilde{G}}_1^{(b)}(\frac{\pi}{2}) = 1.241 > 0,$$

which implies that the conditions (2.18) *and* (2.19) *are satisfied. Furthermore, there exist* $\xi_1^{(b)} = \xi_2^{(b)} = 1$, $b = 0, 1, 2, 3$ *such that with parameters* (2.51)*, the conditions* (2.39a)-(2.39d) *are satisfied. Hence, we can see that all the conditions proposed in Theorem 2.1-2.3 are satisfied for DQVNNs* (2.1)*. Therefore, the DQVNNs* (2.1) *have* 5^8 *EPs, among which* 3^8 *are locally stable. By employing the Quaternion Toolbox for MATLAB and the fourth-order Runge–Kutta method, we perform numerical simulation of the DQVNNs. Since the program may take several days to execute, so we are just executing the program here with one hundred random initial conditions. Next, By setting one hundred random initial conditions, we have obtained one hundred corresponding numerical solutions. It is observed that these numerical solutions converge to* 100 *stable states. Then, we have obtained approximations of the* 100 *EPs, five of which are listed as follows:*

$$\begin{pmatrix} -3.0627 + 2.9627i - 3.0627j + 2.9627k \\ -3.0627 - 3.0123i + 2.9123j - 0.0025k \end{pmatrix}, \quad \begin{pmatrix} -3.0131 + 2.9619i - 3.0619j - 3.0619k \\ -3.0131 - 3.0131i - 0.0075j + 2.9131k \end{pmatrix},$$

$$\begin{pmatrix} 0.0075 - 3.1131i - 3.0131j - 3.0131k \\ -2.9131 - 3.1131i + 2.9619j - 3.0131k \end{pmatrix}, \quad \begin{pmatrix} -3.0375 - 3.0375i - 3.0375j - 3.0375k \\ -3.0375 + 2.9375i + 2.9375j + 2.9375k \end{pmatrix},$$

$$\begin{pmatrix} 3.0375 + 3.0375i - 2.9375j - 2.9375k \\ 3.0375 - 2.9375i + 3.0375j + 3.0375k \end{pmatrix}.$$

Due to the limitations of pages, here we only five EPs are listed. The dynamical behavior of all four components of the considered system is displayed in Figures 2.2(a)-2.2(d), where one hundred random constant quaternion-valued vectors are chosen as the initial conditions. From Figures 2.2(a)-2.2(d), it can be easily seen that each neuron state converges into a stable state.

Remark 2.3. The results obtained in both [7] and [43] are validated with one example each. In both examples, the QVNNs are considered with two neurons with piecewise linear activation functions, and the authors showed that the QVNNs have 3^8 EPs

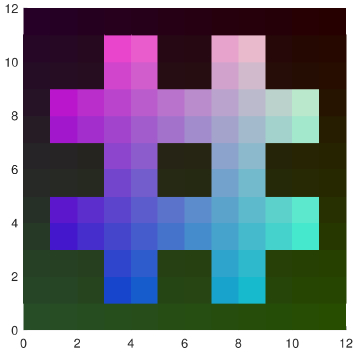


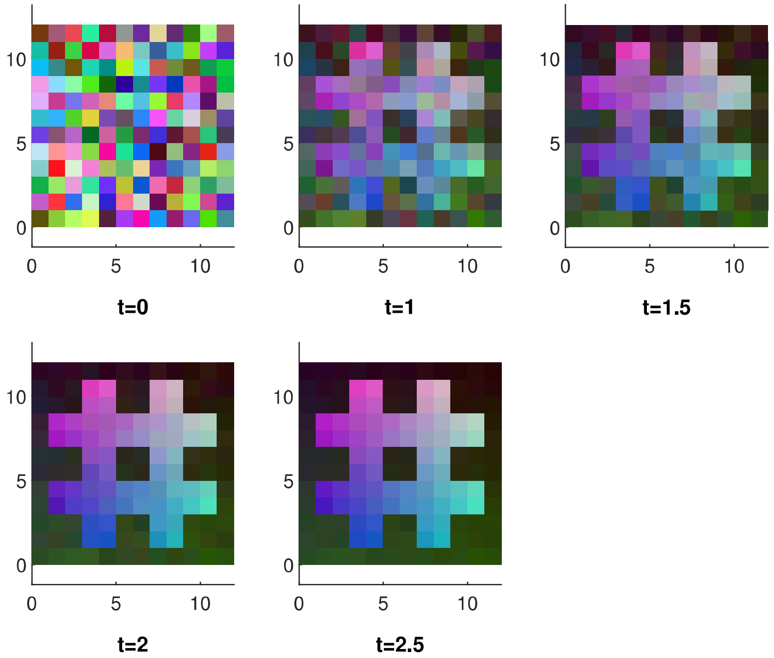
FIGURE 2.3: Plot of the original color image of pattern "#".

.

and 2^8 of them are locally stable. However, the Example 2.1 of the present chapter is also demonstrated with two neurons and piecewise nonlinear activation functions defined in (2.52). Here we have been able to show that the DQVNNs (2.1) with two neurons and with the help of activation functions in (2.52) can have 5^8 EPs out of which 3^8 are locally stable. It is obvious that the number of the EPs of DQVNNs herein is higher than that of QVNNs in [7] and [43]. Thus, the addressed DQVNNs with continuous piecewise nonlinear activation functions are less conservative for the type of activation functions and are better than the QVNNs in [7] and [43] to the associative memory applications.

Example 2.2. *In this example, the color image pattern "#" is considered as shown in Figure 2.3 of size 12×-12 pixels and design the networks in the form of DQVNNs (2.1) for the application of associative memory. Here, a DQVNNs (2.1) consisting of 144 neurons is constructed which contains a 144-dimensional EP storing the image "#". Assume the parameters of DQVNNs (2.1) as follows:*

$$a_p = 125 \times 10^{-2}, \tag{2.53}$$



t=0

t=1

t=1.5

t=2

t=2.5

FIGURE 2.4: Simulation of restoring the pattern "#" with random initial values at $t = 0, 1, 1.5, 2$ and 2.5.

.

$$b_{pq} = \begin{cases} 4 + 4 \times 10^{-1}i - 3 \times 10^{-1}j + 5 \times 10^{-1}k, & p = q \\ 4 \times 10^{-1} - 5 \times 10^{-1}i + 5 \times 10^{-1}j - 3 \times 10^{-1}k, & p \neq q \end{cases} \tag{2.54}$$

$$c_{pq} = \begin{cases} -2 \times 10^{-1} + 2 \times 10^{-1}i - 5 \times 10^{-1}j + 4 \times 10^{-1}k, & p < q \\ 2 + 3 \times 10^{-1}i - 2 \times 10^{-1}j - 3 \times 10^{-1}k, & p = q \\ -1 \times 10^{-1} - 2 \times 10^{-1}i + 3 \times 10^{-1}j - 5 \times 10^{-1}k, & p > q \end{cases} \tag{2.55}$$

with the activation functions (2.2) *where* $p, q = 1, 2, ..., 144$. *To recall the image "#", the EP of the system* (2.1) *must be*

$$h = (h_1, h_2, ..., h_{144}) \in \mathbb{H}^{144},$$

where $h_1 = 0 + 15 \times 10^{-2}i + 3 \times 10^{-1}j + 15 \times 10^{-2}k$, $h_2 = 0 + 15 \times 10^{-2}i + 3 \times$

$10^{-1}j+14\times10^{-2}k, ...,\ h_{144}=0+15\times10^{-2}i+0j+0k$, which correspond to the color $(15\times10^{-2}, 3\times10^{-1}, 15\times10^{-2}), (15\times10^{-2}, 3\times10^{-1}, 14\times10^{-2}), ...,\ (15\times10^{-2}, 0, 0)$ of the pixels in the pattern "#". The external input correspond to the above # is as follows:

$$k=(k_1,k_2,...,k_{144})\in\mathbb{H}^{144}, \tag{2.56}$$

where $k_1=-628\times10^{-1}+2275\times10^{-2}i-905\times10^{-1}j-725\times10^{-2}k, k_2=-63+2095\times10^{-2}i-905\times10^{-1}j-566\times10^{-2}k, ..., k_{144}=-914\times10^{-1}-23465\times10^{-2}i-908\times10^{-1}j+2214\times10^{-1}k$. Here, due to limitations of space, we list only three of the components of h and k. One simulation with random initial conditions performed in Figure 2.4 shows that the considered DQVNNs (2.1) with parameters (2.53)–(2.56) have the ability to retrieve the above true color image "#" reliably.

Remark 2.4. In [44], the authors have discussed that to store a 12 × 12-pixel figure the designed CVNNS needs 432 neurons. Whereas through this chapter we have successfully shown that the DQVNNs (2.1) require only 144 neurons to store a 12 × 12 pixel image. As a result, we have effectively reduced the number of neurons need to store a 12 × 12-pixel color image through the adoption of DQVNNs which is significantly less than is the case with CVNNs.

2.5 Conclusions

In this chapter the coexistence of multistable EPs of n-neuron DQVNNs has been investigated, in which the continuous and piecewise nonlinear functions are used as the activation functions. Since the state variables are quaternions, hence the DQVNNs given in equation (2.1) are decomposed into four equivelent RVNNs. Then by decomposition technique, the state space $\mathbb{H}^n$ is decomposed into 5^{4n} disjoint regions. And then by applying Brouwer's fixed point theorem, and Lagrange's mean value theorem, some novel sufficient conditions have been derived to ensure the coexistence of 5^{4n} EPs in which 3^{4n} of those are locally stable. At the end of this chapter two numerical examples have been given, the first example validated the theoretical results and second example discussed the associative memory application of DQVNNs given in (2.1).

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