Chapter 1

Introduction

Imagine throwing a tiny stone into the water while standing beside a serene lake. Immediately, ripples begin to develop and extend outward. These ripples, which are waves that travel outward in distinct directions and speeds, are a basic daily illustration of how disruptions spread in different media, such as gas, water, or air. Hyperbolic partial differential equations (PDEs) are a mathematical tool used to describe these kinds of wave motions and the associated phenomena.

Let us now add a fascinating twist. Consider yourself on a busy freeway. Like particles or bits of gas or air, cars have restrictions; they can't just show up or disappear on their own. The 'conservation rules', which basically state that something significant, like mass, momentum, or energy, cannot be miraculously generated or destroyed; it can only move or change form, must be followed. This concept—that physical characteristics like density, velocity, and pressure in gases and fluids are conserved as they flow and change—is mathematically represented by hyperbolic PDEs of conservation laws.

However, what would happen if two different states or conditions of air or gas collided suddenly? Imagine a traffic bottleneck, where fast-moving vehicles abruptly collide with slower-moving ones. A comparable circumstance is referred to as a "Riemann problem" in fluid dynamics. It basically poses the following question: "How would a gas or fluid change over time if I know its precise state on two sides? Will it produce waves that are smooth, or will there be abrupt jumps or shock waves?

This brings us to yet another interesting phenomenon: shock wave diffraction and reflection. Imagine a wedge-shaped object being struck by a weak shock wave, similar to a mild but sudden gust of wind. The interesting part is that the wave doesn't simply stop. Instead, it reflects and bends around the wedge's corners. The intricate yet exquisite patterns of shock waves that engineers and scientists see in everyday scenarios, such as supersonic aircraft passing sharp objects, explosions close to buildings, or waves interacting with barriers, are demonstrated by this interaction, which is known as a reflection-diffraction phenomenon.

1.1 Background

In this section, the fundamental tools are introduced for a better understanding of elementary waves and their interactions in Riemann problems and the wedge problem in gas dynamics.

1.1.1 Linear and Nonlinear Waves

Nonlinearity gives rise to a host of intriguing effects, among them the formation of shock waves, complex wave-wave interactions, and the dynamic evolution of nonlinear waveforms. In fact, interest in and development of nonlinear wave phenomena have grown substantially in recent years. These behaviours stand in sharp contrast to "linear" processes like sound, light, or electromagnetic propagation, instead manifesting as intense, often violent disturbances: explosive detonations, supersonic flows through rocket nozzles, and high-velocity impacts on solid materials. Unlike linear waves, which follow linear differential equations and obey principles of superposition, reflection, and refraction, shock waves are described by nonlinear equations that violate these familiar laws. One of their hallmark features is the shock front, a thin layer across which physical properties like temperature, density, and pressure change almost instantaneously. The behaviour of such nonlinear waves has intrigued scientists for well over a century. Early trailblazers Stokes, Earnshaw, Riemann, Rankine, Hugoniot, and Lord Rayleigh, together with later contributors like Hadamard, von Neumann, Courant, Friedrichs, and G. B. Whitham, established the foundational theory through their landmark papers and books. In recent decades, as the boundary between pure and applied science has blurred, the study of nonlinear phenomena, particularly shock and expansion waves, has attracted even greater attention.

Waves can be viewed as disturbances or fluctuations that travel through a medium, carrying energy from one location to another. These fluctuations might appear as elastic deformations or as changes in pressure, electric or magnetic field strength, electric potential, or temperature. Because waves inherently involve both the spatial coordinates $\mathbb{R}^n$ and the temporal variable t, time must be treated separately from the other independent variables.

In the case of linear waves, sound being a prime example, the disturbance travels at a well-defined speed relative to the medium; in a homogeneous medium, this "sound speed" is uniform for all such waves. Nonlinear wave behaviour is also governed by this characteristic speed: small perturbations superimposed on a primary wave propagate at a rate determined by the local state of the medium, which has itself been altered by the main motion. Unlike linear waves, nonlinear waves can feature finite disturbances or discontinuities that are not necessarily infinitesimal.

Waves appear throughout many areas of science and engineering, such as quantum physics, fluid dynamics, optics, electromagnetism, solid mechanics, and structural analysis, and are governed by two broad classes of partial differential equations: linear and nonlinear.

Homogeneous linear PDEs obey the superposition principle, meaning any linear combination of solutions is itself a solution. As a result, their solution set forms a vector space, whose linear structure makes it straightforward to assemble solutions that satisfy various initial and boundary conditions. In contrast, nonlinear PDEs do not admit superposition.

In this thesis, we focus on quasilinear hyperbolic PDEs arising in gas dynamics to study the propagation of nonlinear waves. Under suitable assumptions, these waves satisfy a system of first-order quasilinear equations; linear in the highest derivatives but with coefficients that depend on the dependent variables. When viscosity and heat conduction are ignored, this system reduces to the classical Euler equations, which form a hyperbolic system.

1.1.2 Hyperbolic PDEs

Consider the system of conservation laws:

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x + \mathbf{G}(\mathbf{U})_y = 0, \tag{1.1.1}$$

where $\mathbf{U} = (u_1, u_2, \ldots, u_n)$, $\mathbf{F} = (f_1, f_2, \ldots, f_n)$, $\mathbf{G} = (g_1, g_2, \ldots, g_n)$, $\mathbf{F}(\mathbf{U})$ and $\mathbf{G}(\mathbf{U})$ are given smooth functions of $\mathbf{U}$ in a domain $\Omega \subset \mathbb{R}^n$.

This section generalizes key concepts originally defined for one-dimensional conservation laws, such as hyperbolicity, convexity (genuine nonlinearity), centred simple waves, Rankine–Hugoniot jump conditions, and entropy (stability) conditions to two-dimensional systems. It introduces an initial-value problem involving a linear combination of spatial variables, characterized by a specific initial discontinuity defined by two distinct states separated by a straight line. The system is analyzed by seeking self-similar solutions depending only on the ratio $\lambda = (\mu x + \nu y)/t$, which reduces the original PDEs to a system of ordinary differential equations (ODEs). These ODEs lead to solutions either representing constant states or special singular solutions, highlighting the fundamental wave structures and discontinuities inherent in multidimensional conservation laws.

Definition 1.1.1. A system (1.1.1) is called **hyperbolic** in the direction (μ, ν) if the matrix $\mu\mathbf{F}'(\mathbf{U}) + \nu\mathbf{G}'(\mathbf{U})$ has n real eigenvalues $\lambda_i(\mathbf{U})$ $(i = 1, 2, \ldots, n)$. It is called **strictly hyperbolic** in the direction (μ, ν) if all eigenvalues $\lambda_i(\mathbf{U})$ $(i = 1, 2, \ldots, n)$ are distinct, i.e.,

$$\lambda_1(\mathbf{U}) < \lambda_2(\mathbf{U}) < \cdots < \lambda_n(U).$$

The system (1.1.1) is called **(strictly) hyperbolic** if it is (strictly) hyperbolic in every direction (μ, ν).

Definition 1.1.2. A system (1.1.1) is called *ith* $(1 \leq i \leq n)$ **convex** or **genuinely nonlinear** in the direction (μ, ν) if

$$r_i(\mathbf{U}; \mu, \nu) \cdot \nabla_{\mathbf{U}} \lambda_i(\mathbf{U}; \mu, \nu) \neq 0 \quad \text{for all } \mathbf{U} \in \Omega. \tag{1.1.2}$$

It is called ith convex or genuinely nonlinear if it satisfies the above condition in every direction (μ, ν).

The system is called **convex** or **genuinely nonlinear** if it is ith convex or genuinely nonlinear for each i $(1 \leq i \leq n)$.

Definition 1.1.3. The system (1.1.1) is called ith **linear degenerate** if

$$r_i(\mathbf{U}; \mu, \nu) \cdot \nabla_{\mathbf{U}} \lambda_i(\mathbf{U}; \mu, \nu) \equiv 0 \quad \text{for all } \mathbf{U} \in \Omega. \tag{1.1.3}$$

1.1.3 Riemann Problem

The great mathematician B. Riemann [1] in his inspiring work, solved an initial value problem, usually referred to as the Riemann problem, for one-dimensional isentropic Euler equations and proved the existence of elementary waves (shock and rarefaction waves). Fundamentally, the one-dimensional Riemann problem is an initial value problem for a system of hyperbolic PDEs with the piecewise constant initial data having one arbitrary discontinuity. A shock wave is a discontinuous solution that satisfies Rankine-Hugoniot jump conditions and Lax entropy conditions, while a rarefaction wave is a continuous solution across which corresponding Riemann invariants are constant and characteristics on the left and right diverge. Courant and Friedrichs [2] extended the work of Riemann for adiabatic Euler equations and proved the existence of the third elementary wave called the contact discontinuity. A contact discontinuity is an admissible discontinuity that satisfies Rankine-Hugoniot jump conditions, and the characteristic speed on either side coincides with the speed of the contact discontinuity. Since then, a lot of interesting work in the context of the one-dimensional hyperbolic system of conservation laws has been done by many researchers in the last two centuries. The theory of the nonlinear hyperbolic system of conservation laws in one space dimension usually assumes that the system is strictly hyperbolic with genuinely nonlinear or linearly degenerate characteristic fields. Moreover, general results on the existence of entropy-weak solutions to these systems are established only for initial values with a small total variation. P. D. Lax [3], J. Glimm [4], T. P. Liu [5], etc., established the theory of small solutions to the Riemann problem for the one-dimensional general strictly hyperbolic system of conservation laws. Since then, many mathematicians have been working in the field of hyperbolic conservation laws and have developed several existence and uniqueness results for the corresponding Cauchy problem (viz. see the [6, 7]). Indeed, the Riemann problem has been playing an important role in all three areas of theory, applications, and computation. It is fully recognised that the Riemann problem is the most fundamental problem in the entire field of nonlinear hyperbolic conservation laws. Riemann's work [1], is one of the most significant contributions to the area of mathematical physics. In comparison with the Cauchy problem, it is easier to study the Riemann problem, but it still reveals some basic properties of the Cauchy problem. Furthermore, the solutions to the Riemann problem constitute the basic building blocks for the construction of solutions to the Cauchy problem by using the random choice method. Thus, it serves as a touchstone for numerical schemes as well due to the explicit structures of the Riemann solutions.

The Initial Value Problem in conservation laws corresponds to the Riemann problem for one-dimensional Euler equations with time dependence

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0. \tag{1.1.4}$$

Here,

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \end{bmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{bmatrix} \rho u \\ \rho u^2 + p \end{bmatrix},$$

where $(x,t) \in R \times R_+$ and x, t represents the space and time coordinates, respectively. $u(x,t)$ and $\rho(x,t) > 0$ represents the velocity and density, respectively. The initial conditions of the Riemann problem for the system (1.1.4) are given by

$$\mathbf{U}(x,0) = \begin{cases} (\rho_l, v_l, p_l), & \text{if } x < 0 \\ (\rho_r, v_r, p_r), & \text{if } x > 0 \end{cases}. \tag{1.1.5}$$

Here, (ρ_l, v_l, p_l) and (ρ_r, v_r, p_r) show the left and right constant states, respectively, separated by the jump discontinuity at $x = 0$.

Non-constant initial conditions give rise to the generalized Riemann problem, whereas a standard Riemann problem assumes constant initial data, corresponding to the system's equilibrium. For linear hyperbolic systems, the generalized Riemann problem admits a smooth solution. In contrast, when applied to nonlinear hyperbolic systems, it yields solutions that remain bounded yet may exhibit discontinuities. Typically viewed as a perturbation of the classical Riemann problem, the generalized Riemann problem produces solutions whose behaviour closely mirrors that of the traditional Riemann solution.

The Generalized Riemann problem for governing system (1.1.4) with initial boundary condition

$$\mathbf{U}(x,0) = \begin{cases} (\rho_l(x), u_l(x)), & \text{if } x < 0, \\ (\rho_r(x), u_r(x)), & \text{if } x > 0. \end{cases} \tag{1.1.6}$$

Here $\rho_l(x)$, $\rho_r(x)$, $u_l(x)$ and $u_r(x)$ are smooth arbitrary functions such that $\rho_l(0) \neq \rho_r(0)$, $u_l(0) \neq u_r(0)$.

Definition 1.1.4. Let the system of conservation laws be as in (1.1.4), and a discontinuity (shock) move with speed S, and let u_l and u_r be the left and right states across the shock. Then the **Rankine–Hugoniot condition** is given by

$$\mathbf{F}(u_r) - \mathbf{F}(u_l) = S(u_r - u_l). \tag{1.1.7}$$

Equation (1.1.7) expresses that the net change in flux across the discontinuity equals the rate at which conserved quantities are carried by the moving shock.

Definition 1.1.5. Shock Wave: A shock wave is a type of propagating disturbance or wave that carries a sudden change in pressure, temperature, density, and other physical properties through a medium, such as air, water, or solid materials.
If $\lambda_1(u_l) > \lambda_1(u_r)$, the entropy satisfying weak solution is a shock wave given by

$$\mathbf{U}(x,t) = \begin{cases} u_l \text{ for } \frac{x}{t} < S, \\ u_r \text{ for } \frac{x}{t} \geq S, \end{cases}$$

where S is the shock speed given by the Rankine-Hugoniot condition.

Definition 1.1.6. Rarefaction Wave: In a rarefaction wave, the particles of the medium move apart, leading to a decrease in density and pressure. This occurs when a disturbance or change in a medium causes the particles to spread out, creating lower-density areas.

If $\lambda_1(u_l) < \lambda_1(u_r)$, then the correct entropy solution is a rarefaction wave given by

$$\mathbf{U}(x,t) = \begin{cases} u_l, & \text{for } \lambda_1(u_l) > \frac{x}{t}, \\ u(\frac{x}{t}), & \text{for } \lambda_1(u_l) \leq \frac{x}{t} \leq \lambda_1(u_r), \\ u_r, & \text{for } \lambda_1(u_r) \leq \frac{x}{t}, \end{cases}$$

where $\mathbf{U}(\frac{x}{t})$ is the solution of $\mathbf{F}'(\mathbf{U}(\frac{x}{t})) = \frac{x}{t}$.

1.1.4 Chaplygin Gas

We know that the universe isn't just expanding, but it's also speeding up. To explain this, scientists propose a mysterious "dark energy" that pushes space outward by acting like a fluid with negative pressure. Among the various ideas for dark energy, the Chaplygin gas equation stands out as a simple way to describe such negative pressure, and many cosmic models build on it [8, 9]. To match what telescopes actually see, researchers replaced it with the generalized Chaplygin gas equation [10]. Later, in 2014, Naji suggested yet another version, the extended Chaplygin gas, whose equation of state looks like this [11]:

$$p(\rho) = \mathcal{A}\rho^n - \frac{\mathcal{B}}{\rho^\alpha} \tag{1.1.8}$$

where $\mathcal{A}$ and $\mathcal{B}$ are either constants or functions of entropy S only. For $n = 1$, (1.1.8) reduces to the modified Chaplygin gas equation, which was proposed by Benaoum [12]. A more realistic version of the extended Chaplygin gas model, whose equation of state is

$$p(\rho) = \mathcal{A}\left(\frac{\rho}{1-a\rho}\right)^n - \frac{\mathcal{B}}{\rho^\alpha} \tag{1.1.9}$$

where a is a constant called the van der Waals excluded volume, $\mathcal{A}$ and $\mathcal{B}$ are either constants or functions of entropy S only, $1 \leq n \leq 3$, and $0 < \alpha \leq 1$. When $\mathcal{B}(S) = 0$, this equation becomes the co-volume equation of state [13]. For $\mathcal{B}(S) = 0$ and $a = 0$, this gas behaves as a polytropic perfect gas, and for $\mathcal{A}(S) = 0$, this gas behaves as a generalized Chaplygin gas. For different Chaplygin gas models, we refer [14–17].

1.2 Motivation

Many physical phenomena can be modelled by systems of partial differential equations (PDEs), and among these, hyperbolic systems of conservation laws play a central role in the theory of continuum mechanics. In continuum mechanics, one begins with the fundamental conservation laws of mass, momentum, and energy and then specifies the material's behaviour through constitutive relations. Under natural assumptions, these conservation laws and constitutive equations combine to yield field equations, typically nonlinear and nonhomogeneous PDEs that reflect the complexity of real-world behaviour.

For linear systems, a rich variety of solution techniques exists, and the superposition principle allows one to construct general solutions from simpler building blocks. In contrast, nonlinear hyperbolic problems are far less tractable: neither their solution methods nor the qualitative features of their solutions are as well understood. In fact, even with smooth initial data, solutions of hyperbolic conservation laws can develop singularities in finite time. Because they arise from the divergence form of balance laws, quasilinear first-order PDEs are often called conservation laws. Physically, each law asserts that a particular measurable quantity remains constant in an isolated system. The study of hyperbolic conservation laws is a vibrant area of research, driven by applications across many fields, from multiphase flows, traffic dynamics, and gas dynamics to kinetic theory, fluid mechanics, chemistry, biology, geology, and beyond.

Wave Propagation: Hyperbolic PDEs offer a powerful framework for modelling wave propagation, an essential feature of many natural processes, from acoustics and optics to water and electromagnetic waves. By expressing wave dynamics through hyperbolic PDEs, researchers can rigorously analyze how these disturbances evolve, interact, and transmit energy as they travel through various media.

Shock Waves and Discontinuities: Hyperbolic PDEs play a pivotal role in modelling shock waves and other discontinuities in fluid and gas dynamics, as well as various physical systems. Shock waves form when sudden changes in properties, such as pressure or density, travel through a medium. Accurately describing their generation and propagation is essential for predicting and controlling these dynamic processes.

Fluid Dynamics and Gas Dynamics: Hyperbolic PDEs are essential for modelling fluid and gas dynamics, offering a rigorous framework to analyze intricate flow structures, turbulence, and the transitions between subsonic, transonic, and supersonic regimes.

Aerospace and Aerodynamics: In aerospace engineering, hyperbolic PDEs model the behaviour of high-speed vehicles, such as aircraft and rockets, enabling researchers to predict aerodynamic forces like lift and drag and to assess how shock waves influence overall performance.

Nonlinear Phenomena: Contemporary mathematical research grapples with numerous challenges posed by nonlinear hyperbolic conservation laws. Even when initial data are smooth, solutions of hyperbolic PDE systems inevitably develop discontinuities, such as shock waves or slip surfaces, making exact solutions difficult to obtain. To address this, analysts and computational scientists have pursued both analytical and numerical strategies. Over the past seventy-eight years, a rich array of analytical techniques, among them the method of characteristics, progressive wave approach, wave-front analysis, perturbation methods, and the differential constraint method, have been devised to probe the physical behaviour of quasilinear hyperbolic waves. In parallel, numerical frameworks like the finite element and finite volume methods have been refined to approximate these phenomena across a broad class of problems. Comprehensive treatments of these mathematical properties, solution techniques, and applications can be found in the literature, including the foundational works by Courant and Friedrichs [2], Jeffrey [18], Zheng [19], Sharma [20], Smoller [21], and Whitham [22].

The motivation of this work arises from the lack of rigorous analytical results in areas where numerical methods dominate. While classical theories address ideal gases and homogeneous systems, little is known analytically for non-ideal gases like Chaplygin-type models, magnetohydrodynamics flows, and non-homogeneous generalized Riemann problems (GRPs).

Chaplygin and extended Chaplygin gases are chosen because of their dual importance: in cosmology (modeling dark matter-dark energy unification) and in aerodynamics (compressibility effects). Studying simple waves in such settings extends the foundational theory of hyperbolic PDEs. Similarly, solving GRPs with source terms provides exact benchmarks for validating numerical schemes.

The study of weak shock diffraction in extended Chaplygin gas is motivated by its relevance in aerospace flows and astrophysical shocks, where non-ideal effects play a decisive role. Overall, this thesis aims to fill an analytical gap and provide benchmark solutions for problems of both theoretical and practical significance.

1.3 Literature Review

In this section, a brief review of the literature on the study of the existence of simple waves in various states, shock wave propagation, Riemann problems, and weak shock reflection-diffraction phenomenon is presented.

Simple waves exhibit remarkable versatility, finding applications in diverse areas such as medical treatments (e.g., lithotripsy), traffic flow modelling, and oceanographic research. Their broad utility lies in enabling the analysis and prediction of wave behaviour across numerous scientific and engineering disciplines. In fluid dynamics, simple waves are especially valuable for examining compressible flows characterized by significant variations in pressure and density. By definition, a simple wave is a flow that depends on a single independent parameter, commonly referred to as the "phase." Exploiting this property allows for tractable solutions to otherwise complex flow problems [2, 23–29].

Courant and Friedrichs [2] showed the existence of simple waves in a reducible system by proving the invariance of Riemann invariants. It was shown that the Riemann invariant need not exist for more than 2×2 systems [30]. To overcome this, Dai and Zhang [31] were the first to introduce the characteristic decomposition method for the pressure gradient system. In the context of two-dimensional Euler equations for compressible flows, Li et al. [32] extended the analysis to ideal gases, Zafar and Sharma [33, 34] addressed non-ideal gases, and Xiao and Li [35] studied a class of pressure laws. For further developments on multidimensional conservation laws, see [36–38]. In the setting of quasilinear strictly hyperbolic systems, Hu and Sheng [27, 28] provided a useful sufficient condition to guarantee characteristic decompositions. For non-reducible systems, Čanic and Keyfitz [39] generalized the fundamental theorem of [2]. More recently, using Hu and Sheng's sufficient condition

[27], Chen and Sheng [40], and Barthwal and Sekhar [41] extended these results to magnetohydrodynamics systems.

Shock waves are most prominently encountered in aerospace engineering, especially during supersonic flight. This field's early development dates back to 1746, when the mathematician Benjamin Robins used a ballistic pendulum to measure a bullet's speed and observed that aerodynamic drag sharply increased as velocities approached the speed of sound. Yet, well into the nineteenth century, shock waves remained poorly understood. In 1759, Euler, though never explicitly using the term "shock wave", referred to the "size of disturbance" in a sound wave, intending to describe its amplitude. His hypothesis that wave speed would decrease with increasing amplitude, however, proved incorrect.

In 1808, Poisson [42] became the first researcher to obtain exact solutions of the one-dimensional unsteady Euler equations for fluid flow. Fifteen years later, in 1823, he achieved a milestone in nonlinear wave theory by formulating the isentropic gas law for infinitesimal-amplitude sound waves. In 1848, Stokes [43] introduced the term "surface of discontinuity" and extended the theory to finite-amplitude acoustic waves by examining wave steepening. Uncertain about whether truly discontinuous motion could occur, he invoked the isentropic relation—governing the interplay of dissipation and energy conservation in shock formation—instead of employing the full energy equation. Nevertheless, Stokes derived the conservation laws for mass and momentum that remain fundamental in modern fluid-dynamics analysis. In 1889, Hugoniot [44] independently established the correct jump conditions for shock waves. The resulting Rankine–Hugoniot theory remains the fundamental model for shock wave propagation even today.

B. Riemann's 1859 "theory of waves of finite amplitude"—which treated planar waves propagating in both directions, provided the basis for the Riemann problem. In 1860, Riemann [1] formulated this problem for gas-dynamic conservation laws, prescribing initial data as two constant states U_*^1 and U_*^2 separated by a discontinuity at $x = 0$. When $\|U_*^1 - U_*^2\|$ was small, Lax constructed solutions by piecing together shocks, rarefactions, and contact discontinuities. In the Euler-equation context, this corresponds to the classical shock-tube problem; for a detailed exposition, see Courant and Friedrichs [2].

The Riemann problem's exact solution is crucial, as it underlies many general initial-value methods, such as Glimm's random choice technique [4]. Detailed solution strategies for hyperbolic conservation laws can be found in [13, 21, 45, 46]. Chorin also introduced a novel approach for solving the Riemann problem [47, 48], and Van Leer enhanced Godunov's original solver [49]. We reference Liu [6] and the treatments by Dafermos [50], Bressan [51], and LeVeque [52] for the Riemann problem. In the Euler equations, a simple wave keeps one Riemann invariant constant, but breaking necessitates shocks. Whitham [22] noted that entropy and invariants change very little for weak-to-moderate shocks; Courant and Friedrichs [2] used this to replace such shocks with equivalent simple compression waves.

When a compressible fluid's speed exceeds the sound speed, shocks form and particles concentrate at the shock front, behaving "sticky" (Chang et al. [53]). In rarefaction regions, particles separate, leading to cavitation. Chen and Liu [54] rigorously explained these density concentration and cavitation phenomena for isentropic fluids via the vanishing-pressure limit.

A classical Riemann problem uses constant initial data corresponding to equilibrium states, whereas a generalized Riemann problem (GRP) involves non-constant initial conditions with a discontinuity at a single point, making exact solutions considerably more difficult. Over recent decades, techniques such as similarity transformations, perturbation methods, and especially the differential constraint approach have yielded exact or approximate solutions. Curro et al. [55–57] employed systematic reduction and differential constraints to solve GRPs in quasilinear hyperbolic systems (e.g., traffic flow), while Curro and Manganaro [58] applied these ideas to a rate-type material model. In gas dynamics, Janenko [59] pioneered this approach, and subsequent work has established existence and uniqueness results, showing that near the origin, GRP solutions behave like classical Riemann solutions. For further details, see [60–65].

Shock waves were recognised as a natural phenomenon over a century ago, yet they remain not fully understood. In gas dynamics, the problem of shock reflection and diffraction by wedges has attracted sustained interest over the past fifty years (see [2, 66, 67]). When a weak incident shock impinges on a wedge, it can produce different configurations, most notably regular and Mach reflections (see [68–71]). In the case of sufficiently large wedge angles, regular reflection occurs: a plane shock striking the wedge head-on is reflected off the wedge surface while the post-shock flow is diffracted by the wedge's compressive corner, generating circular waves emanating from that corner. These diffracted circular waves propagate at the local sound speed within the medium enclosed by the wedge. This remarkable reflection-diffraction configuration has been extensively studied for an ideal gas by [72–80].

The ideal gas law assumes gases consist of point masses undergoing perfectly elastic collisions, but at low temperatures or high pressures, real gases deviate from this behaviour and follow a van der Waals–type equation of state [81, 82]. Gupta and Sharma [83, 84] applied an asymptotic approach to the shock-diffraction problem that incorporates non-ideal gas effects via a van der Waals–type equation of state.

1.4 Thesis Objectives

The present thesis aims to solve selected initial and boundary value problems for multidimensional quasilinear hyperbolic systems by combining several analytical techniques such as generalized characteristic analysis in the self-similar plane, construction of solutions via implicit functions corresponding to initial discontinuities, the characteristic decomposition method, the differential constraint method, and asymptotic analysis.

A survey of the literature highlights many open questions in the broad field of multidimensional conservation laws. Motivated by these challenges, our goal is to address a representative subset of these problems and thereby advance the mathematical theory of hyperbolic systems of conservation laws in one or two dimensions. To this end, the thesis is organised around the following objectives.

- To prove the existence of simple waves in a non-ideal magnetohydrodynamic system using the characteristic decomposition of the 2D compressible flow.
- To prove the existence of simple waves in a non-ideal magnetohydrodynamic system using the sufficient condition for the characteristic decomposition of the 2D compressible flow.
- To find non-trivial closed-form solution of non-homogeneous quasi-linear hyperbolic system for a genralised Chaplygin gas. Further, we characterize generalised Riemann problem solutions in full for piecewise-linear initial data, showing their explicit form. Also, analyze a nonlinear generalized Riemann problem and provide a closed-form rarefaction wave solution for the generalized Chaplygin gas dynamics.

- To study the reflection-diffraction phenomenon when a weak shock hits a rigid right-angled wedge using asymptotic analysis focused on analysing the nonlinear behaviour of diffracted shock and expansion wave profiles on different boundaries in extended Chaplygin gas.

- Employs asymptotic analysis to investigate the reflection–diffraction process that occurs when a weak shock wave strikes a rigid wedge of arbitrary angle, with particular emphasis on the nonlinear evolution of the diffracted shock and expansion wave profiles along the wedge boundaries in extended Chaplygin gas.

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