Chapter 6

Appendix

6.1 Almost operational matrices for integration

6.1.1 Legendre wavelets operational matrix of integration for variable t, M=M'=3

$$\int_0^t \frac{\psi^L(s,x)}{(t-s)^\eta} ds \approx H_t^L \psi^L(t,x) \tag{6.1}$$

where, $H_t^L = V^L \otimes I$, I is identity matrix and V^L as follows:

$$V^L = \begin{pmatrix} -\frac{1}{(2-\eta)(-1+\eta)} & \frac{\sqrt{3}}{6-5\eta+\eta^2} & \frac{\sqrt{5}\eta}{(-4+\eta)(-3+\eta)(-2+\eta)} \\ \frac{\sqrt{3}(-1-\eta)}{(-3+\eta)A} & \frac{-3\eta}{(-4+\eta)A} & \frac{\sqrt{15}(1-\eta^2)}{(15-8\eta+\eta^2)A} \\ \frac{\sqrt{5}\eta(-1+\eta)}{(-4+\eta)B} & -\frac{\sqrt{15}(1-\eta^2)}{(-5+\eta)B} & -\frac{5\eta(6+\eta+\eta^2)}{(-6+\eta)(-4+\eta)B} \end{pmatrix},$$

here, $A_1 = 2 - 3\eta + \eta^2$ and $B_1 = -6 + 11\eta - 6\eta^2 + \eta^3$.

So,

$$H_t^L = V^L \otimes I = \begin{pmatrix} -\frac{1}{(2-\eta)(-1+\eta)}I & \frac{\sqrt{3}}{6-5\eta+\eta^2}I & \frac{\sqrt{5}\eta}{(-4+\eta)(-3+\eta)(-2+\eta)}I \\ \\ \frac{\sqrt{3}(-1-\eta)}{(-3+\eta)A_1}I & \frac{-3\eta}{(-4+\eta)A_1}I & \frac{\sqrt{15}(1-\eta^2)}{(15-8\eta+\eta^2)A_1}I \\ \\ \frac{\sqrt{5}\eta(-1+\eta)}{(-4+\eta)B_1}I & -\frac{\sqrt{15}(1-\eta^2)}{(-5+\eta)B_1}I & -\frac{5\eta(6+\eta+\eta^2)}{(-6+\eta)(-4+\eta)B_1}I \\ \\ \end{pmatrix}$$

Similarlly,

$$Q_t^L = W^L \otimes I = \begin{pmatrix} -\frac{1}{(2-\nu)(-1+\nu)}I & \frac{\sqrt{3}}{6-5\nu+\nu^2}I & \frac{\sqrt{5}\nu}{(-4+\nu)(-3+\nu)(-2+\nu)}I \\ \\ \frac{\sqrt{3}(-1-\nu)}{(-3+\nu)A_2}I & \frac{-3\nu}{(-4+\nu)A_2}I & \frac{\sqrt{15}(1-\nu^2)}{(15-8\nu+\nu^2)A_2}I \\ \\ \frac{\sqrt{5}\nu(-1+\nu)}{(-4+\nu)B_2}I & -\frac{\sqrt{15}(1-\nu^2)}{(-5+\nu)B_2}I & -\frac{5\nu(6+\nu+\nu^2)}{(-6+\nu)(-4+\nu)B_2}I \\ \\ \end{pmatrix},$$

where, $A_2 = 2 - 3\nu + \nu^2$ and $B_2 = -6 + 11\nu - 6\nu^2 + \nu^3$.

6.1.2 Chebyshev wavelets operational matrix of integration for variable t, M=M'=3

$$\int_0^t \frac{\phi^C(s,x)}{(t-s)^\varsigma} ds \approx H_t^C \phi^C(t,x) \tag{6.2}$$

where, $H_t^C = V^C \otimes I$, I is identity matrix and V^C as follows:

$$V^C = \begin{pmatrix} \frac{1}{(2-\varsigma)A_3} & \frac{2}{\pi B_3} & \frac{2C_3}{A_3} \\ \frac{2}{\pi B_3} & \frac{4}{A_3}\left(\frac{1}{2-\varsigma} + \frac{4}{-3+\varsigma} - \frac{4}{-3+\varsigma}\right) & \frac{4D_3}{A_3} \\ \frac{2C_3}{A_3} & \frac{4D_3}{A_3} & \frac{4}{A_3}\left(\frac{1}{2-\varsigma} + \frac{16}{-3+\varsigma} - \frac{80}{-4+\varsigma} + \frac{128}{-5+\varsigma} - \frac{64}{-6+\varsigma}\right) \end{pmatrix},$$

where,

$$A_3 = \pi(1-\varsigma),$$

$$B_3 = 6 - 5\varsigma + \varsigma^2,$$

$$C_3 = \frac{1}{2-\varsigma} + \frac{8}{-3+\varsigma} - \frac{8}{-4+\varsigma},$$

and

$$D_3 = \frac{1}{-2+\varsigma} - \frac{10}{-3+\varsigma} + \frac{24}{-4+\varsigma} - \frac{16}{-5+\varsigma}.$$

So,

$$H_t^C = V^C \otimes I$$

$$= \begin{pmatrix} \frac{1}{(2-\varsigma)A_3}I & \frac{2}{\pi B_3}I & \frac{2C_3}{A_3}I \\ \frac{2}{\pi B_3}I & \frac{4}{A_3}\left(\frac{1}{2-\varsigma} + \frac{4}{-3+\varsigma} - \frac{4}{-3+\varsigma}\right)I & \frac{4D_3}{A_3}I \\ \frac{2C_3}{A_3}I & \frac{4D_3}{A_3}I & \frac{4}{A_3}\left(\frac{1}{2-\varsigma} + \frac{16}{-3+\varsigma} - \frac{80}{-4+\varsigma} + \frac{128}{-5+\varsigma} - \frac{64}{-6+\varsigma}\right)I \end{pmatrix}.$$

Similarly,

$$Q_t^C = W^C \otimes I$$

$$= \begin{pmatrix} \frac{1}{(2-\delta)A_4}I & \frac{2}{\pi B_4}I & \frac{2C_4}{A_4}I \\ \frac{2}{\pi B_4}I & \frac{4}{A_4}\left(\frac{1}{2-\delta}+\frac{4}{-3+\delta}-\frac{4}{-3+\delta}\right)I & \frac{4D_4}{A_4}I \\ \frac{2C_4}{A_4}I & \frac{4D_4}{A_4}I & \frac{4}{A_4}\left(\frac{1}{2-\delta}+\frac{16}{-3+\delta}-\frac{80}{-4+\delta}+\frac{128}{-5+\delta}-\frac{64}{-6+\delta}\right)I \end{pmatrix},$$

where,

$$A_4 = \pi(1-\delta),$$

$$B_4 = 6 - 5\delta + \delta^2,$$

$$C_4 = \frac{1}{2-\delta} + \frac{8}{-3+\delta} - \frac{8}{-4+\delta},$$

and

$$D_4 = \frac{1}{-2+\delta} - \frac{10}{-3+\delta} + \frac{24}{-4+\delta} - \frac{16}{-5+\delta}.$$

6.2 Piecewise almost operational matrices for integration

6.2.1 *For Legendre wavelets*

<u>$Case(i)$</u> For $k = 1, n = 1$:

$$V = \begin{pmatrix} \frac{1}{(1-\alpha)(2-\alpha)} & \frac{\sqrt{3}(1-\alpha)}{(-3+\alpha)L_1} & \frac{-\sqrt{5}\alpha(-1+\alpha)}{(-4+\alpha)L_2} \\ \frac{\sqrt{3}}{6-5\alpha+\alpha^2} & \frac{-3\alpha}{(-4+\alpha)L_1} & \frac{\sqrt{15}(1-\alpha^2)}{(-5+\alpha)L_2} \\ \frac{\sqrt{5}\alpha(-1+\alpha)}{(-1+\alpha)(-1+\alpha)L_3} & \frac{\sqrt{15}(1-\alpha^2)}{(15-8\alpha+\alpha^2)L_1} & \frac{5\alpha}{(-4+\alpha)(-6+\alpha)L_2} \end{pmatrix},$$

where,

$$L_1 = 2 - 3\alpha + \alpha^2,$$

$$L_2 = 6 - 11\alpha + 6\alpha^2 - \alpha^3,$$

and

$$L_3 = (-24 + 26\alpha - 9\alpha^2 + \alpha^3).$$

Case(ii) For $k = 2, n = 1$:

$$V = \begin{pmatrix} \frac{2}{(-2+\alpha)(-1+\alpha)} & \frac{2\sqrt{3}(-5+3\alpha)}{L_2} & \frac{2\sqrt{5}(60-55\alpha+13\alpha^2)}{(-1+\alpha)L_3} \\ \frac{2\sqrt{3}(-5+3\alpha)}{L_2} & \frac{6(44-39\alpha+\alpha^2)}{(-1+\alpha)(-2+\alpha)(-3+\alpha)(-4+\alpha)} & \frac{2\sqrt{15}L_4}{L_5} \\ \frac{2\sqrt{5}(60-55\alpha+13\alpha^2)}{(-1+\alpha)L_3} & \frac{2\sqrt{15}L_4}{L_5} & \frac{10(15336-20406\alpha+9911\alpha^2-2106\alpha^3+169\alpha^4)}{(-6+\alpha)L_5} \end{pmatrix},$$

where,

$$L_4 = (-724 + 825\alpha - 308\alpha^2 + 39\alpha^3),$$

$$L_5 = (-1+\alpha)(-2+\alpha)(-3+\alpha)(-4+\alpha)(-5+\alpha).$$

Case(iii) For Legendre wavelet at $k = 2, n = 2$:

$$V = \begin{pmatrix} \frac{2}{(-2+\alpha)(-1+\alpha)} & \frac{2\sqrt{3}(1+\alpha)}{L_2} & \frac{2\sqrt{5}(12+5\alpha+\alpha^2)}{(-1+\alpha)L_3} \\ \frac{2\sqrt{3}(1+\alpha)}{L_2} & \frac{6(12+\alpha+\alpha^2)}{(-1+\alpha)L_3} & \frac{2\sqrt{15}(116+\alpha(47+\alpha(4+\alpha)))}{L_5} \\ \frac{2\sqrt{5}(12+5\alpha+\alpha^2)}{(-1+\alpha)L_3} & \frac{2\sqrt{15}(116+\alpha(47+\alpha(4+\alpha)))}{L_5} & \frac{10(2088+\alpha(666+\alpha(143+\alpha(6+\alpha))))}{(-6+\alpha)L_5} \end{pmatrix},$$

Case(iv) For $k = 3, n = 1$:

$$V = \begin{pmatrix} \frac{4}{(-2+\alpha)(-1+\alpha)} & \frac{4\sqrt{3}(-13+7\alpha)}{L_2} & \frac{4\sqrt{5}(396-343\alpha+73\alpha^2)}{(-1+\alpha)L_3} \\ \frac{4\sqrt{3}(-13+7\alpha)}{L_2} & \frac{12(268-231\alpha+49\alpha^2)}{L_3} & \frac{4\sqrt{15}L_6}{L_5} \\ \frac{4\sqrt{5}(396-343\alpha+73\alpha^2)}{(-1+\alpha)L_3} & \frac{4\sqrt{15}L_6}{L_5} & \frac{20(569448-743190\alpha+350711\alpha^2-71394\alpha^3+5329\alpha^4)}{(-6+\alpha)L_5} \end{pmatrix},$$

where,

$$L_6 = -25852 + 27727\alpha - 9452\alpha^2 + 1025\alpha^3.$$

Case(v) For $k = 3, n = 2$:

$$V = \begin{pmatrix} \frac{4}{(-2+\alpha)(-1+\alpha)} & \frac{4\sqrt{3}(-7+5\alpha)}{L_2} & \frac{4\sqrt{5}(156-139\alpha+37\alpha^2)}{(-1+\alpha)L_3} \\ \frac{4\sqrt{3}(-7+5\alpha)}{L_2} & \frac{12(108-95\alpha+25\alpha^2)}{(-1+\alpha)L_3} & \frac{4\sqrt{15}L_7}{L_5} \\ \frac{4\sqrt{5}(156-139\alpha+37\alpha^2)}{(-1+\alpha)L_3} & \frac{4\sqrt{15}L_7}{L_5} & \frac{20(109224-145926\alpha+72719\alpha^2-15762\alpha^3+1369\alpha^4)}{(-6+\alpha)L_5} \end{pmatrix},$$

where,

$$L_7 = -2972 + 3511\alpha - 1324\alpha^2 + 185\alpha^3.$$

$\underline{Case(vi)}$ For Legendre wavelet at $k = 3, n = 3$:

$$V = \begin{pmatrix} \frac{4}{(-2+\alpha)(-1+\alpha)} & \frac{4\sqrt{3}(-1+3\alpha)}{L_2} & \frac{4\sqrt{5}(60-19\alpha+13\alpha^2)}{(-1+\alpha)L_3} \\ \frac{4\sqrt{3}(-1+3\alpha)}{L_2} & \frac{12(44-15\alpha+9\alpha^2)}{(-1+\alpha)L_3} & \frac{4\sqrt{15}L_8}{L_5} \\ \frac{4\sqrt{5}(60-19\alpha+13\alpha^2)}{(-1+\alpha)L_3} & \frac{4\sqrt{15}L_8}{L_5} & \frac{20(20520-5286\alpha+7391\alpha^2-1170\alpha^3+169\alpha^4)}{(-6+\alpha)L_5} \end{pmatrix},$$

where,

$$L_8 = 28 + 681\alpha - 148\alpha^2 + 39\alpha^3.$$

$\underline{Case(vii)}$ For $k = 3, n = 4$:

$$V = \begin{pmatrix} \frac{4}{(-2+\alpha)(-1+\alpha)} & \frac{4\sqrt{3}(5+\alpha)}{L_2} & \frac{4\sqrt{5}(108+17\alpha+\alpha^2)}{(-1+\alpha)L_3} \\ \frac{4\sqrt{3}(5+\alpha)}{L_2} & \frac{12(76+9\alpha+\alpha+\alpha^2)}{L_3} & \frac{4\sqrt{15}L_9}{L_5} \\ \frac{4\sqrt{5}(108+17\alpha+\alpha^2)}{(-1+\alpha)L_3} & \frac{4\sqrt{15}L_9}{L_5} & \frac{20(95926-\alpha(13962-\alpha(935-\alpha(30-\alpha))))}{(-6+\alpha)L_5} \end{pmatrix},$$

where,

$$L_9 = 2308 - \alpha(335 - \alpha(20 - \alpha)).$$

6.2.2 For Chebyshev wavelets

$\underline{Case(i)}$ For $k = 1, n = 1$:

$$V = \begin{pmatrix} \frac{2}{\pi(1-\alpha)(2-\alpha)} & \frac{2\sqrt{2}}{\pi C_1} & \frac{2\sqrt{2}(4-\alpha-\alpha^2)}{\pi(1-\alpha)C_2} \\ \frac{2\sqrt{2}(4-\alpha-\alpha^2)}{\pi C_1} & \frac{4(4-3\alpha+\alpha^2)}{C_2} & \frac{4(4-\alpha-\alpha^2)}{\pi C3} \\ \frac{2\sqrt{2}(4-\alpha-\alpha^2)}{\pi(1-\alpha)C_2} & \frac{4(4-\alpha-\alpha^2)}{\pi C_3} & \frac{4(-168+150\alpha-39\alpha^2+2\alpha^3-\alpha^4)}{\pi(1-\alpha)(-6+\alpha)C_3} \end{pmatrix}.$$

where,

$$C_1 = 6 - 5\alpha + \alpha^2,$$

$$C_2 = -24 + 26\alpha - 9\alpha^2 + \alpha^3,$$

$$C_3 = (-2 + \alpha)(-3 + \alpha)(-4 + \alpha)(-5 + \alpha).$$

<u>*Case(ii)*</u> For $k = 2, n = 1$:

$$V = \begin{pmatrix} \frac{4}{\pi(1-\alpha)(2-\alpha)} & \frac{4\sqrt{2}(-5+3\alpha)}{\pi(-1+\alpha)C_1} & \frac{4\sqrt{2}C_4}{\pi(1-\alpha)C_2} \\ \frac{4\sqrt{2}(-5+3\alpha)}{\pi(-1+\alpha)C_1} & \frac{8(-44+39\alpha-9\alpha^2)}{\pi(-1+\alpha)C_2} & \frac{8C_5}{\pi(1-\alpha)C_3} \\ \frac{4\sqrt{2}C_4}{\pi(1-\alpha)C_2} & \frac{8C_5}{\pi(1-\alpha)} & \frac{8(25704-34262\alpha+16695\alpha^2-3570\alpha^3+289\alpha^4)}{\pi(-1+\alpha)(-6+\alpha)C_3} \end{pmatrix},$$

where,

$$C_4 = -76 + 71\alpha - 17\alpha^2,$$

$$C_5 = 932 - 1065\alpha + 400\alpha^2 - 51\alpha^3.$$

<u>*Case(iii)*</u> For $k = 2, n = 2$:

$$V = \begin{pmatrix} \frac{4}{\pi(1-\alpha)(2-\alpha)} & \frac{4\sqrt{2}(1+\alpha)}{\pi(-1+\alpha)C_1} & \frac{4\sqrt{2}C_6}{\pi(-1+\alpha)C_2} \\ \frac{4\sqrt{2}(1+\alpha)}{\pi(-1+\alpha)C_1} & \frac{8(12+\alpha+\alpha^2)}{\pi(-1+\alpha)C_2} & \frac{8C_7}{\pi(-1+\alpha)C_3} \\ \frac{4\sqrt{2}C_6}{\pi(-1+\alpha)C_2} & \frac{8C_7}{\pi(-1+\alpha)C_3} & \frac{8(3432+\alpha(1130+\alpha(279+\alpha(14+\alpha))))}{\pi(-1+\alpha)(-6+\alpha)C_3} \end{pmatrix},$$

where,

$$C_6 = 12 + 9\alpha + \alpha^2,$$

$$C_7 = 148 + \alpha(59 + \alpha(8 + \alpha)).$$

<u>*Case(iv)*</u> For $k = 3, n = 1$:

$$V = \begin{pmatrix} \frac{8}{\pi(1-\alpha)(2-\alpha)} & \frac{8\sqrt{2}(-13+7\alpha)}{\pi(-1+\alpha)C_1} & \frac{8\sqrt{2}C_8}{\pi(-1+\alpha)C_2} \\ \frac{8\sqrt{2}(-13+7\alpha)}{\pi(-1+\alpha)C_1} & \frac{16(268-231\alpha+49\alpha^2)}{\pi(-1+\alpha)C_2} & \frac{16C_9}{\pi(-1+\alpha)C_3} \\ \frac{8\sqrt{2}C_8}{\pi(-1+\alpha)C_2} & \frac{16C_9}{\pi(-1+\alpha)C_3} & \frac{16(1001832-1308246\alpha+617847\alpha^2-125906\alpha^3+9409\alpha^4)}{\pi(-1+\alpha)(-6+\alpha)C_3} \end{pmatrix},$$

where,

$$C_8 = 524 - 455\alpha + 97\alpha^2,$$

$$C_9 = -14598 + 16193\alpha - 5804\alpha^2 + 679\alpha^3.$$

Case(v) For $k = 3, n = 2$:

$$V = \begin{pmatrix} \frac{8}{\pi(1-\alpha)(2-\alpha)} & \frac{8\sqrt{2}(-7+5\alpha)}{\pi(-1+\alpha)C_1} & \frac{8\sqrt{2}C_{10}}{\pi(-1+\alpha)C_2} \\ \frac{8\sqrt{2}(-7+5\alpha)}{\pi(-1+\alpha)C_1} & \frac{16(108-95\alpha+25\alpha^2)}{\pi(-1+\alpha)C_2} & \frac{16C_{11}}{\pi(-1+\alpha)C_3} \\ \frac{8\sqrt{2}C_{10}}{\pi(-1+\alpha)C_2} & \frac{16C_{11}}{\pi(-1+\alpha)C_3} & \frac{16(190056-25423\alpha+12687\alpha^2-27538\alpha^3+2401\alpha^4)}{\pi(-1+\alpha)(-6+\alpha)C_3} \end{pmatrix},$$

where,

$$C_{10} = 204 - 183\alpha + 49\alpha^2,$$

$$C_{11} = -3916 + 4627\alpha - 1748\alpha^2 + 245\alpha^3.$$

Case(vi) For at $k = 3, n = 3$:

$$V = \begin{pmatrix} \frac{8}{\pi(1-\alpha)(2-\alpha)} & \frac{8\sqrt{2}(-1+3\alpha)}{\pi(-1+\alpha)C1} & \frac{8\sqrt{2}C_{12}}{\pi(-1+\alpha)C_2} \\ \frac{8\sqrt{2}(-1+3\alpha)}{\pi(-1+\alpha)C1} & \frac{16(44-15\alpha+9\alpha^2)}{\pi(-1+\alpha)C_2} & \frac{16C_{13}}{\pi(-1+\alpha)C_3} \\ \frac{8\sqrt{2}C_{12}}{\pi(-1+\alpha)C_2} & \frac{16C_{13}}{\pi(-1+\alpha)C_3} & \frac{16(34920-8342\alpha+12567\alpha^2-1938\alpha^3+289\alpha^4)}{\pi(-1+\alpha)(-6+\alpha)C_3} \end{pmatrix},$$

where,

$$C_{12} = 76 - 23\alpha + 17\alpha^2,$$

$$C_{13} = 44 + 885\alpha - 188\alpha^2 + 51\alpha^3.$$

Case(vii) For at $k = 3, n = 4$:

$$V = \begin{pmatrix} \frac{8}{\pi(1-\alpha)(2-\alpha)} & \frac{8\sqrt{2}(5+\alpha)}{\pi(-1+\alpha)C1} & \frac{8\sqrt{2}C_{14}}{\pi(-1+\alpha)C_2} \\ \frac{8\sqrt{2}(5+\alpha)}{\pi(-1+\alpha)C1} & \frac{16(76+9\alpha+\alpha^2)}{\pi(-1+\alpha)C_2} & \frac{16C_{15}}{\pi(-1+\alpha)C_3} \\ \frac{8\sqrt{2}C_{14}}{\pi(-1+\alpha)C_2} & \frac{16C_{15}}{\pi(-1+\alpha)C_3} & \frac{16(167784+\alpha(25386+\alpha(1719+\alpha(46+\alpha))))}{\pi(-1+\alpha)(-6+\alpha)C_3} \end{pmatrix},$$

where,

$$C_{14} = 140 + 25\alpha + \alpha^2,$$

$$C_{15} = 3044 + \alpha(455 + \alpha(28 + \alpha)).$$

Remark 10 *We noticed that in the all the derived almost operational matrices are symmetric.*

Remark 11 *All case (i)-(xiv) have used for piecewise approximation in Example 4.1-4.6.*

6.3 Almost operational matrices for CPDEs

6.3.1 Almost operational matrix of integration I

Let $\Phi(a,b)$ be two-dimensional Legendre and Bernoulli wavelets vector defined in section 2 then

$$\begin{aligned} a\int_0^1\left(\int_0^a r\Phi(r,b)dr\right)da &= a\int_0^1\left(\int_0^a r\Phi(r)\otimes\Phi(b)dr\right)da, \\ &= \left(a\int_0^1\left(\int_0^a r\Phi(r)\otimes\Phi(b)dr\right)da\right)\otimes\Phi(b), \\ &= (L_r\Phi(a))\otimes(I\Phi(b)) \\ &= (L_r\otimes I)(\Phi(a)\otimes\Phi(b)) \\ &= L_{r1}\Phi(a,b) \end{aligned}$$

where, $L_{r1} = L_r \otimes I$ is the matrix of order $2^{k-1}2^{k'-1}MM'$.

6.3.2 Almost operational matrix of integration II

Let $\Phi(a,b)$ be two-dimensional Legendre and Bernoulli wavelets vector defined in section 2 then

$$\begin{aligned} a\int_0^1\left(\int_0^a \Phi(r,b)dr\right)da = a\int_0^1\left(\int_0^a \Phi(r)\otimes\Phi(b)dr\right)da,\\ =\left(a\int_0^1\left(\int_0^a \Phi(r)\otimes\Phi(b)dr\right)da\right)\otimes\Phi(b),\\ =(L\Phi(a))\otimes(I\Phi(b))\\ =(L\otimes I)(\Phi(a)\otimes\Phi(b))\\ =L_1\Phi(a,b)\end{aligned}$$

where, $L_1 = L\otimes I$ is the matrix of order $2^{k-1}2^{k'-1}MM'$.

6.3.3 Almost operational matrix of integration III

Let $\Phi(a,b)$ be two-dimensional Legendre and Bernoulli wavelets vector defined in section 2 then

$$\begin{aligned} a\int_0^1\left(\int_0^a r^2\Phi(r,b)dr\right)da = a\int_0^1\left(\int_0^a r^2\Phi(r)\otimes\Phi(b)dr\right)da,\\ =\left(a\int_0^1\left(\int_0^a r^2\Phi(r)\otimes\Phi(b)dr\right)da\right)\otimes\Phi(b),\\ =(L_{r^2}\Phi(a))\otimes(I\Phi(b))\\ =(L_{r^2}\otimes I)(\Phi(a)\otimes\Phi(b))\\ =L_{r^21}\Phi(a,b)\end{aligned}$$

where, $L_{r^21} = L_{r^2}\otimes I$ is the matrix of order $2^{k-1}2^{k'-1}MM'$.

6.3.4 Almost operational matrix of integration IV

Let $\Phi(a,b)$ be two-dimensional Legendre and Bernoulli wavelets vector defined in section 2 then

$$\int_0^a r\Phi(r,b)dr = \int_0^a r\Phi(r) \otimes \Phi(b)dr,$$
$$= \int_0^a r\Phi(r)dr \otimes \Phi(b),$$
$$= (Q_r\Phi(a)) \otimes (I\Phi(b))$$
$$= (Q_r \otimes I)(\Phi(a) \otimes \Phi(b))$$
$$= Q_{ra}\Phi(a,b)$$

where, $Q_{ra} = Q_r \otimes I$ is the matrix of order $2^{k-1}2^{k^{'}-1}MM^{'}$.

6.3.5 Almost operational matrix of integration V

Let $\Phi(a,b)$ be two-dimensional Legendre and Bernoulli wavelets vector defined in section 2 then

$$\int_0^a \Phi(r,b)dr = \int_0^a \Phi(r) \otimes \Phi(b)dr,$$
$$= \int_0^a \Phi(r)dr \otimes \Phi(b),$$
$$= (Q\Phi(a)) \otimes (I\Phi(b))$$
$$= (Q \otimes I)(\Phi(a) \otimes \Phi(b))$$
$$= Q_a\Phi(a,b)$$

where, $Q_a = Q \otimes I$ is the matrix of order $2^{k-1}2^{k^{'}-1}MM^{'}$.

6.3.6 Almost operational matrix of integration VI

Let $\Phi(a,b)$ be two-dimensional Legendre and Bernoulli wavelets vector defined in section 2 then

$$
\begin{aligned}
\int_0^a r^2\Phi(r,b)dr &= \int_0^a r^2\Phi(r)\otimes\Phi(b)dr,\\
&= \int_0^a r^2\Phi(r)dr\otimes\Phi(b),\\
&= (Q_{r^2}\Phi(a))\otimes(I\Phi(b))\\
&= (Q_{r^2}\otimes I)(\Phi(a)\otimes\Phi(b))\\
&= Q_{r^2a}\Phi^L(a,b)
\end{aligned}
$$

where, $Q_{r^2a} = Q_{r^2}\otimes I$ is the matrix of order $2^{k-1}2^{k'-1}MM'$.

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