

## Chapter 6

### **Function projective synchronization of fractional order satellite systems and its stability analysis for incommensurate case**

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#### **6.1 Introduction**

In nonlinear science, study of nonlinear phenomena occurring in various areas of scientific fields has gained immense popularity amongst the scientists and engineers who have delivered tireless efforts towards the step by step development of the models using nonlinear differential equations, during the last few decades. Introduction of fractional calculus in nonlinear models has rendered a new dimension to the existing problems.

The fractional calculus is becoming a popular field of research since fractional order system response ultimately converges to the integer order system and also due to the non-local behaviours of differential operator while integer order counterpart is local. For high exactness, fractional derivatives are used to describe the dynamics of systems. A wide range of problems in different branches of engineering and biology have already been studied by a number of researchers of different parts of the world to explore the potential of the fractional derivative. The characteristic of fractional order systems for which they have gained popularity in the investigation of dynamical systems is that it allows a greater flexibility in the model. During last few years, The applications of fractional calculus have played very important role in various fields such as dielectric polarization, viscoelastic systems, electrode-electrolyte polarization, quantitative finance, the control of fractional-order dynamic systems etc. (Sun et al. (1984), Koeller (1984), Ichise et al.

(1971), Laskin (2000), Hartley and Lorenzo (2002)). In fractional calculus, chaos theory is one of the emerging areas of application and this contributes to a noteworthy amount of the ongoing research nowadays. The chaotic systems are extremely sensitive to initial condition, which is known as butterfly effect (Alligood et al. (1997)).

Study of chaos in nonlinear science has received a great deal of interest amongst the scientists and engineers in the last few decades in various fields and considerable research efforts have been devoted to chaos control and chaos synchronization problems in many dynamical systems. Pecora and Carroll (1990) were showed first time that it is possible to synchronize chaotic systems through a simple coupling, which are considered as the most important achievement in the research of chaos.

Synchronization of two systems means that trajectories of systems converges and remain in step with each other. The synchronization of chaotic dynamical systems has been intensively studied by many researchers and has attracted a great deal of interest in various fields due to its important applications in secure communications, physical system, chemical system, ecological system etc. (Murali and Lakshmanan (2003), Lakshmanan and Murali (1996)). A variety of approaches of synchronization phenomena have been proposed to investigate chaos synchronization such as adaptive control, active control, and so on, which have been effectively applied to applications of chaos synchronization problem. Chaos control and chaotic synchronization of fractional order differential systems have become interesting phenomena in chaos theory. Chaos control refers to manipulate the dynamical behavior of a chaotic system, in which the purpose is suppressed or to enhance or create chaos when it is needed (Miranda (2004)). Recently, many researchers have been successfully applied to achieve chaos control and synchronization of chaotic dynamical systems in standard and fractional orders.

Synchronization of fractional order chaotic systems is one of the potent research areas due to its extensive applications in communication theory and control processing.

In the present article, the necessary conditions for the existence of chaotic attractors have been investigated in the commensurate and incommensurate fractional order satellite systems applying the stability theory of fractional order systems and also applying chaos control of those systems with proper feedback control method. The function projective synchronization (FPS) between identical fractional order satellite systems have been achieved through the study. In FPS system, the drive and response systems could be synchronized up to a scaling function, but not a constant. Thus, FPS is considered to be more general definition of projective synchronization (PS), and it is used to obtain higher unpredictability of the error dynamical system which can enhance the security of communications. This proportional feature can be used to extend binary digital to M-nary digital communication for achieving fast communication compared to projective synchronization (Chee and Xu (2005)). Here numerical simulations are carried out using Adams Bashforth Moulton method (Diethelm and Ford (2004a), Diethelm and Ford (2004b)) for different fractional order derivatives which are displayed graphically to demonstrate the efficiency of the proposed approach for different particular cases. But to the best of authors' knowledge the function projective synchronization between identical fractional order satellite systems have not yet been studied by any researcher. There have been many synchronization results about integer order satellite system in the past few decades those could not be extended and applied easily to the fractional order case. Therefore, to establish some theoretically sufficient criteria for the synchronization of fractional order satellite systems are necessary and challenging. The objective will be

improving the efficiency of satellite communication systems so that the receiver is able to recover rapidly the synchronization with transmitter.

Keeping in view towards application of formations of satellites, the researchers should have objectives towards point measuring instruments in same direction. Therefore the slave satellite systems would be chosen so that the measurements of their trajectories will be according to altitudes of master satellite systems. In space technology progresses, the need for improved satellite systems by better understanding of satellite dynamics has continuously kept attention (Kuang et al. (2003)). Recently, non-linear dynamics, especially the chaotic attitude dynamics of a satellite have attracted the attention of many scientists. For instance it is possible to achieve synchronization among the systems, so that chaos-based communication can be implemented properly on a satellite formulation flying environment (Grzybowski et al. (2010)). Also both the chaos and optimal control of the steady rotations of a satellite-gyrostad on a circular orbit (Gohary (2009)).

In this chapter, a sincere attempt has been taken to study function projective synchronization between two identical fractional order satellite systems. This article has been organized as follows. Some definitions are introduced in Section 6.2. In Section 6.3, the system description of satellite system, stability analysis of fractional order satellite system, control of chaos using feedback control method and stabilizing the critical points are discussed. Section 6.4 contains function projective synchronization of fractional order satellite systems in incommensurate order and section 6.5 provides numerical simulation and results. In section 6.6, conclusion of the overall work is provided.

## 6.2 Some Preliminaries

### 6.2.1 Function projective synchronization

Consider the drive system and response system as

$$D^\alpha x = Ax + f(x)$$

$$D^\alpha y = By + g(y) + u(x, y),$$

where  $0 < \alpha \leq 1$  is the order of the fractional derivative,  $x = [x_1, x_2, \dots, x_n]^T$ ,

$y = [y_1, y_2, \dots, y_n]^T$  are the state vectors,  $f(x) \in R^n$ ,  $g(x) \in R^n$  are the nonlinear terms of

the systems,  $A$  and  $B$  are the constant matrices and  $u(x, y)$  are the control functions.

Now defining the error function as

$$e = y - k(t)x,$$

where  $k(t) = \text{diag}(k_1(t), k_2(t), \dots, k_n(t))$  and  $k_i(t)$ ,  $i = 1, 2, \dots, n$  are the continuous differentiable functions with bounded,  $k_i(t) \neq 0$  for all  $t$ .

**Definition 6.1:** The considered drive system and master system are said to be function projective synchronization if  $\lim_{t \rightarrow \infty} \|e(t)\| = 0$ .

## 6.3 System's description, stability analysis and control of chaos

### 6.3.1 Satellite system

The dynamic equations of a satellite (Sadaoui et al. (2011)) are given as

$$I_x \dot{\omega}_x = \omega_y \omega_z (I_y - I_z) + h_x + u_x,$$

$$I_y \dot{\omega}_y = \omega_z \omega_x (I_z - I_x) + h_y + u_y, \tag{6.1}$$

$$I_z \dot{\omega}_z = \omega_x \omega_y (I_x - I_y) + h_z + u_z,$$

where  $\omega_x, \omega_y, \omega_z$  are the angular velocities and  $I_x, I_y, I_z$  are the principal moments of inertia of the satellite,  $h_x, h_y, h_z$  are perturbing torques and  $u_x, u_y, u_z$  are the three control torques.

Taking  $I_x = 3, I_y = 2, I_z = 1$ , the perturbing torques become

$$\begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = \begin{bmatrix} -1.2 & 0 & \sqrt{6}/2 \\ 0 & 0.35 & 0 \\ -\sqrt{6} & 0 & -0.4 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}. \quad (6.2)$$

These torques are chosen in such a way that the satellite shows the chaotic motion (Kung et al. (2003)).

### 6.3.2 Stability analysis of fractional order satellite system

In this chapter stability analysis of fractional order system has been thoroughly investigated where it's necessary and sufficient conditions are derived (Grzybowski et al. (2010), Aslanov and Doroshin (2010), Gohary (2009), Tsui and Jones (2009)). For this purpose, we consider the following  $n$  dimensional fractional order system

$$D_t^{\alpha_i} x_i = f_i(x_1, x_2, \dots, x_n), \quad (6.3)$$

where  $\alpha_i$  is rational numbers between 0 and 1, and  $D_t^{\alpha_i} x_i$  is the Caputo fractional order derivative of order  $\alpha_i$ , for  $i = 1, 2, \dots, n$ .

Assume that the system (6.3) is an autonomous system as  $[f_1(x), f_2(x), \dots, f_n(x)]^T = [a_{ij}]_{i,j=1}^n x = Ax$ , where  $x \in R^n$ , then we have the following results:

**Theorem 6.1:** If  $\alpha = \alpha_1 = \alpha_2 = \dots = \alpha_n$ , then the fractional order system is asymptotically stable iff  $|\arg(\text{spec}(K))| > \pi\alpha/2$  and the system is stable iff  $|\arg(\lambda)| \geq \pi\alpha/2$  for all eigenvalues  $\lambda$  of the matrix  $K$  with those critical eigenvalues satisfying  $|\arg(\lambda)| \geq \pi\alpha/2$

having geometric multiplicity of one. In this case the components of the state decay towards 0 like  $t^{-\alpha}$ .

**Theorem 6.2:** If  $\alpha_i$ 's are rational numbers between 0 and 1, for  $i=1,2,\dots,n$ . Let  $\gamma = 1/m$  and  $m$  be the least common multiple of the denominators  $m_i$ 's of  $\alpha_i$ 's, where  $\alpha_i = k_i/m_i$ ,  $(k_i, m_i) = 1$ ,  $m_i, k_i \in N$ ,  $i=1,2,\dots,n$ . Then the system (7.3) is asymptotically stable if all roots of the equation  $\det(\text{diag}(\lambda^{m\alpha_1}, \lambda^{m\alpha_2}, \dots, \lambda^{m\alpha_n}) - K) = 0$  satisfy  $|\arg(\lambda)| > \gamma \pi / 2$ .

Again if the function  $f_i$  has second continuous partial derivatives in a ball centered at an equilibrium point  $x^* = (x_1, x_2, \dots, x_n)$ , that is  $f_i(x_1, x_2, \dots, x_n) = 0$ , for  $i=1,2,\dots,n$ , then we have the following results.

**Theorem 6.3:** If  $\alpha = \alpha_1 = \alpha_2 = \dots = \alpha_n$ , then the equilibrium point  $x^*$  of the system (7.3) is asymptotically stable iff  $|\arg(\text{spec}(J/x^*))| > \pi\alpha/2$ , where the matrix  $J$  is Jacobian

matrix of the system (6.3) which is defined by  $j = \left[ \frac{\partial f_i}{\partial x_j} \right]_{i,j=1}^n$ .

**Theorem 6.4:** If  $\alpha_i$ 's are rational numbers between 0 and 1, then the equilibrium point  $x^*$  of the system (6.3) is asymptotically stable if all the roots of the equation

$$\det(\text{diag}(\lambda^{m\alpha_1}, \lambda^{m\alpha_2}, \dots, \lambda^{m\alpha_n}) - J/x^*) = 0, \tag{6.4}$$

satisfy  $|\arg(\lambda)| > \gamma \pi / 2$ , where  $\gamma = 1/m$ . We can write inequality of this condition in another

$$\text{way as } \frac{\pi}{2m} - \min_i \{|\arg(\lambda_i)|\} < 0. \tag{6.5}$$

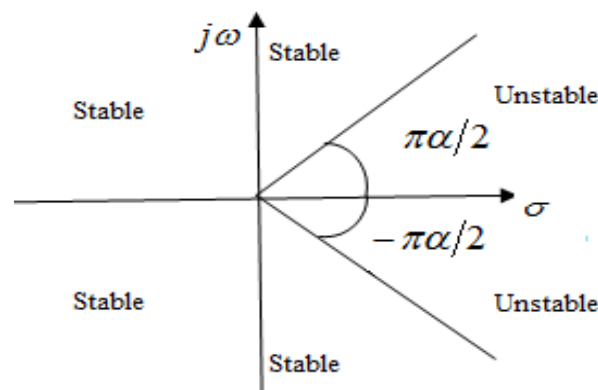
Thus, equilibrium point of the system (6.3) is asymptotically stable if the condition (6.5) is satisfied.

Hence, a necessary condition for fractional order system to exhibit a chaotic attractor is

$$\frac{\pi}{2m} - \min_i \{|\arg(\lambda_i)|\} \geq 0. \quad (6.6)$$

In studying the existence of chaotic attractors and the synchronization between fractional order systems, the previous stability results play an important role.

If all the eigen values of the Jacobian matrix are evaluated at equilibrium point are outside of the sector region, then the fractional differential system is stable. According to the above results, stable and unstable regions with order  $\alpha$  are shown in Fig. 6.1.



**Fig. 6.1** Stability region of linear fractional-order system with order  $\alpha$  .

The three-dimension autonomous chaotic dynamical satellite system is presented in (Farid and Moghaddam (2014), Sadaoui et al. (2011)), which is described as

$$\begin{aligned} D_t x &= \frac{yz}{3} - 0.4x + \frac{z}{\sqrt{6}}, \\ D_t y &= -xz + 0.175y, \\ D_t z &= -\sqrt{6}x - 0.4z + xy, \end{aligned} \quad (6.7)$$

where  $x, y, z$  are state variables of the system. The fractional-order satellite system can be defined as

$$\begin{aligned} D_t^{\alpha_1} x &= \frac{yz}{3} - 0.4x + \frac{z}{\sqrt{6}}, \\ D_t^{\alpha_2} y &= -xz + 0.175y, \\ D_t^{\alpha_3} z &= -\sqrt{6}x - 0.4z + xy, \quad 0 < \alpha_1, \alpha_2, \alpha_3 < 1. \end{aligned} \tag{6.8}$$

Next it is my job to discuss the necessary conditions for the existence of chaotic attractors in fractional-order system. The equilibrium points of the system (6.8) are obtained through

$$\frac{yz}{3} - 0.4x + \frac{z}{\sqrt{6}} = 0, \quad -xz + 0.175y = 0 \quad \text{and} \quad -\sqrt{6}x - 0.4z + xy = 0.$$

Five equilibrium points of the system (6.8) as

$$\begin{aligned} E_0 &= (0, 0, 0), E_1 = (-0.1577, -1.3510, 1.4988), E_2 = (0.1577, -1.3510, -1.4988), E_3 = (-1.1948, \\ &2.5758, -0.3773), E_4 = (1.1948, 2.5758, 0.3773) . \end{aligned}$$

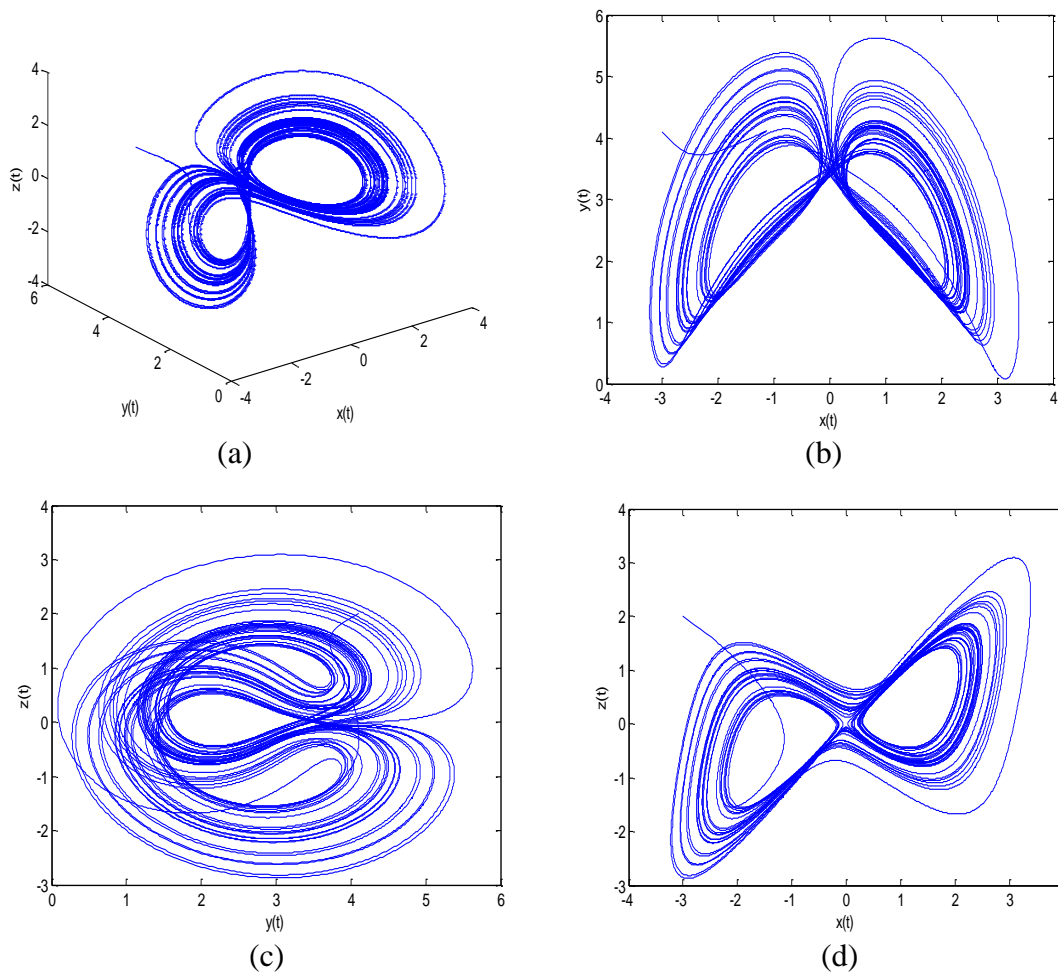
For five equilibrium points the corresponding eigen values of Jacobian matrix are  $(-0.4+i, -0.4-i, 0.175)$ ,  $(0.0875+0.8752i, 0.0875-0.8752i, -0.8)$ ,  $(0.0875+0.8752i, 0.0875-0.8752i, -0.8)$ ,  $(-0.8, 0.0875+1.2114i, 0.0875-1.2114i)$ ,  $(-0.8, 0.0875+1.2114i, 0.0875-1.2114i)$

respectively. We found that equilibrium point  $E_0 = (0, 0, 0)$  is a saddle point of index 1 whereas  $E_1, E_2, E_3$  and  $E_4$  are saddle points of index 2. In chaotic systems, it is proved that scrolls are generated only around the saddle points of index 2. The saddle point of index 1 is responsible only for connecting scrolls (Matignon (1996), Deng et al. (2007)).

The necessary condition that the chaos in fractional-order system exists when the eigen

value  $\lambda$  is in the unstable region  $|\arg(\lambda)| < \frac{\alpha\pi}{2}$  i.e.,  $\alpha > \frac{2}{\pi}|\arg(\lambda)|$ .

Therefore, the lowest fractional order of satellite system where chaos occurs is 0.94. Thus for a commensurate order system at  $\alpha_1 = \alpha_2 = \alpha_3 = \alpha = 0.94$ , the system shows the regular chaotic behavior. The chaotic attractors of satellite system for this case are shown through Fig. 6.2.



**Fig. 6.2** Phase portraits of the fractional-order satellite system with commensurate orders  $(\alpha_1, \alpha_2, \alpha_3) = (0.94, 0.94, 0.94)$  in (a) x-y-z space (b) x-y plane (c) y-z plane and (d) x-z plane.

In the case of incommensurate fractional order system, we consider four cases for fractional order satellite system.

**Case 1:** If  $\alpha_1 = 0.96$ ,  $\alpha_2 = 0.95$ ,  $\alpha_3 = 1$ , then from equation (6.4), we get

$$\det(\text{diag}(\lambda^{96} \lambda^{95} \lambda^{100}) - J(E_1)) = 0, \quad \text{where } m = 100$$

$$\lambda^{29} + 0.6\lambda^{20} + 2.1\lambda^{19} + 18\lambda^{10} - 0.000063\lambda^9 + 70.3094 = 0.$$

This with the help of equation (6.4) gives

$$\frac{\pi}{20} - \min_i \{|\arg(\lambda_i)|\} = 0.000395057 > 0,$$

which shows that the system is chaotic in given fractional order, which is depicted through Fig. 6.4(a).

**Case 2:** If  $\alpha_1 = 0.96$ ,  $\alpha_2 = 0.95$ ,  $\alpha_3 = 0.94$ , we obtain from equation (6.4),

$$\det(\text{diag}(\lambda^{96} \lambda^{95} \lambda^{94}) - J(E_1)) = 0, \quad \text{where } m = 100$$

$$\lambda^{283} + 0.6\lambda^{192} + 2.1\lambda^{190} + 1.26\lambda^{99} + 16.7403\lambda^{93} - 0.000063\lambda^{91} + 70.3094 = 0.$$

This with equation (6.4) gives

$$\frac{\pi}{200} - \min_i \{|\arg(\lambda_i)|\} = 0.000256924 > 0.$$

For this reason, the chaotic attractor occurs are shown through Fig. 6.4(b).

**Case 3:** If  $\alpha_1 = 0.97$ ,  $\alpha_2 = 0.93$ ,  $\alpha_3 = 0.98$ , according to the previous case we have

$$\lambda^{284} + 0.6\lambda^{188} + 2.1\lambda^{191} + 1.26\lambda^{95} + 16.7403\lambda^{93} - 0.000063\lambda^{96} + 70.3094 = 0$$

$$\text{Thus, } \frac{\pi}{200} - \min_i \{|\arg(\lambda_i)|\} = 0.000241663 > 0.$$

The chaotic attractor for this case is shown through Fig. 6.4(c).

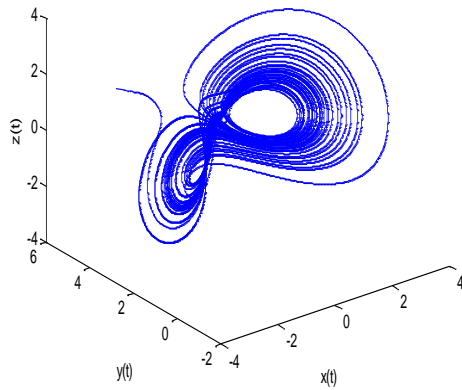
**Case 4:** If  $\alpha_1 = 0.95$ ,  $\alpha_2 = 0.92$ ,  $\alpha_3 = 0.93$ , then from (6.4), we have

$$\lambda^{27} + 0.6\lambda^{20} + 2.1\lambda^{17} + 18\lambda^{10} - 0.000063\lambda^7 + 70.3094 = 0$$

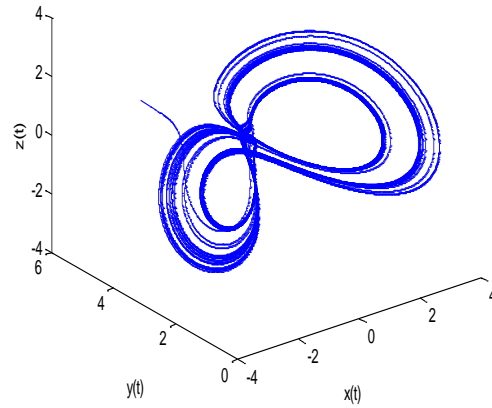
Now, necessary condition of the existence of chaos

$$\frac{\pi}{20} - \min_i \{|\arg(\lambda_i)|\} = -0.00007618 < 0.$$

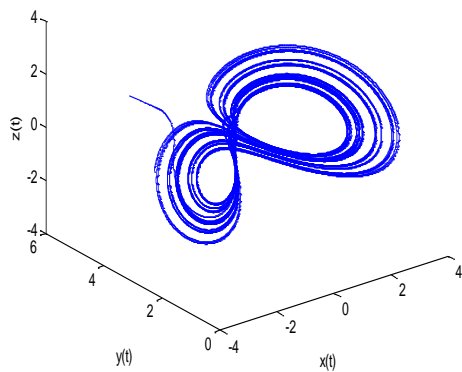
This implies that system does not exhibit chaos, which is found through Fig. 6.4(d).



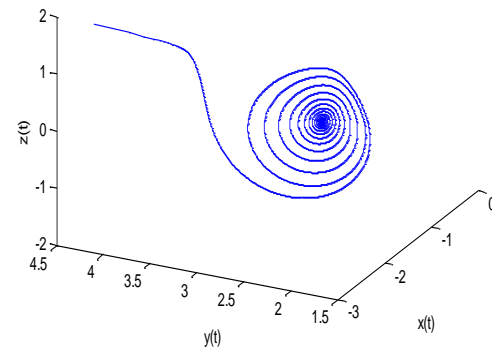
(a) For  $\alpha_1 = 0.96$ ,  $\alpha_2 = 0.95$ ,  $\alpha_3 = 1$ .



(b) For  $\alpha_1 = 0.96$ ,  $\alpha_2 = 0.95$ ,  $\alpha_3 = 0.94$ .



(c) For  $\alpha_1 = 0.97$ ,  $\alpha_2 = 0.93$ ,  $\alpha_3 = 0.98$ .



(d) For  $\alpha_1 = 0.95$ ,  $\alpha_2 = 0.92$ ,  $\alpha_3 = 0.93$ .

**Fig. 6.3** Phase portraits of fractional-order satellite system in incommensurate order.

### 6.3.3 Control of Chaos

Let the fractional order satellite system is taken as the controlled system with control functions  $u_i(t)$ ,  $i = 1, 2, 3$  to stabilize the unstable periodic orbit or fixed point as given

by

$$D_t^{\alpha_1} x = \frac{yz}{3} - 0.4x + \frac{z}{\sqrt{6}} + U_1(t),$$

$$D_t^{\alpha_2} y = -x z + 0.175 y + U_2(t), \quad (6.9)$$

$$D_t^{\alpha_3} z = -\sqrt{6} x - 0.4 z + x y + U_3(t).$$

Let  $(\bar{x}, \bar{y}, \bar{z})$  is the solution of the system (6.8), we have

$$D_t^{\alpha_1} \bar{x} = \frac{\bar{y} \bar{z}}{3} - 0.4 \bar{x} + \frac{\bar{z}}{\sqrt{6}},$$

$$D_t^{\alpha_2} \bar{y} = -\bar{x} \bar{z} + 0.175 \bar{y}, \quad (6.10)$$

$$D_t^{\alpha_3} \bar{z} = -\sqrt{6} \bar{x} - 0.4 \bar{z} + \bar{x} \bar{y},$$

Defining the error functions as  $e_1 = x - \bar{x}$ ,  $e_2 = y - \bar{y}$ ,  $e_3 = z - \bar{z}$ , we obtain the following error system as

$$D_t^{\alpha_1} e_1 = -0.4 e_1 + \frac{e_3}{\sqrt{6}} + \frac{y z}{3} - \frac{\bar{y} \bar{z}}{3} + U_1(t),$$

$$D_t^{\alpha_2} e_2 = 0.175 e_2 - x z + \bar{x} \bar{z} + U_2(t), \quad (6.11)$$

$$D_t^{\alpha_3} e_3 = -\sqrt{6} e_1 - 0.4 e_3 + x y - \bar{x} \bar{y} + U_3(t).$$

Let us define the active control functions  $U_i(t)$ ,  $i = 1, 2, 3$  as

$$U_1(t) = V_1(t) - \frac{y z}{3} + \frac{\bar{y} \bar{z}}{3},$$

$$U_2(t) = V_2(t) + x z - \bar{x} \bar{z}, \quad (6.12)$$

$$U_3(t) = V_3(t) - x y + \bar{x} \bar{y},$$

where the terms  $V_i(t)$ ,  $i = 1, 2, 3$  are linear functions in the error terms  $e_i(t)$ . Now the error system (6.11) will be

$$D_t^{\alpha_1} e_1 = -0.4 e_1 + \frac{e_3}{\sqrt{6}} + V_1(t),$$

$$D_t^{\alpha_2} e_2 = 0.175e_2 + V_2(t), \quad (6.13)$$

$$D_t^{\alpha_3} e_3 = -\sqrt{6}e_1 - 0.4e_3 + V_3(t).$$

Let us design an appropriate feedback control to stabilize the system. There are many possible choices for the control inputs  $V_i(t)$ . We choose

$$\begin{bmatrix} V_1(t) \\ V_2(t) \\ V_3(t) \end{bmatrix} = A \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \quad (6.14)$$

where  $A$  is  $3 \times 3$  constant matrix. In order to make the closed loop system stable, the matrix is selected in such a way that the feedback system has eigen values  $\lambda_i$  of  $A$ , which satisfy the conditions  $|\arg(\lambda_i)| > \alpha\pi/2$ ,  $i = 1, 2, 3$ . There is no unique choice for matrix  $A$ .

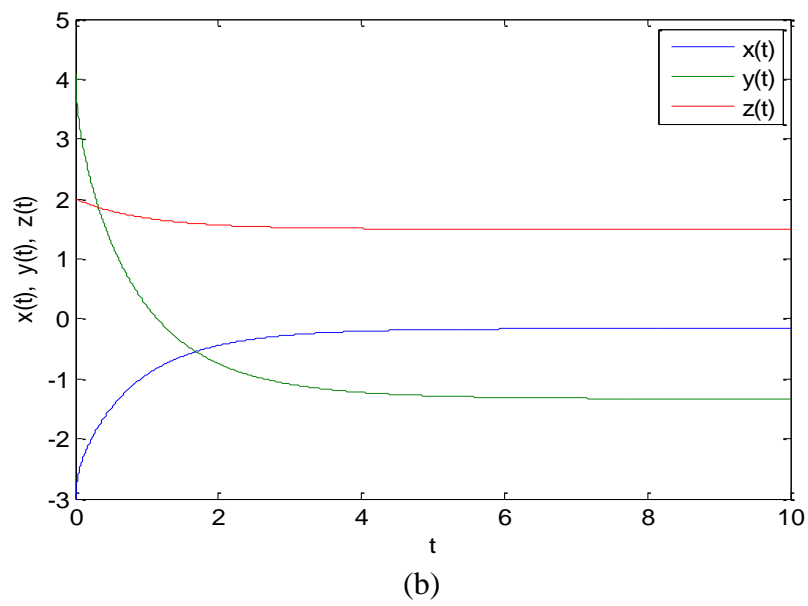
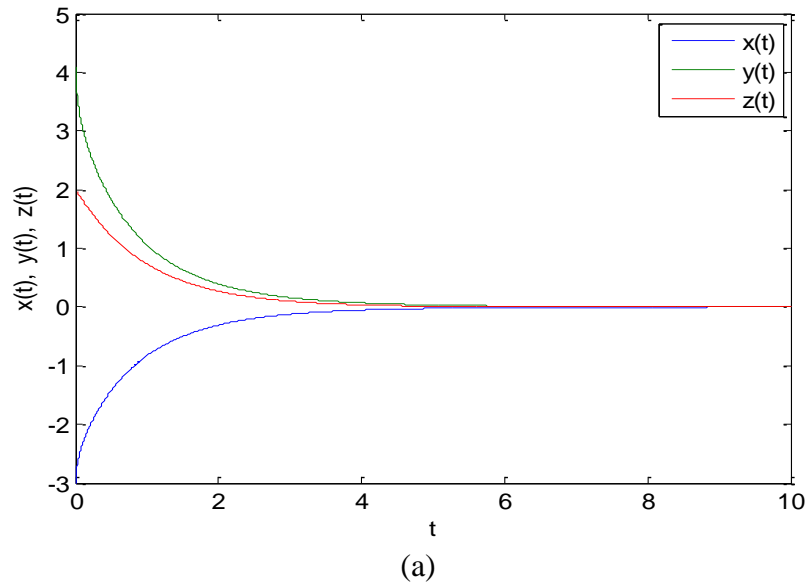
Let us choose the matrix  $A$  in the form

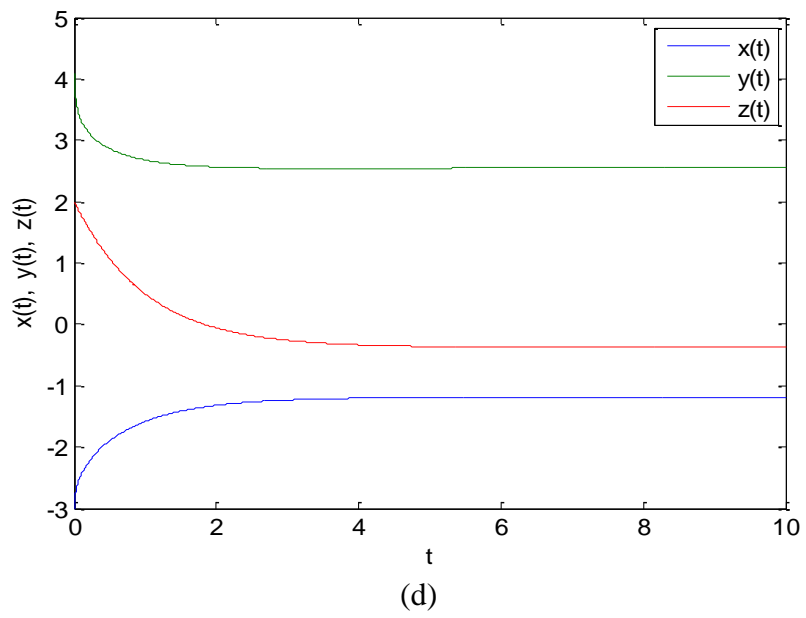
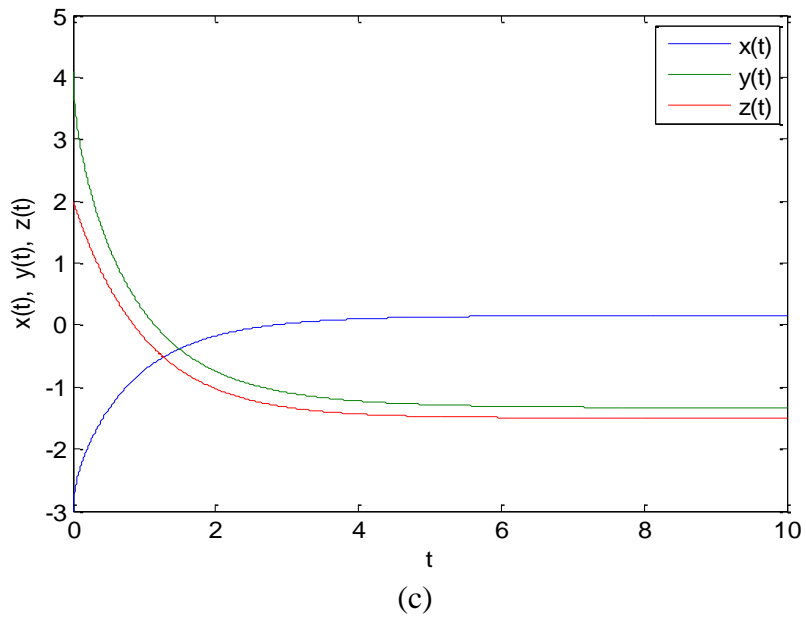
$$A = \begin{bmatrix} -0.6 & 0 & -1/\sqrt{6} \\ 0 & -1.175 & 0 \\ \sqrt{6} & 0 & -0.6 \end{bmatrix}.$$

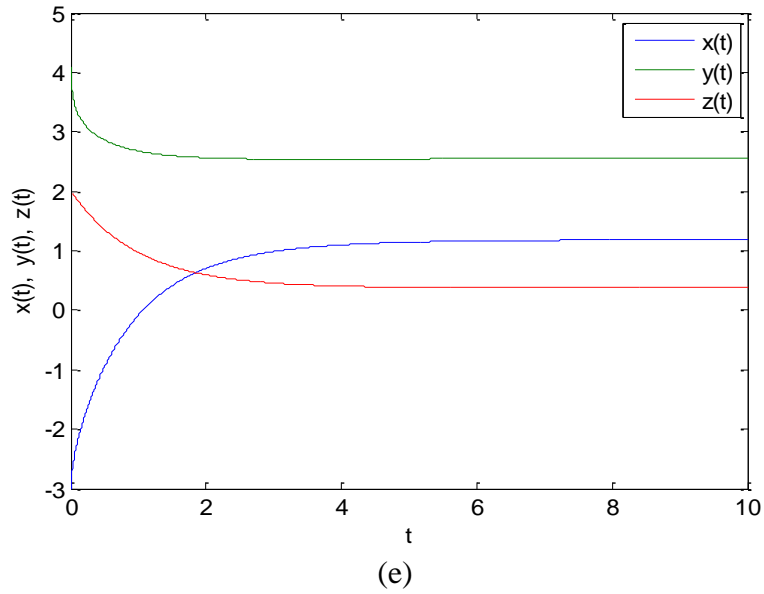
With this choice of matrix, the closed loop system will have the negative eigen values. This choice will lead to the error states  $e_i$ ,  $i = 1, 2, 3$  converges to zero as time becomes large and thus the trajectories  $(x(t), y(t), z(t))$  of satellite system converge to the point  $(\bar{x}, \bar{y}, \bar{z})$ .

### 6.3.3.1 Stabilizing the points $E_0, E_1, E_2, E_3$ and $E_4$

It is clear from the Fig. 6.4(a)-(e) at the point  $E_0$ , the system is stable for the order  $0 < \alpha_1 \leq 1$ ,  $0 < \alpha_2 \leq 1$ ,  $0 < \alpha_3 \leq 1$ . Similarly at the points  $E_1, E_2, E_3$  and  $E_4$ , the system is also stable for the order  $0 < \alpha_1 \leq 1$ ,  $0 < \alpha_2 \leq 1$ ,  $0 < \alpha_3 \leq 1$ .







**Fig. 6.4** Plots of  $x(t)$ ,  $y(t)$ ,  $z(t)$  of the controlled system (8): (a) at equilibrium point  $E_0$ , (b) at the equilibrium  $E_1$ , (c) at the equilibrium  $E_2$ , (d) at the equilibrium  $E_3$ , (e) at the equilibrium  $E_4$ .

#### 6.4 Function projective synchronization of incommensurate fractional order satellite system

To perform the function projective synchronization of the identical fractional order satellite system in incommensurate order, let us consider the satellite system as the master system as

$$\begin{aligned}
 D_t^{\alpha_1} x_1 &= \frac{y_1 z_1}{3} - 0.4 x_1 + \frac{z_1}{\sqrt{6}}, \\
 D_t^{\alpha_2} y_1 &= -x_1 z_1 + 0.175 y_1, \\
 D_t^{\alpha_3} z_1 &= -\sqrt{6} x_1 - 0.4 z_1 + x_1 y_1,
 \end{aligned} \tag{6.15}$$

and the slave system is taken as

$$\begin{aligned}
 D_t^{\alpha_1} x_2 &= \frac{y_2 z_2}{3} - 0.4 x_2 + \frac{z_2}{\sqrt{6}} + u_1(t), \\
 D_t^{\alpha_2} y_2 &= -x_2 z_2 + 0.175 y_2 + u_2(t), \\
 D_t^{\alpha_3} z_2 &= -\sqrt{6} x_2 - 0.4 z_2 + x_2 y_2 + u_3(t).
 \end{aligned} \tag{6.16}$$

Consider that the error functions  $e_1 = x_2 - k_1 x_1$ ,  $e_2 = y_2 - k_2 y_1$ ,  $e_3 = z_2 - k_3 z_1$  are characterizing the divergence of the master and slave solutions from each other, where  $k_1$ ,  $k_2$ , and  $k_3$  are scaling functions.

Then the error dynamical system becomes

$$\begin{aligned}
 D_t^{\alpha_1} e_1 &= -0.4 e_1 + \frac{e_3}{\sqrt{6}} + \frac{y_2 z_2}{3} - \frac{k_1 y_1 z_1}{3} + (k_3 - k_1) \frac{z_1}{\sqrt{6}} + u_1(t), \\
 D_t^{\alpha_2} e_2 &= 0.175 e_2 - x_2 z_2 + k_2 x_1 z_1 + u_2(t) \\
 D_t^{\alpha_3} e_3 &= -\sqrt{6} e_1 - 0.4 e_3 + \sqrt{6}(k_3 - k_1)x_1 + x_2 y_2 - k_3 x_1 y_1 + u_3(t).
 \end{aligned} \tag{6.17}$$

Defining the active control functions  $u_i(t)$ ,  $i = 1, 2, 3$  as

$$\begin{aligned}
 u_1(t) &= v_1(t) - \frac{y_2 z_2}{3} + \frac{k_1 y_1 z_1}{3} - (k_3 - k_1) \frac{z_1}{\sqrt{6}}, \\
 u_2(t) &= v_2(t) + x_2 z_2 - k_2 x_1 z_1, \\
 u_3(t) &= v_3 - \sqrt{6}(k_3 - k_1)x_1 - x_2 y_2 + k_3 x_1 y_1,
 \end{aligned} \tag{6.18}$$

where  $v_i(t)$ ,  $i = 1, 2, 3$  are linear functions of the error terms  $e_i(t)$ , The error system (6.17) is reduced to

$$\begin{aligned}
 D_t^{\alpha_1} e_1 &= -0.4 e_1 + \frac{e_3}{\sqrt{6}} + v_1(t), \\
 D_t^{\alpha_2} e_2 &= 0.175 e_2 + v_2(t), \\
 D_t^{\alpha_3} e_3 &= -\sqrt{6} e_1 - 0.4 e_3 + v_3(t).
 \end{aligned} \tag{6.19}$$

Now we design an appropriate feedback control which stabilizes the system so that  $e_i(t)$ ,  $i=1,2,3$  converge to zero as time  $t$  becomes large. There are many possible choices for the control inputs  $v_i(t)$ . We choose

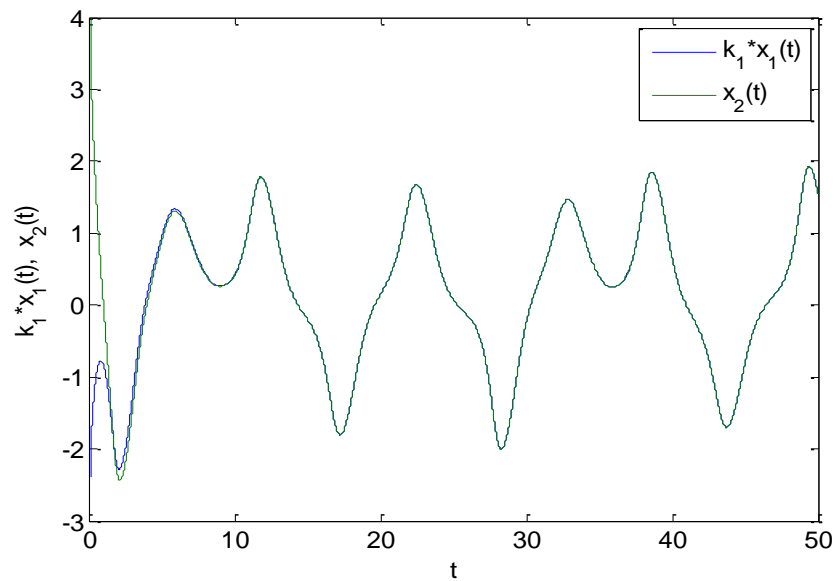
$$v(t) = Ae(t),$$

where  $v(t) = [v_1(t), v_2(t), v_3(t)]^T$ ,  $e(t) = [e_1(t), e_2(t), e_3(t)]^T$  and  $A$  is the  $3 \times 3$  matrix. In order to make the closed loop system stable, the matrix  $A$  is selected as

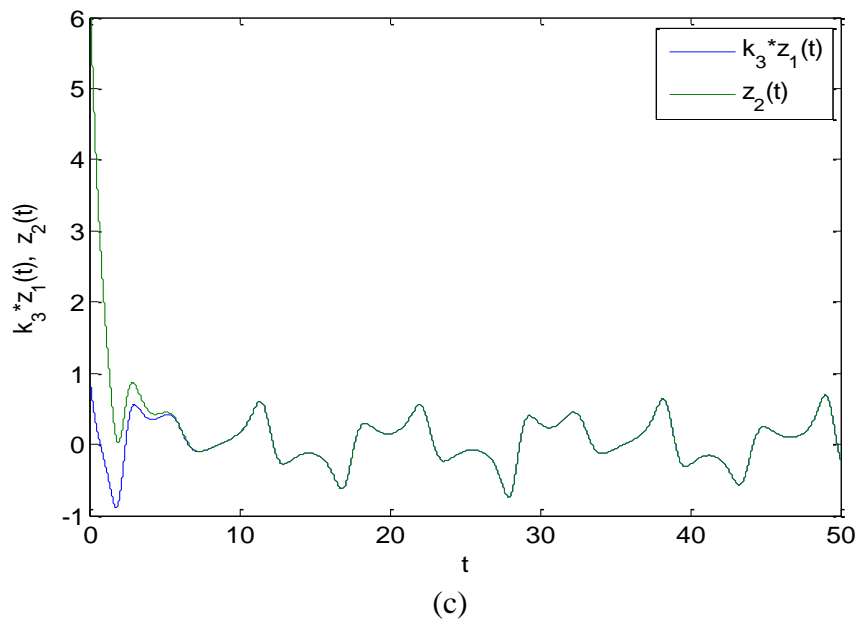
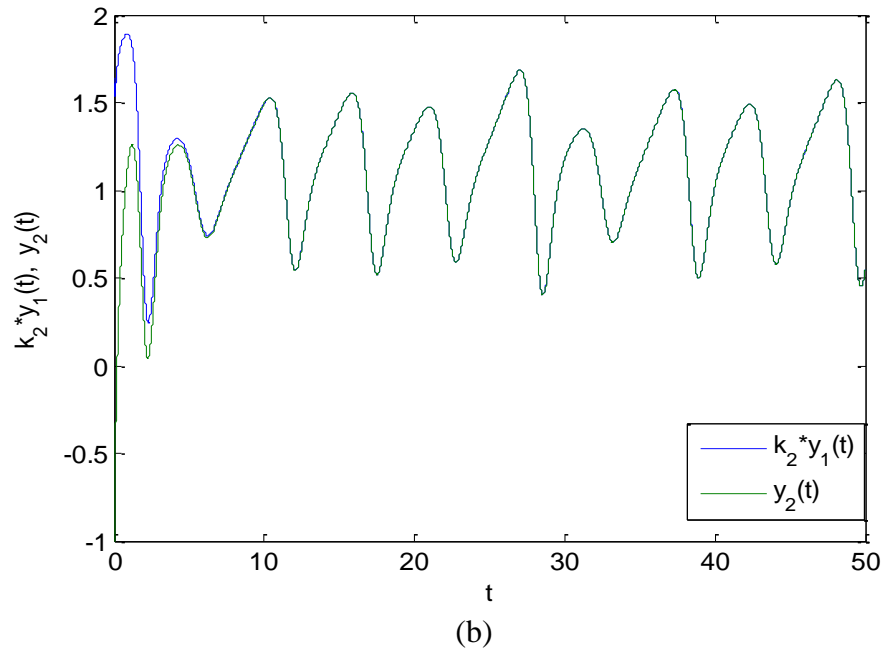
$$A = \begin{bmatrix} -0.6 & 0 & -1/\sqrt{6} \\ 0 & -1.175 & 0 \\ \sqrt{6} & 0 & -0.6 \end{bmatrix}, \quad (6.20)$$

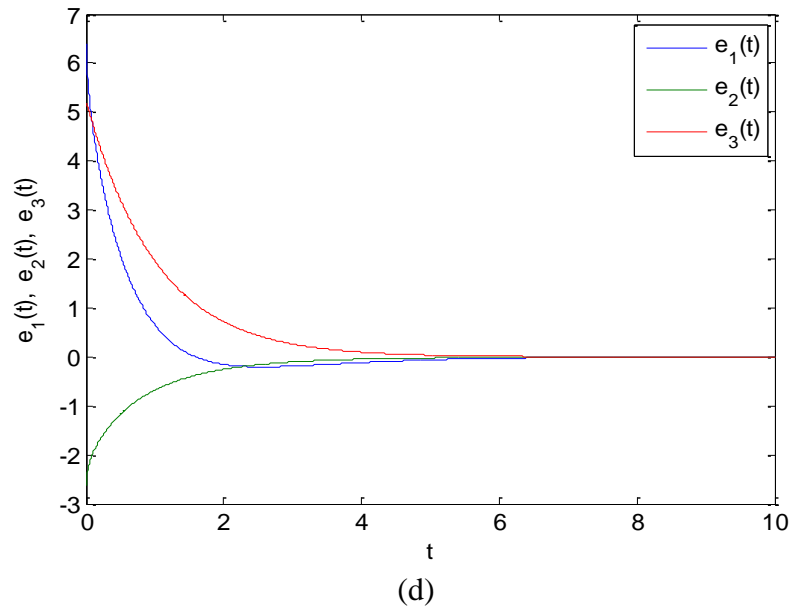
so that the feedback system has eigen values  $\lambda_i$  of  $A$  satisfy the conditions  $|\arg(\lambda_i)| > \alpha\pi/2$ ,  $i=1,2,3$ .

In the above closed loop system the eigen values are obtained as  $-1$ ,  $-1$  and  $-1$  and as a result the error states  $e_i$ ,  $i=1,2,3$  converges to zero as time tends to infinity and thus the required synchronization of the considered systems is achieved.

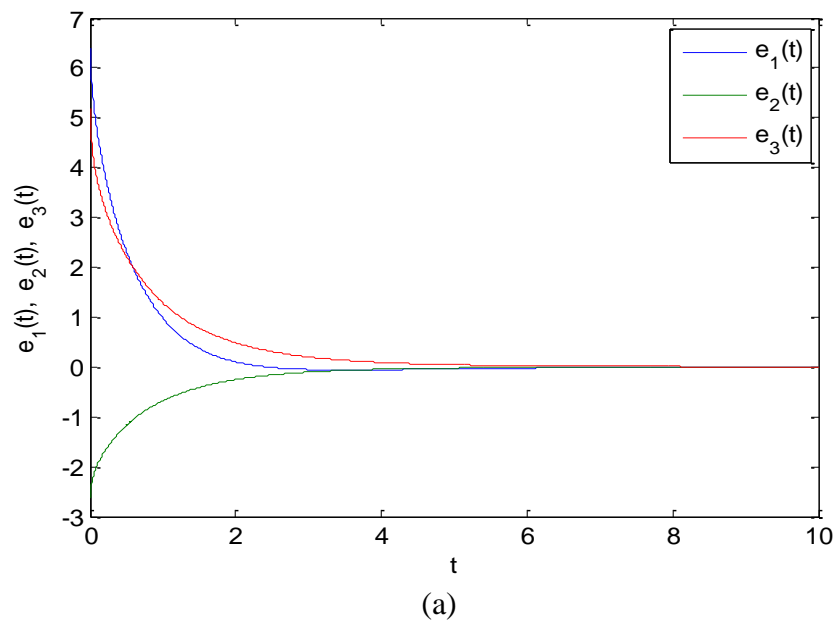


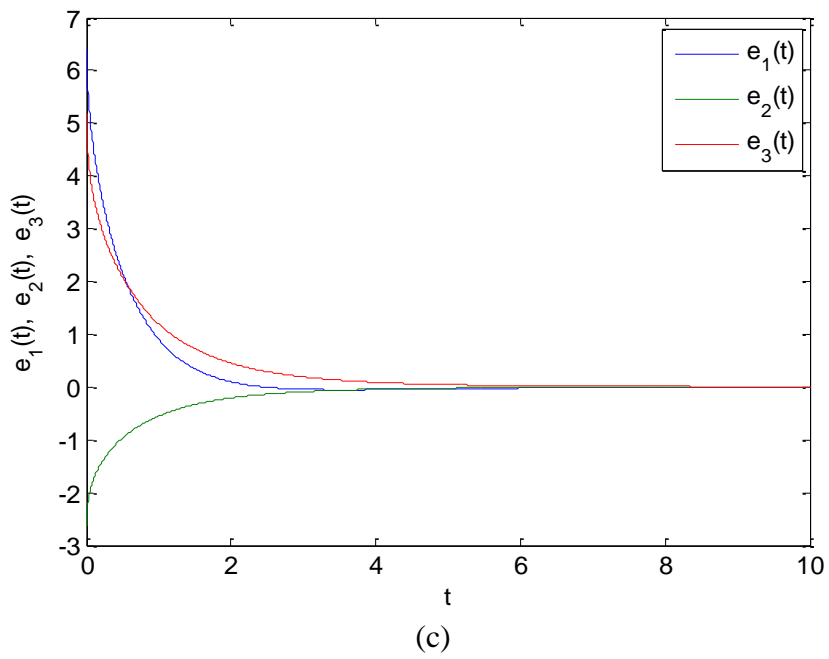
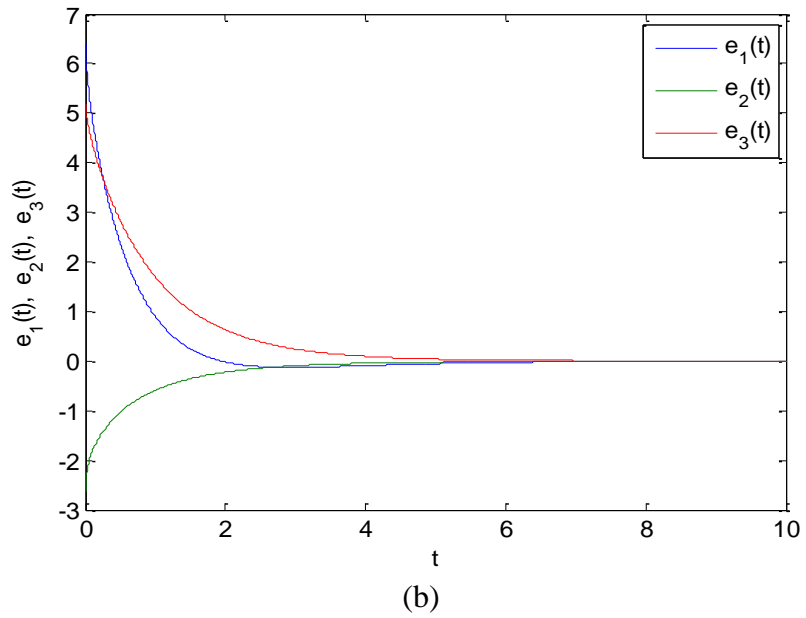
(a)

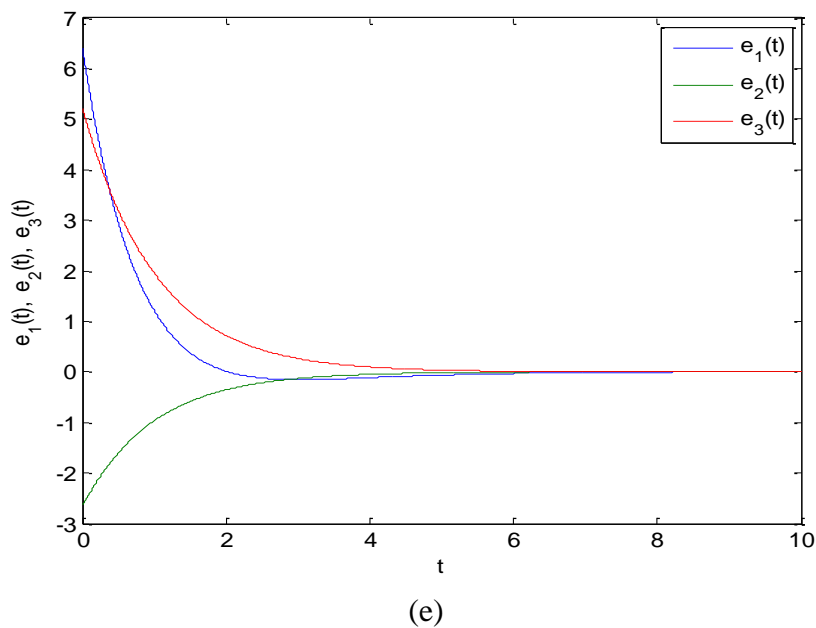
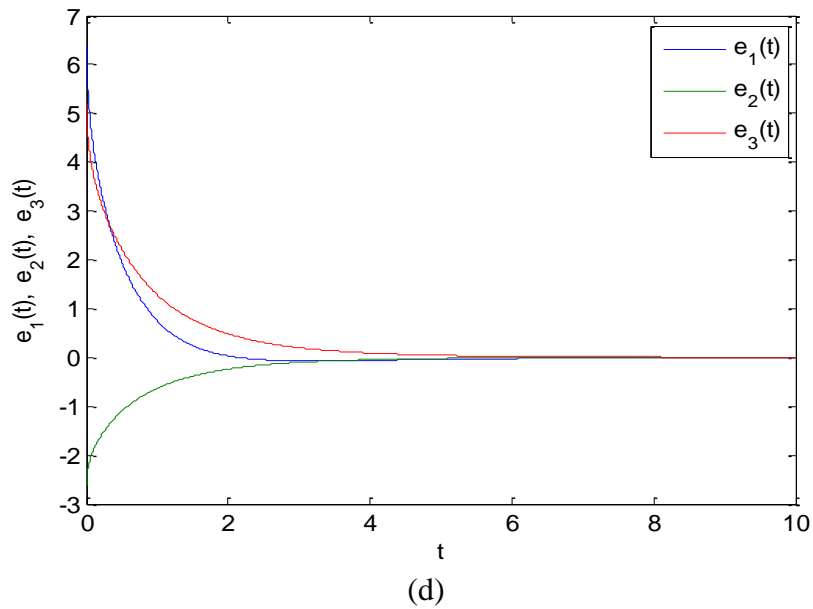




**Fig. 6.5** State trajectories of the master systems (15) and response systems (16) for order  $(\alpha_1, \alpha_2, \alpha_3) = (0.96, 0.95, 1)$ : (a) synchronization between  $k_1 * x_1(t)$  and  $x_2(t)$ ; (b) synchronization between  $k_2 * y_1(t)$  and  $y_2(t)$ ; (c) synchronization between  $k_3 * z_1(t)$  and  $z_2(t)$ ; (d) The evolution of the error functions  $e_i(t), i = 1, 2, 3$ .







**Fig. 6.6** The evolution of the error functions  $e_i(t), i=1, 2, 3$  : (a) for  $(\alpha_1, \alpha_2, \alpha_3) = (0.96, 0.95, 0.94)$  ; (b) for  $(\alpha_1, \alpha_2, \alpha_3) = (0.97, 0.93, 0.98)$  ; (c) for  $(\alpha_1, \alpha_2, \alpha_3) = (0.95, 0.92, 0.93)$  ; (d) for  $(\alpha_1, \alpha_2, \alpha_3) = (0.94, 0.94, 0.94)$  , (e) for  $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 1)$ .

### 6.5 Numerical simulation and results

In numerical simulation, the initial values of the master and slave systems are taken as  $(x_1(0), y_1(0), z_1(0)) = (-3, 4.1, 2)$ ,  $(x_2(0), y_2(0), z_2(0)) = (4, -1, 6)$ . Consequently, error is  $(e_1(0), e_2(0), e_3(0)) = (6.4, -2.6396, 5.2)$ . The time step is taken as 0.005. We choose the scaling functions as  $k_1 = a_1 \cos(b_1 x_1) + d_1$ ,  $k_2 = a_2 \cos(b_2 y_1) + d_2$ ,  $k_3 = a_3 \cos(b_3 z_1) + d_3$ , with parameters  $a_1 = 0.6$ ,  $a_2 = 0.3$ ,  $a_3 = 0.1$ ,  $b_1 = 0.2$ ,  $b_2 = 0.4$ ,  $b_3 = 0.3$ ,  $d_1 = 0.2$ ,  $d_2 = 0.1$ ,  $d_3 = 0.3$ . The function projective synchronization of the satellite system between the trajectories are displayed through Fig. 6.5(a)-(c) for  $(\alpha_1, \alpha_2, \alpha_3) = (0.96, 0.95, 1)$ , which shows the time responses of state variables of the satellite systems. Fig. 6.5(d) and Figs. 6.6(a)-(e) show that error states converge to zero when time becomes large for the orders  $(\alpha_1, \alpha_2, \alpha_3) = (0.96, 0.95, 1)$ ,  $(\alpha_1, \alpha_2, \alpha_3) = (0.96, 0.95, 0.94)$ ,  $(\alpha_1, \alpha_2, \alpha_3) = (0.97, 0.93, 0.98)$ ,  $(\alpha_1, \alpha_2, \alpha_3) = (0.95, 0.92, 0.93)$ ,  $(\alpha_1, \alpha_2, \alpha_3) = (0.94, 0.94, 0.94)$  and  $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 1)$  respectively. From the simulation results, it is found that the attitude trajectory of the slave system track the attitude of the master system. Thus active control works well and the master and slave satellites are indeed achieving function projective synchronization.

### 6.6 Conclusion

Through this research contribution achieved three goals have been achieved. First one is the stability criteria of proposed commensurate and incommensurate fractional order satellite systems are obtained based on the stability theory of fractional order systems. Second one, instead of generating reference trajectories for each satellite system, synchronization of two fractional order satellite systems is done using control parameters to ensure proper coordination between those as a result of which the attitude of the

response would track the attitude of the drive system. This process helps to achieve improvement in efficiency of satellite communication systems. Thirdly, function projective synchronization between identical fractional order chaotic systems based on fractional order stability theory has been investigated. The graphical presentations of synchronization of fractional order chaotic systems between identical satellite system using numerical simulation show that the proposed method is very much effective and reliable.

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