Chapter 5

Sufficient Optimality Conditions and Duality for a Nonsmooth Interval-Valued Optimization Problem with Generalized Convexity via gH-Clarke Subgradients

5.1 Introduction

In many real word optimization problems the assumption of convexity rarely holds. In such a scenario, a possible direction is to define and study the notion of generalized convexity for nonconvex functions so that we can borrow well established concepts from convex optimization and apply them to solve nonconvex optimization. In this regard, to define the generalized convexity of functions we require generalized subdifferentials, similar to how subdifferential is used to characterize convexity. Among the generalized subdifferentials already proposed in the literature, Clarke subdifferential [33] is especially useful for functions that exhibit local Lipschitz-ness and lower semicontinuousness. It is also very frequently employed in variational analysis and optimization theory to study optimality conditions for nonsmooth optimization problems. Further, Clarke subdifferential plays a major role in duality theory (see [1, 10, 13, 178]), where it is used to establish connections between the primal and dual problems so that we can identify the necessary and sufficient optimality conditions. The theory of duality by itself is extremely valuable in the field of optimization, primarily because it provides a tool to study a minimization problem from the perspective of a corresponding maximization problem and vice versa. It also provides a better understanding of the nature of the optimization problem, along with the nature of its constraints and optimal solutions. Furthermore, it helps to evaluate the quality of the obtained solution.

Therefore, because of the already known importance of duality theory and the role of generalized convexity and Clarke subdifferential for nonsmooth optmization, it becomes vitally important for the case of interval-valued optimization as well that we define and develop such concepts. Further, to develop duality theory and identify sufficient optimality conditions for IOPs, we need the two nonsmooth generalizations of gH-strong convexity, namely gH-pseudoconvexity and gH-quasiconvexity (along with their two types: type I and type II), to be defined.

5.2 Motivation

At present, there does not exist any literature on duality theory defined in terms of type I and II strong pseudoconvexity and type I and II strong quasiconvexity for the kind of nonsmooth IOP considered in this chapter. For locally Lipscitz and nonsmooth IVFs, study of generalized directional derivative (gH-Clarke subdifferential) is also missing from the literature. Further, the development of the concept of Clarke subdifferential with respect to generalized convexity for nonsmooth IVFs can offer a new direction for characterizing optimality conditions through duality theory for nonsmooth IOPs.

5.3 Contributions

The main contributions of this chapter are as follows:

(i) We introduce two nonsmooth generalizations of strong convexity for gH-locally Lipschitz interval-valued functions: gH-pseudoconvexity and gH-quasiconvexity, using the proposed idea of gH-Clarke subdifferential.

(ii) With the help of these generalized convex interval-valued functions, we establish sufficient optimality conditions for a type of nonsmooth IOP whose objective function involves a support function of a given compact and convex set.

(iii) We provide weak and strong duality theorems for the IOP under consideration.

(iv) Using the proposed gH-Clarke subdifferential, we derive a sufficient optimality condition for strict minimizer, weak duality, and strong duality for strict efficiency for our nonsmooth IOP.

5.4 gH-Clarke subdifferential and gH-generalized convex IVFs

In this section, we introduce the concept of gH-Clarke directional derivative for gH-locally Lipschitz set-valued mappings (IVFs). Additionally, we need to define the gH-Clarke subdifferential for these mappings in order to establish optimality conditions in subsequent sections.

Definition 5.1 (gH-Clarke directional derivative for IVF). *We define the gH-Clarke directional derivative of a gH-locally Lipschitz IVF $\boldsymbol{T} : \mathbb{R}^n \to I(\mathbb{R})$ at $y^* \in \mathbb{R}^n$ in the direction $h \in \mathbb{R}^n$ by $\boldsymbol{T}^c(y^*; h)$ as follows*

$$\boldsymbol{T}^c(y^*; h) = \limsup_{\substack{y \to y^* \\ \beta \to 0^+}} \frac{1}{\beta} \odot \left(\boldsymbol{T}(y + \beta h) \ominus_{gH} \boldsymbol{T}(y) \right).$$

Definition 5.2 (gH-Clarke subdifferential for IVF). *We define the gH-Clarke subdifferential of $\boldsymbol{T}$ at $y^* \in \mathbb{R}^n$ by $\partial^c \boldsymbol{T}(y)$ as follows*

$$\partial^c \boldsymbol{T}(y^*) = \{\widehat{\boldsymbol{K}} \in I(\mathbb{R})^n : h^\top \odot \widehat{\boldsymbol{K}} \preceq \boldsymbol{T}^c(y^*; h)\}.$$

Example 5.4.1 *Let $\mathcal{Y} = (0, \infty)$. We define an IVF $\boldsymbol{T} : \mathcal{Y} \to I(\mathbb{R})$ by $\boldsymbol{T}(y) = [\ln y, y - 1]$. Let us evaluate gH-Clarke subdifferential of $\boldsymbol{T}$ at 1 along h. First, we evaluate the gH-Clarke directional derivative at $y^* = 1$ along h.*

$$\begin{aligned}
\boldsymbol{T}^c(1; h) &= \limsup_{\substack{y \to 1 \\ \beta \to 0^+}} \frac{1}{\beta} \odot \left\{ \boldsymbol{T}(y + \beta h) \ominus_{gH} \boldsymbol{T}(y) \right\} \\
&= \limsup_{\substack{y \to 1 \\ \beta \to 0^+}} \frac{1}{\beta} \odot \left\{ [\ln(y + \beta h), y + \beta h - 1] \ominus_{gH} [\ln y, y - 1] \right\} \\
&= \limsup_{\substack{y \to 1 \\ \beta \to 0^+}} \frac{1}{\beta} \odot \left\{ [\ln(y + \beta h) - \ln y, \beta h] \right\}.
\end{aligned}$$

Since

$$\lim_{\substack{y \to 1 \\ \beta \to 0^+}} \frac{1}{\beta} \odot \left\{ [\ln(y + \beta h) - \ln y, \beta h] \right\} = [h, h] = h \odot \boldsymbol{1},$$

$\boldsymbol{T}^c(1; h) = h \odot \boldsymbol{1}$. Hence, the gH-Clarke subdifferntial is $\partial^c \boldsymbol{T}(1) = \{\widehat{\boldsymbol{K}} : \widehat{\boldsymbol{K}} \preceq \boldsymbol{1}\}$.

Lemma 5.1 *If $\boldsymbol{T} : \mathcal{Y} \to I(\mathbb{R})$ is gH-locally lipschitz on $\mathcal{Y}$, then for any scalar $\lambda \geq 0$,*

$$\partial^c(\lambda \odot \boldsymbol{T})(y^*) = \lambda \odot \partial^c \boldsymbol{T}(y^*),$$

where dom $(\lambda \odot \boldsymbol{T})=$ *dom* $(\boldsymbol{T})$.

Proof: Let $\widehat{\mathbf{K}} \in \partial^c \mathbf{T}(y^*)$. Then, for any $y \in \text{dom}(\mathbf{T})$,

$$
\begin{aligned}
& h^\top \odot \widehat{\mathbf{K}} \preceq \limsup_{\substack{y \to y^* \\ \beta \to 0^+}} \frac{1}{\beta} \odot (\mathbf{T}(y+\beta h) \ominus_{gH} \mathbf{T}(y)) \\
\Longleftrightarrow\ & \lambda \odot (h^\top \odot \widehat{\mathbf{K}}) \preceq \limsup_{\substack{y \to y^* \\ \beta \to 0^+}} \frac{1}{\beta} (\lambda \odot (\mathbf{T}(y+\beta h) \ominus_{gH} \mathbf{T}(y))) \text{ since } \lambda \geq 0 \\
\Longleftrightarrow\ & h^\top \odot (\lambda \odot \widehat{\mathbf{K}}) \preceq \limsup_{\substack{y \to y^* \\ \beta \to 0^+}} \frac{1}{\beta} \odot ((\lambda \odot \mathbf{T})(y+\beta h) \ominus_{gH} (\lambda \odot \mathbf{T})(y)) \\
\Longleftrightarrow\ & \lambda \odot \widehat{\mathbf{K}} \in \partial^c (\lambda \odot \widehat{\mathbf{K}})(y^*).
\end{aligned}
$$

Hence, the result follows. □

Definition 5.3 (Support function) [145]. *Let* D *be a compact and convex subset of* $\mathbb{R}^n$*. Then, we define the support function* $\boldsymbol{\sigma}_D : \mathbb{R}^n \to \overline{\mathbb{R}}$ *for any* $y \in \mathbb{R}^n$ *as*

$$\boldsymbol{\sigma}_D(y) = \sup_{w \in D} y^\top w.$$

This support function is shown to be convex. Thus, its gH-Clarke subdifferential is not difficult to calculate. The Clarke subdifferential of $\boldsymbol{\sigma}_D(y)$ at y is given by

$$\partial^c \boldsymbol{\sigma}_D(y) = \left\{ w \in \mathbf{D} : y^\top w = \boldsymbol{\sigma}_D(y) \right\}.$$

Definition 5.4 (gH-Strongly convex IVF). *An IVF* $\boldsymbol{T} : \mathcal{Y} \to \mathbb{R}$ *is said to be a strongly convex function of order* m *if there exists a constant* $\alpha > 0$ *such that*

$$\boldsymbol{T}(\lambda y + (1-\lambda) z) \preceq \lambda \odot \boldsymbol{T}(y) \oplus (1-\lambda) \boldsymbol{T}(z) \ominus_{gH} \alpha \lambda (1-\lambda) \|y - z\|^m$$

for any $y, z \in \mathcal{Y}, \lambda \in [0,1]$.

Note 5.1 *Since* $\boldsymbol{T}$ *is an IVF with* $\boldsymbol{T}(y) = [\underline{t}(y), \overline{t}(y)]$*, where* $\underline{t}, \overline{t}$ *are real-valued functions, a* gH*-strongly convex IVF (Definition 5.4) directly reduces to conventional strongly convex functions [130].*

Lemma 5.2 *If each* $g_j : \mathcal{Y} \to \mathbb{R}, j = 1, 2, \ldots, q$ *is a type II strongly pseudoconvex of order* m *on a convex set* $\mathcal{Y}$*, then for* $\mu_j \geq 0, j = 1, 2, \ldots, q$*, the function* $\sum_{j=1}^{q} \mu_j g_j$ *is also a type II strongly pseudoconvex of order* m *on* $\mathcal{Y}$*.*

Proof: Let $\sum_{j=1}^{q} \mu_j \xi_j \in \partial^c \left(\sum_{j=1}^{q} \mu_j g_j(z^*)\right)$, where $z^* \in \mathcal{Y}$ and $\mu_j \xi_j \in \partial^c \left(\mu_j g_j(z^*)\right),\ j = 1, 2, \ldots, q$. Then, $\xi_j \in \partial^c g_j(z^*),\ j = 1, 2, \ldots, q$. Using the formula of Clarke subdifferential (see [34]) of $g_j, j = 1, 2, \ldots, q$ at z^* along h, one has

$$
\begin{aligned}
h^\top \xi_j &\leq \limsup_{\substack{z \to z^* \\ \beta \to 0^+}} \frac{g_j(z + \beta h) - g_j(z)}{\beta} \\
\Longrightarrow h^\top \xi_j &\leq \inf_{\epsilon \in (0, \epsilon_o)} \sup_{\substack{z \in \mathcal{B}(z^*, \delta\epsilon) \\ \beta \in (0, \epsilon)}} \frac{g_j(z + \beta h) - g_j(z)}{\beta}
\end{aligned}
$$

if and only if for any $\epsilon \geq 0$ there exists a $\delta \geq 0$ such that in a open neighbourhood $\mathcal{B}(z^*, \delta\epsilon)$,

$$
\begin{aligned}
& h^\top \xi_j + \epsilon \leq \frac{g_j(z + \beta h) - g_j(z)}{\beta},\ j = 1, 2, \ldots, q,\ \forall z \in \mathcal{B}(z^*, \delta\epsilon) \text{ and } \beta \in (0, \epsilon). \\
\Longrightarrow\ & (h\beta)^\top \xi_j + \beta\epsilon \leq g_j(z + \beta h) - g_j(z),\ j = 1, 2, \ldots, q.
\end{aligned}
$$

For simplicity, let us consider $y = z + \beta h$ and due to arbitrariness of ϵ, we obtain c and m such that $\beta\epsilon = c\|y - z\|^m$. Therefore,

$$
c\|y - z\|^m + (y - z)^\top \xi_j \leq g_j(y) - g_j(z),\ j = 1, 2, \ldots, q. \tag{5.1}
$$

Since each $g_j, j = 1, 2, \ldots, q$ is type II strongly pseudoconvex of order m on a convex set $\mathcal{Y}$, so for $c\|y - z\|^m + (y - z)^\top \xi_j \geq 0, j = 1, 2, \ldots, q$, from (5.1), we have

$$
\begin{aligned}
& g_j(z) \leq g_j(y),\ j = 1, 2, \ldots, q \\
\Longrightarrow\ & \mu_j g_j(z) \leq \mu_j g_j(y),\ \mu_j \geq 0,\ j = 1, 2, \ldots, q \\
\Longrightarrow\ & \sum_{j=1}^{q} \mu_j g_j(z) \leq \sum_{j=1}^{q} \mu_j g_j(y),\ j = 1, 2, \ldots, q.
\end{aligned}
$$

Therefore, from the arbitrariness of y and z, $\sum_{j=1}^{q} \mu_j g_j$ is a type II strongly pseudoconvex function of order m on $\mathcal{Y}$. □

In the following sections, we present two nonsmooth extensions of the notion of gH-strongly convexity: gH-pseudoconvexity and gH-quasiconvexity for IVFs. These categories are subdivided into type I and type II. In the remaining part, we assume $\mathcal{Y}$ to be a nonempty convex subset of $\mathbb{R}^n$.

Definition 5.5 (gH-Pseudoconvex IVF). *A gH-locally Lipschitz $\boldsymbol{T} : \mathcal{Y} \to I(\mathbb{R})$ on $\mathcal{Y}$*

is referred to as gH-pseudoconvex IVF if for all $y, z \in \mathcal{Y}$, the following result is true:

$$\boldsymbol{0} \preceq (y-z)^\top \odot \widehat{\boldsymbol{K}} \text{ for some } \widehat{\boldsymbol{K}} \in \partial^c \boldsymbol{T}(z) \implies \boldsymbol{T}(z) \preceq \boldsymbol{T}(y).$$

Definition 5.6 (gH-Type I strongly pseudoconvex IVF). *A gH-locally Lipschitz $\boldsymbol{T} : \mathcal{Y} \to I(\mathbb{R})$ on $\mathcal{Y}$ is referred to as gH-type I strongly pseudoconvex IVF of order m on $\mathcal{Y}$ if there exists a positive constant α satisfying*

$$\boldsymbol{0} \preceq (y-z)^\top \odot \widehat{\boldsymbol{K}} \text{ for some } \widehat{\boldsymbol{K}} \in \partial^c \boldsymbol{T}(z) \implies \boldsymbol{T}(z) \oplus \alpha \|y-z\|^m \preceq \boldsymbol{T}(y) \ \forall \ y, z \ \in \mathcal{Y}.$$

Remark 5.4.1 *Every gH-type I strongly pseudoconvex IVF of order m is also a gH-pseudoconvex IVF, but the converse is not true. For instance, consider the IVF $\boldsymbol{T} : (0, 2) \to I(\mathbb{R})$ which is given by*

$$\boldsymbol{T}(y) = \begin{cases} \ln y \odot \boldsymbol{C}, & 1 < y < 2 \\ \boldsymbol{0}, & 0 \leq y \leq 1. \end{cases}$$

where $0 \preceq \boldsymbol{C} \in I(\mathbb{R})$. Here, $\boldsymbol{T}$ is gH-pseudoconvex IVF but is not of gH-type I strongly pseudoconvex of any order: for $y = \frac{1}{2}, z = 1$ and $\boldsymbol{0} \in \partial^c \boldsymbol{T}(1)$, we have $(y-z) \odot \boldsymbol{K} = \boldsymbol{0}$, but $\boldsymbol{T}(1) \oplus \alpha \|y-1\|^m \preceq \boldsymbol{T}(y)$ is not true for any $\alpha > 0$.

Definition 5.7 (gH-Type II strongly pseudoconvex IVF). *A gH-locally Lipschitz $\boldsymbol{T}$: $\mathcal{Y} \to I(\mathbb{R})$ on $\mathcal{Y}$ is referred to as gH-type II strongly pseudoconvex IVF of order m on $\mathcal{Y}$ if there exists a positive constant α such that*

$$\boldsymbol{0} \preceq (y-z)^\top \odot \widehat{\boldsymbol{K}} \oplus \alpha \|y-z\|^m \text{ for some } \widehat{\boldsymbol{K}} \in \partial^c \boldsymbol{T}(z) \implies \boldsymbol{T}(z) \preceq \ \boldsymbol{T}(y) \ \forall \ y, z \in \mathcal{Y}.$$

Remark 5.4.2 *Every gH-type II strongly pseudoconvex IVF of order m is also gH-pseudoconvex IVF but converse does not hold. For instance, consider the IVF $\boldsymbol{T}$: $(-2, 2) \to I(\mathbb{R})$ as*

$$\boldsymbol{T}(y) = \begin{cases} y \odot [3, 4], & y > 0 \\ -y^2 \odot [1, 2], & y \leq 0. \end{cases}$$

Here, $\boldsymbol{T}$ is a gH-pseudoconvex IVF but is not of gH-type II strongly pseudoconvex of any order: for $y = -1, z = 0$, we have $\boldsymbol{0} \preceq (y-z) \odot \boldsymbol{K} \oplus \alpha \|y-z\|^m$ for $\boldsymbol{K} = \boldsymbol{0} \in \partial^c \boldsymbol{T}(0) = \{\boldsymbol{K} : \boldsymbol{0} \preceq \boldsymbol{K} \preceq [3, 4]\}$ although $\boldsymbol{T}(y) \prec \boldsymbol{T}(z)$.

Definition 5.8 (gH-quasiconvex IVF). *A gH-locally Lipschitz $\boldsymbol{T} : \mathcal{Y} \to I(\mathbb{R})$ on $\mathcal{Y}$ is* *said to be gH-quasiconvex if for all $y, z \in \mathcal{Y}$, the below result is true:*

$$\boldsymbol{T}(y) \preceq \boldsymbol{T}(z) \implies (y-z)^\top \odot \widehat{\boldsymbol{K}} \preceq \boldsymbol{0}, \text{ for all } \widehat{\boldsymbol{K}} \in \partial^c \boldsymbol{T}(z).$$

The following definitions are other nonsmooth extensions of the concepts of gH-quasiconvex type I and type II IVFs.

Definition 5.9 (gH-Type I strongly quasiconvex IVF). *A gH-locally Lipschitz $\boldsymbol{T}: \mathcal{Y} \to I(\mathbb{R})$ on $\mathcal{Y}$ is referred to as gH-type I strongly quasiconvex with order m on $\mathcal{Y}$ if there exists a positive constant α such that*

$$\boldsymbol{T}(y) \preceq \boldsymbol{T}(z) \implies (y-z)^\top \odot \widehat{\boldsymbol{K}} \oplus \alpha\|y-z\|^m \preceq \boldsymbol{0}, \ \forall\ y, z \in \mathcal{Y} \text{ and } \widehat{\boldsymbol{K}} \in \partial^c \boldsymbol{T}(z).$$

Remark 5.4.3 *Every gH-type I strongly quasiconvex IVF of order m is gH-quasiconvex IVF but not conversely. For instance, consider the IVF $\boldsymbol{T}: \mathbb{R} \to I(\mathbb{R})$, given by*

$$\boldsymbol{T}(y) = \begin{cases} y \odot [3,4], & y \geq 0 \\ 2y \odot [1,2], & 0 < y < 1 \\ [5,6], & y \geq 1, \end{cases}$$

is a gH-quasiconvex IVF but is not of gH-type I strongly quasiconvex IVF: for $y = 2, z = 1$, we have $\boldsymbol{T}(y) \preceq \boldsymbol{T}(z)$, but for $[2,4] \in \partial^c \boldsymbol{T}(1) = \{\boldsymbol{K} : \boldsymbol{0} \preceq \boldsymbol{K} \preceq [2,4]\}$, we have $\boldsymbol{0} \preceq (y-z) \odot \boldsymbol{K} \oplus \alpha\|y-z\|^m$ for any m.

Definition 5.10 (gH-type II strongly quasiconvex IVF). *A gH-locally Lipschitz IVF $\boldsymbol{T}: \mathcal{Y} \to I(\mathbb{R})$ on $\mathcal{Y}$ is referred to as gH-type II strongly quasiconvex of order m on $\mathcal{Y}$ if there exists a positive constant α such that*

$$\boldsymbol{T}(y) \preceq \boldsymbol{T}(z) \oplus \alpha\|y-z\|^m \implies (y-z)^\top \odot \widehat{\boldsymbol{K}} \preceq \boldsymbol{0}, \ \forall\ y, z \in \mathcal{Y} \text{ and } \widehat{\boldsymbol{K}} \in \partial^c \boldsymbol{T}(z).$$

Remark 5.4.4 *Every gH-type II strongly quasiconvex IVF of order m is also gH-quasiconvex IVF but not conversely. For instance, consider the IVF $\boldsymbol{T}: \mathbb{R} \to I(\mathbb{R})$, given by*

$$\boldsymbol{T}(y) = \begin{cases} y \odot [1,2], & y \leq 0 \\ \boldsymbol{0}, & 0 < y \leq 1 \\ (y-1) \odot [3,4], & y > 1, \end{cases}$$

is gH-quasiconvex IVF but is not of gH-type II strongly quasiconvex IVF of any order:

for $y = \frac{1}{2}, z = 0$ *and for every* $\alpha > 0$, *we have* $\boldsymbol{T}(y) \preceq \boldsymbol{T}(z) \oplus \alpha\|y - z\|^m$ *but for* $\boldsymbol{K} = 1 \in \partial^c \boldsymbol{T}(0) = \{\boldsymbol{K} : \boldsymbol{0} \preceq \boldsymbol{K} \preceq [3, 4]\}$, *we have* $\boldsymbol{0} \prec (y - z) \odot \boldsymbol{K}$.

5.5 Optimality conditions for a nonsmooth optimization problem

In this section, we will establish a sufficient optimality condition for the following Nonsmooth Interval Optimization Problem (NIOP) which includes a real-valued support function with its objective IVF. Let the IVFs $\mathbf{T} : \mathcal{Y} \to I(\mathbb{R})$ and $g_j : \mathcal{Y} \to \mathbb{R}, j = 1, 2, \ldots, q$ are gH-locally Lipschitz functions on $\mathcal{Y}$ and D is a compact convex subset of $\mathbb{R}^n$. Consider the NIOP:

$$\text{(NIOP)} \quad \begin{cases} \text{min} & \mathbf{T}(y) \oplus \boldsymbol{\sigma}_D(y) \\ \text{subject to} & g_j(y) \leq 0, j = 1, 2, \ldots, q. \end{cases}$$

Let $\mathcal{M} = \{y \in \mathcal{Y} : g_j(y) \leq 0, j = 1, 2, \ldots, q\}$ be the feasible set for (NIOP). We denote $J(y^*) = \{j : g_j(y^*) = 0, j = 1, 2, \ldots, q\}$ and refer it to the index set of active constraints at y^*.

To support our main optimality theorems, we rely on the following basic regularity condition.

Definition 5.11 (Basic regularity assumption for (NIOP)). *Let* y^* *be a feasible solution of the (NIOP). Then, (NIOP) satisfies the basic regularity assumption at* y^* *if there exist* $\lambda \geq 0$, $\mu_j \geq 0, j \in J(y^*)$ *and* $\mu_j = 0, j \notin J(y^*)$, *which satisfy*

$$\boldsymbol{0} \in \lambda \odot (\partial^c \boldsymbol{T}(y^*) \oplus w^*) \oplus \sum_{j=1}^{q} \mu_j^* \partial^c g_j(y^*), \tag{5.2}$$

$$\textit{where} \begin{cases} \lambda > 0 & \textit{if } 0 \in \partial^c \boldsymbol{T}(y^*) \oplus w^* \\ \lambda = 0 & \textit{if } 0 \notin \partial^c \boldsymbol{T}(y^*) \oplus w^* \end{cases}.$$

Definition 5.12 (Efficient solution of order m). *Let* $m \geq 1$ *be an integer. Then,* $y^* \in \mathcal{M}$ *is an efficient solution with order* m *for (NIOP) if there exists a positive constant* $\alpha > 0$ *such that for all* $y \in \mathcal{M}$,

$$\boldsymbol{T}(y) \oplus \boldsymbol{\sigma}_D(y) \not\preceq \boldsymbol{T}(y^*) \oplus \boldsymbol{\sigma}_D(y^*) \oplus \alpha\|y - y^*\|^m.$$

Definition 5.13 (Strict efficient solution of order m for (NIOP)). *Let* $m \geq 1$ *be an integer. Then,* $y^* \in \mathcal{M}$ *is a strict efficient solution with order* m *for the (NIOP) if* *there exists a positive constant* $\alpha > 0$ *such that for all* $y \in \mathcal{M}$,

$$\boldsymbol{T}(y) \oplus \boldsymbol{\sigma}_D(y) \not\preceq \boldsymbol{T}(y^*) \oplus \boldsymbol{\sigma}_D(y^*) \oplus \alpha\|y - y^*\|^m.$$

Definition 5.14 (Karush-Kuhn-Tucker point). *Let the IVF* $\boldsymbol{T}$ *and* $g_j, j = 1, 2, \ldots, q$ *are gH-locally Lipschitz at* $y^* \in \mathcal{M}$. *Assume that* y^* *is a strict efficient solution of order m. Assume, further, the basic regularity assumption (5.2) satisfies at* y^*, *then there exist* $\lambda \in \mathbb{R}_+, w^* \in D, \mu^* \in \mathbb{R}^q_+$, *which satisfy*

$$\boldsymbol{0} \in \lambda \odot (\partial^c \boldsymbol{T}(y^*) \oplus w^*) \oplus \sum_{j=1}^{q} \mu_j^* \partial^c g_j(y^*), \tag{5.3}$$

$$\boldsymbol{\sigma}_D(y^*) = y^* w^*, \tag{5.4}$$

$$\text{and } \mu_j^* g_j(y^*) = 0, j = 1, 2, \ldots, q. \tag{5.5}$$

Theorem 5.1 *Suppose that the conditions (5.3)–(5.5) of Definition 5.14 hold at* $y^* \in \mathcal{Y}$. *Let* $\boldsymbol{T}(y) \oplus y^\top w$, *be gH-type II strongly quasiconvex with order m on* $\mathcal{Y}$, *and* $\sum_{j=1}^{q} \mu_j^* g_j(y)$ *be type II strictly strongly pseudoconvex with order m on* $\mathcal{Y}$, *then* y^* *is a strict efficient solution of order m for (NIOP).*

Proof: By the way of contradiction that that y^* is not an strict efficient solution with order m for nonsmooth interval optimization problem (NIOP). Then for any $\alpha > 0$, we have that

$$\mathbf{T}(y) \oplus \boldsymbol{\sigma}_D(y) \preceq \mathbf{T}(y^*) \oplus \boldsymbol{\sigma}_D(y^*) \oplus \alpha\|y - y^*\|^m. \tag{5.6}$$

Since $\mathbf{T}(y) \oplus y^\top w$ is gH-type II strongly quasiconvex with order m at y^*, the inequality (5.6) implies that

$$(y - y^*)^\top \ \odot \ (\widehat{\mathbf{K}} \oplus w) \preceq \mathbf{0} \ \forall \ \widehat{\mathbf{K}} \in \partial^c \mathbf{T}(y^*) \text{ and } w \in D.$$

Then, from (5.3), we have

$$0 \leq (y - y^*)^\top \sum_{j=1}^{q} \mu_j^* h_j \ \forall \ h_j \in \partial^c g_j(y^*)$$

$$\Longrightarrow 0 \leq (y - y^*)^\top \sum_{j=1}^{q} \mu_j^* h_j + \alpha\|y - y^*\|^m, \ \forall \ \alpha \geq 0 \text{ and } h_j \in \partial^c g_j(y^*).$$

Since $\sum_{j=1}^{q} \mu_j^* g_j(y)$ is type II strictly strongly pseudoconvex with order m, from the above inequality, we get

$$0 = \sum_{j=1}^{q} \mu_j^* g_j(y^*) \leq \sum_{j=1}^{q} \mu_j^* g_j(y),$$

which is impossible. Hence, y^* is a strict efficient solution with order m for (NIOP). $\square$

Example 5.5.1 *Consider the following NIOP:*

$$\min\left\{\boldsymbol{T}(y)\oplus\boldsymbol{\sigma}_D(y): g_j(y)\leq 0,\ j=1,2\right\}, \tag{5.7}$$

where $D=[-1,1]$, and $\boldsymbol{T}:[-3,3]\to I(\mathbb{R})$ and $\boldsymbol{\sigma}_D:[-3,3]\to\mathbb{R}$ are given by

$$\boldsymbol{T}(y)=y^2\odot[1,2] \text{ and } \boldsymbol{\sigma}_D(y)=|y|, \text{ respectively.}$$

Since y^2 and $2y^2$ are locally Lipschitz functions on $[-3,3]$, $\boldsymbol{T}$ is a gH-locally Lipschitz function by (ii) of Lemma 2.4 of [26]. $\boldsymbol{\sigma}_D$ is also a well-known locally Lipschitz function on $[-1,1]$. Hence, we have

$$(\boldsymbol{T}\oplus\boldsymbol{\sigma}_D)(y)=\begin{cases}[2y^2+y,3y^2+y], & y\geq 0\\ [2y^2-y,3y^2-y], & y<0.\end{cases}$$

For all nonzero h in $[-3,3]$, we see that

$$\limsup_{\substack{y\to 0\\ \beta\to 0^+}}\frac{1}{\beta}\odot\{(\boldsymbol{T}\oplus\boldsymbol{\sigma}_D)(y+\beta h)\ominus_{gH}(\boldsymbol{T}\oplus\boldsymbol{\sigma}_D)(y)\}=\begin{cases}h\odot\boldsymbol{1}, & y\geq 0\\ -h\odot\boldsymbol{1}, & y<0.\end{cases}$$

At $y^*=0$, $\partial^c(\boldsymbol{T}(y^*)\oplus\boldsymbol{\sigma}_D(y^*))=\{\boldsymbol{K}:-\boldsymbol{1}\preceq\boldsymbol{K}\preceq\boldsymbol{1}\}$.
Here

$$g_1(y)=\begin{cases}y^2+2y-1, & y>0\\ y^2-\frac{1}{2}, & y\leq 0\end{cases} \quad \text{and} \quad g_2(y)=\begin{cases}e^{2y}-1, & y>0\\ y^3, & y\leq 0.\end{cases}$$

Clearly, g_1 and g_2 are locally Lipschitz on $[-3,3]$. Note that the set of feasible solutions of (5.7) is $\mathcal{M}=\{y\in[-3,3]:-0.7\leq y\leq 0\}$. Notice that for $\lambda=1$, and $\mu_1=0$, $\mu_2=1$ with $J(y^*)=\{2\}$,

$$\boldsymbol{0}\in\lambda\odot(\partial^c\boldsymbol{T}(y^*)\oplus w)\oplus\sum_{j=1}^{q}\mu_j^*\partial^c g_j(y^*).$$

Hence, that the basic regularity assumption (5.2) and the conditions (5.3)–(5.5) are satisfied at $y^*=0$. It is simple to verify that IVF $\boldsymbol{T}(y)\oplus\boldsymbol{\sigma}_D(y)$ are gH-type II strongly quasiconvex with order 2 for $\alpha=1$. For $\mu=(0,1)\in\mathbb{R}^2_+$, the function $\sum_{j=1}^q\mu_j^*g_j(y)$ is type II strictly strongly pseudoconvex with order 2 on $[-3,3]$ and $\alpha=1$. Thus, we get that $y^*=0$ is an efficient solution of (5.7) with order 2 and $\alpha=1$.

5.6 Duality theory for NIOP

In this section, we establish the link between the primal (NIOP) and its dual under the newly defined generalized strong convexity concepts with order m.

Let us separate the index set $\mathbf{P} = \{1, 2, \ldots, q\}$ into two disjoint sets $\mathbf{L} = \{1\}$ and $\mathbf{J} = \{2, 3, \ldots, q\}$. We define the dual of (NIOP) as the following problem:

$$\text{(NIOD)} \begin{cases} \max & \mathbf{T}(v) \oplus v^\top w \oplus \sum_{j \in \mathbf{J}} \mu_j g_j(v) \\ \text{subject to} & \mathbf{0} \in \partial^c \mathbf{T}(v) \oplus w \oplus \sum_{j=1}^{q} \partial^c \mu_j g_j(v) \\ & 0 \leq \mu_1 g_1, \mu = (\mu_1, \mu_2, \ldots, \mu_q), \quad w \in D, \quad \mu_j \geq 0, \ \forall j = 1, 2, \ldots, q, \end{cases}$$

where $\mathbf{T} : \mathcal{Y} \to I(\mathbb{R})$ be an IVF and $g_1, g_2, \ldots, g_q : \mathcal{Y} \to \mathbb{R}$ be real-valued functions. Let $\mathcal{M}_D$ be the feasible set to the problem (NIOD).

Theorem 5.2 (Weak duality). *Let y^* and (v^*, w, μ) be feasible solutions for the primal (NIOP) and the dual (NIOD), respectively. Suppose that $\boldsymbol{T}(v) \oplus v^\top w \oplus \sum_{j \in \boldsymbol{J}} \mu_j g_j(v)$ is a gH-type I strongly pseudoconvex function with order m at v^* and $\mu_1 g_1(v)$ is a type I strongly quasiconvex with order m at v^*, then the following holds:*

$$\boldsymbol{T}(y^*) \oplus {y^*}^\top w \not\prec \boldsymbol{T}(v^*) \oplus {v^*}^\top w \oplus \sum_{j \in \boldsymbol{J}} \mu_j g_j(v^*). \tag{5.8}$$

Proof: Since (v^*, w, μ) is a feasible solution to (NIOD), there exist $\widehat{\mathbf{K}} \in \partial^c \mathbf{T}(v^*)$, $w \in D$ and $h_j \in \partial^c \mu_j g_j(v^*)$, $j = 1, 2, \ldots, q$ such that

$$\widehat{\mathbf{K}} \oplus w \oplus \sum_{j=1}^{q} h_j = \mathbf{0} \tag{5.9}$$

and $\mu_1 g_1(v^*) \geq 0$. Further, as y^* is feasible for (NIOP), $g_1(y^*) \leq 0$ and also $\mu_1 \geq 0$, we have $\mu_1 g_1(y^*) \leq \mu_1 g_1(v^*)$. Since $\mu_1 g_1(v)$ is type I strongly quasiconvex with order m at v^*, there exists a constant $\alpha \geq 0$, such that

$$(y^* - v^*)^\top h_1 + \alpha \|y^* - v^*\|^m \leq 0, \ \forall \ h_1 \in \partial^c \mu_1 g_1(v^*). \tag{5.10}$$

From (5.9), we have

$$(y^* - v^*)^\top \odot \widehat{\mathbf{K}} \oplus (y^* - v^*)^\top w \oplus \sum_{j=1}^{q} (y^* - v^*)^\top h_j = \mathbf{0}. \tag{5.11}$$

Taking the difference between (5.11) and (5.10),

$$\mathbf{0} \in (y^* - v^*)^\top \odot \widehat{\mathbf{K}} \oplus (y^* - v^*)^\top w \oplus \sum_{j\in\mathbf{J}} (y^* - v^*)^\top h_j \ominus_{gH} \alpha\|y^* - v^*\|^m \tag{5.12}$$

for some $\widehat{\mathbf{K}} \in \partial^c \mathbf{T}(v^*)$, $w \in D$ and $h_j \in \partial^c \mu_j g_j(v^*), j \in \mathbf{J}$. Thus,

$$\alpha\|y^* - v^*\|^m \preceq (y^* - v^*)^\top \odot \widehat{\mathbf{K}} \oplus (y^* - v^*)^\top w \oplus \sum_{j\in\mathbf{J}} (y^* - v^*)^\top h_j$$
$$\Longrightarrow \mathbf{0} \preceq (y^* - v^*)^\top \odot \left(\widehat{\mathbf{K}} \oplus w \oplus \sum_{j\in\mathbf{J}} h_j\right).$$

If possible, let

$$\mathbf{T}(y^*) \oplus {y^*}^\top w \prec \mathbf{T}(v^*) \oplus {v^*}^\top w \oplus \sum_{j\in\mathbf{J}} \mu_j g_j(v^*).$$

Since $g_j(y^*) \leq 0$, $\mu_j \geq 0, j \in \mathbf{J}$, we obtain

$$\mathbf{T}(y^*) \oplus {y^*}^\top w \ \oplus \sum_{j\in\mathbf{J}} \mu_j g_j(y^*) \preceq \mathbf{T}(v^*) \oplus {v^*}^\top w \oplus \sum_{j\in\mathbf{J}} \mu_j g_j(v^*) \oplus \alpha\|y^* - v^*\|^m.$$

This contradicts the assumption that $\mathbf{T}(v) \oplus v^\top w \oplus \sum_{j\in\mathbf{J}} \mu_j g_j(v)$ is gH-type I strongly pseudoconvex with order m at v^*, and the result follows. □

The following example illustrates that the necessity of gH-type I strongly pseudoconvexity assumption imposed on $\mathbf{T}(v) \ \oplus v^\top w \oplus \sum_{j\in\mathbf{J}} \mu_j g_j(v)$ for Theorem 5.2.

Example 5.6.1 *Consider the following NIOP:*

$$\min \{ \boldsymbol{T}(y) \oplus \boldsymbol{\sigma}_D(y) : g_j(y) \leq 0, j = 1, 2 \}, \tag{5.13}$$

where $D = [-1, 1]$, and $\boldsymbol{T} : [-1, 1] \to I(\mathbb{R})$ and the support function $\boldsymbol{\sigma}_D : [-1, 1] \to \mathbb{R}$ are given by

$$\boldsymbol{T}(y) = \begin{cases} (y - y^3) \odot [1, 2], & y > 0 \\ y^2 \odot [1, 2] \oplus [2, 2], & y \leq 0 \end{cases} \quad \text{and} \quad \boldsymbol{\sigma}_D(y) = |y|, \ \text{respectively.}$$

Clearly, $\boldsymbol{T}$ is gH-locally Lipschitz on $[-1, 1]$ and $\boldsymbol{\sigma}_D$ is locally Lipschitz on $[-1, 1]$. We see that

$$(\boldsymbol{T} \oplus \boldsymbol{\sigma}_D)(y) = \begin{cases} [2y - y^3, 3y - 2y^3], & y > 0 \\ [y^2 - y + 2, 2y^2 - y + 2], & y \leq 0. \end{cases}$$

Assume, further that the the constraint functions are defined by

$$g_1(y) = \begin{cases} -e^{3y}, & y > 0 \\ -y^2 + 2y, & y \leq 0 \end{cases} \quad \textit{and} \quad g_2(y) = \begin{cases} -y^2, & y > 0 \\ y^3, & y \leq 0. \end{cases}$$

It is clear that g_1 and g_2 are locally Lipschitz on $[-1, 1]$.

The set of feasible solutions of (5.13) is $\mathcal{M} = \{y^* \in \mathbb{R} : -1 \leq y^* \leq 1\}$. Let us check that $v^* = 0$ is a feasible solution to the dual (NIOD) of (5.13). Since $g_j(v^*) = 0, j = 1, 2$, we can select $\mu_j = 1, j = 1, 2$. For all $h \neq 0$ in $[-1, 1]$, we see that

$$\limsup_{\substack{v \to v^* \\ \beta \to 0^+}} \frac{1}{\beta} \odot \{(\boldsymbol{T} \oplus \boldsymbol{\sigma}_D \oplus \sum_{j=1}^{2} g_j)(v + \beta h) \ominus_{gH} (\boldsymbol{T} \oplus \boldsymbol{\sigma}_D \oplus \sum_{j=1}^{2} g_j)(v)\} = \begin{cases} h \odot [-1, 0], & v \geq 0 \\ h \odot \boldsymbol{1}, & v < 0. \end{cases}$$

At $v^* = 0$,

$$\partial^c(\boldsymbol{T}(v^*) \oplus \boldsymbol{\sigma}_D(v^*) \oplus \sum_{j=1}^{2} g_j(v^*)) = \{\boldsymbol{K} : [-1, 0] \preceq \boldsymbol{K} \preceq \boldsymbol{1}\}.$$

Since $\boldsymbol{K} = \boldsymbol{0} \in \partial^c(\boldsymbol{T}(v^*) \oplus \boldsymbol{\sigma}_D(v^*) \oplus \sum_{j=1}^{2} g_j(v^*))$, $v^* = 0 \in \mathcal{M}_D$. Now, index set $\boldsymbol{T} = \{1, 2\}$ is divided into $\boldsymbol{L} = \{1\}$ and $\boldsymbol{J} = \{2\}$.

Notice that $(\boldsymbol{T} \oplus \boldsymbol{\sigma}_D \oplus g_2)$ is not gH-type I strongly pseudoconvex of any order at $v^* = 0$, as for $y^* = 0.5 \in \mathcal{M}, v^* = 0 \in \mathcal{M}_D$, we have $(y^* - v^*)^\top \odot \boldsymbol{K}' = \boldsymbol{0}$, where $\boldsymbol{K}' = \boldsymbol{0} \in \partial^c(\boldsymbol{T} \oplus \boldsymbol{\sigma}_D \oplus g_2)(v)$; but, $(\boldsymbol{T} \oplus \boldsymbol{\sigma}_D \oplus g_2)(v^*) \oplus \alpha \|y^* - v^*\|^m \preceq (\boldsymbol{T} \oplus \boldsymbol{\sigma}_D \oplus g_2)(y^*)$ is not true for any $\alpha \geq 0$ and for any m. Clearly, that the function g_1 is strongly quasiconvex type I of order 1 with $\alpha = 1$. In fact, for $v^* = 0 \in \mathcal{M}_D, y = 1 \in \mathcal{M}$, we observe that

$$\begin{aligned} \boldsymbol{T}(1) \; \oplus 1 &\preceq \boldsymbol{T}(0) \oplus 0 \oplus g_2(0) \\ \textit{i.e.,} \quad \boldsymbol{T}(1) \; \oplus 1 &\not\preceq \boldsymbol{T}(0) \oplus 0 \oplus g_2(0). \end{aligned}$$

may not be true as $\boldsymbol{T}(v) \; \oplus v^\top w \oplus \mu_2 g_2(v)$ is not gH-type I strongly pseudoconvex.

Theorem 5.3 (Weak duality). *Let y^* and (v^*, w, μ) be feasible solution to the problem (NIOP) and (NIOD), respectively. Suppose that $(\boldsymbol{T}(v) \oplus v^\top w \oplus \sum_{j \in \boldsymbol{J}} \mu_j g_j(v))$ is gH-type II strong pseudoconvex with order m at v^* and $\mu_1 g_1(v)$ is type II strong quasiconvex with order m at v^*, then the following holds:*

$$\boldsymbol{T}(y^*) \oplus {y^*}^\top w \not\preceq \boldsymbol{T}(v^*) \oplus {v^*}^\top w \oplus \sum_{j \in \boldsymbol{J}} \mu_j g_j(v^*).$$

Proof: *Since (v^*, w, μ) is a feasible solution to (NIOD),*

$$\widehat{\boldsymbol{K}} \oplus w \oplus \sum_{j=1}^{q} h_j = 0 \tag{5.14}$$

for some $\widehat{\boldsymbol{K}} \in \partial^c \boldsymbol{T}(v^*), w \in D$ and $h_j \in \partial^c \mu_j g_j(v^*), j = 1, 2, \ldots, q$. Further, as y^* is feasible for (NIOP), $g_1(y^*) \leq 0$ and also $\mu_1 \geq 0$. Therefore,

$$\mu_1 g_1(y^*) \leq \mu_1 g_1(v^*).$$

Then, for all $\alpha > 0$, we have

$$\mu_1 g_1(y^*) \leq \mu_1 g_1(v^*) \oplus \alpha \|y^* - v^*\|^m. \tag{5.15}$$

Since $\mu_1 g_1(v)$ is type II strongly quasiconvex with order m at v^*, we obtain from (5.15) that

$$(y^* - v^*)^\top h_1 \leq 0, \ \forall h_1 \in \partial^c \mu_1 g_1(v^*). \tag{5.16}$$

On using (5.14) and (5.16), we have that

$$\boldsymbol{0} \preceq (y^* - v^*)^\top \odot \widehat{\boldsymbol{K}} \oplus (y^* - v^*)^\top w \oplus \sum_{j \in \boldsymbol{J}} (y^* - v^*)^\top h_j \oplus \alpha \|y^* - v^*\|^m,$$

for all $\alpha \geq 0$ and for some $\widehat{\boldsymbol{K}} \in \partial^c \boldsymbol{T}(v^*), w \in D$, and $h_j \in \partial^c \mu_j g_j(v^*), j \in \boldsymbol{J}$. Since $\boldsymbol{T}(v) \oplus v^\top w \oplus \sum_{j \in \boldsymbol{J}} \mu_j g_j(v)$ is gH-type II strongly pseudoconvex with order m at v^*, we get

$$\boldsymbol{T}(v^*) \oplus {v^*}^\top w \oplus \sum_{j \in \boldsymbol{J}} \mu_j g_j(v^*) \preceq \boldsymbol{T}(y^*) \oplus {y^*}^\top w \oplus \sum_{j \in \boldsymbol{J}} \mu_j g_j(y^*). \tag{5.17}$$

As $g_j(y^*) \leq 0, \mu_j \geq 0, j \in \boldsymbol{J}$, we get

$$\boldsymbol{T}(v^*) \oplus {v^*}^\top w \oplus \sum_{j \in \boldsymbol{J}} \mu_j g_j(v^*) \preceq \boldsymbol{T}(y^*) \oplus {y^*}^\top w.$$

Therefore, $\boldsymbol{T}(y^*) \oplus {y^*}^\top w \not\prec \boldsymbol{T}(v^*) \oplus {v^*}^\top w \oplus \sum_{j \in \boldsymbol{J}} \mu_j g_j(v^*)$. □

The following example illustrates that the necessity of type II strongly pseudoconvexity assumption imposed on $\mathbf{T}(v) \oplus v^\top w \oplus \sum_{j \in \mathbf{J}} \mu_j g_j(v)$ for Theorem 5.3.

Example 5.6.2 *Consider the following NIOP:*

$$\min \left\{ \boldsymbol{T}(y) \oplus \boldsymbol{\sigma}_{\boldsymbol{D}}(y) : g_j(y) \leq 0, j = 1, 2 \right\}, \tag{5.18}$$

where $D = [-1, 1]$, *and* $\boldsymbol{T} : [-1, 1] \to I(\mathbb{R})$ *and support function* $\boldsymbol{\sigma}_D : [-1, 1] \to \mathbb{R}$ *by*

$$\boldsymbol{T}(y) = \begin{cases} (y - y^3) \odot [1, 2], & y > 0 \\ y^2 \odot [1, 2] \oplus [2, 2], & y \leq 0, \end{cases} \quad \text{and} \quad \boldsymbol{\sigma}_D(y) = |y|, \text{ respectively.}$$

Clearly, $\boldsymbol{T}$ *is gH-locally Lipschitz on* $[-1, 1]$ *and* $\boldsymbol{\sigma}_D$ *is locally Lipschitz on* $[-1, 1]$. *We see that* $(\boldsymbol{T} \oplus \boldsymbol{\sigma}_D)(y) = \begin{cases} [2y - y^3, 3y - 2y^3], & y > 0 \\ [y^2 - y + 2, 2y^2 - y + 2], & y \leq 0. \end{cases}$
Assume, further, real-valued constraints $g_j, j{=}1, 2$ *are defined by,*

$$g_1(y) = \begin{cases} -e^y, & y > 0 \\ y^3, & y \leq 0 \end{cases} \quad \text{and} \quad g_2(y) = \begin{cases} -y^2, & y > 0 \\ y^3, & y \leq 0. \end{cases}$$

It is easily verified that $g_j, j = 1, 2$ *are locally Lipschitz on* $[-1, 1]$.

The set of feasible solution of (5.18) is $\mathcal{M} = \{y^* \in [-1, 1] : -1 \leq y^* \leq 1\}$. *Next, we verify if* $v^* = 0$ *is feasible solution of dual (NIOD) of (5.18). Since* $g_j(v^*) = 0, j = 1, 2$, *we can select* $\mu_j = 1, j = 1, 2$. *For all nonzero* h *in* $[-1, 1]$, *we see that*

$$\limsup_{\substack{v \to v^* \\ \beta \to 0^+}} \frac{1}{\beta} \odot \{(\boldsymbol{T} \oplus \boldsymbol{\sigma}_D \oplus \sum_{j=1}^{2} g_j)(v + \beta h) \ominus_{gH} (\boldsymbol{T} \oplus \boldsymbol{\sigma}_D \oplus \sum_{j=1}^{2} g_j)(v)\} = \begin{cases} h \odot [1, 2], & v \geq 0 \\ h \odot -\boldsymbol{1}, & v < 0. \end{cases}$$

At $v^* = 0$,

$$\partial^c(\boldsymbol{T}(v^*) \oplus \boldsymbol{\sigma}_D(v^*) \oplus \sum_{j=1}^{2} g_j(v^*)) = \{\boldsymbol{K} : -\boldsymbol{1} \preceq \boldsymbol{K} \preceq [1, 2]\}.$$

Since $\boldsymbol{K} = \boldsymbol{0} \in \partial^c(\boldsymbol{T}(v) \oplus \boldsymbol{\sigma}_D(v) \oplus \sum_{j=1}^{2} g_j(v))$, $v^* = 0 \in \mathcal{M}_D$. *Here, index set* $\boldsymbol{T} = \{1, 2\}$ *is divided into* $\boldsymbol{L} = \{1\}$ *and* $\boldsymbol{J} = \{2\}$.

Notice that $(\boldsymbol{T} \oplus \boldsymbol{\sigma}_D \oplus g_2)$ *is not gH-type II strongly pseudoconvex of any order, as for* $y^* = 0.5 \in \mathcal{M}, v^* = 0 \in \mathcal{M}_D$, *we have* $\boldsymbol{0} \preceq (y^* - v^*)^\top \odot \boldsymbol{K}' \oplus \alpha \|y^* - v^*\|^m$, *where* $\boldsymbol{K}' = \boldsymbol{0} \in \partial^c(\boldsymbol{T} \oplus \boldsymbol{\sigma}_D \oplus g_2)(v^*)$; *but,* $(\boldsymbol{T} \oplus \boldsymbol{\sigma}_D \oplus g_2)(v^*) \preceq (\boldsymbol{T} \oplus \boldsymbol{\sigma}_D \oplus g_2)(y^*)$ *is not true for any* $\alpha \geq 0$ *and for any* m. *It is easy to see that the function* g_1 *is strongly quasiconvex type II of order 1 with* $\alpha = 1$. *In fact, for* $v^* = 0 \in \mathcal{M}_D, y^* = 1 \in \mathcal{M}$, *we* *observe that*

$$\boldsymbol{T}(1)\ \oplus 1 \preceq \boldsymbol{T}(0) \oplus 0 \oplus g_2(0)$$
$$i.e.,\ \boldsymbol{T}(1)\ \oplus 1 \not\prec \boldsymbol{T}(0) \oplus 0 \oplus g_j(0)$$

cannot be true as $\boldsymbol{T}(v) \oplus v^\top w \oplus \mu_2 g_2(v)$ *is not gH-type II strongly pseudoconvex.*

Definition 5.15 (Strictly efficient of order m for (NIOD)). *Let $m \geq 1$ be an integer. Then, $(y^*, w^*, \mu) \in \mathcal{M}_D$ is strictly efficient with order m for (NIOD) if there exists a positive constant α such that for all $(y, w, \mu) \in \mathcal{M}_D$,*

$$\boldsymbol{T}(y^*) \oplus \boldsymbol{\sigma}_D(y^*) \oplus \sum_{j\in \boldsymbol{J}} \mu_j g_j(y^*) \oplus \alpha \|y - y^*\|^m \not\prec \boldsymbol{T}(y) \oplus \boldsymbol{\sigma}_D(y) \oplus \sum_{j\in \boldsymbol{J}} \mu_j g_j(y).$$

Theorem 5.4 (Strong duality). *Let y^* be a strict efficient solution of order m for the nonsmooth interval optimization primal problem (NIOP) and the basic regularity assumption (5.2) is satisfied at y^*, then there exist $w^* \in D$ and $\mu \in \mathbb{R}^q_+$, such that (y^*, w^*, μ) is a feasible solution for (NIOD) and ${y^*}^\top w^* = \boldsymbol{\sigma}_D(y^*)$. Moreover, if the supposition of weak duality theorem (either Theorem 5.2 or Theorem 5.3) are satisfied, (y^*, w^*, μ) is a strictly efficient of order m for the problem (NIOD).*

Proof: Let y^* is a strict efficient solution of order m for the problem (NIOP) at which the basic regularity assumption (5.2) hold. From Definition 5.14, there exist $\lambda \in \mathbb{R}_+, w^* \in D, \mu = (\mu_1, \mu_2, \ldots, \mu_q) \in \mathbb{R}^q_+$, such that the conditions (5.3)-(5.5) are satisfied at y^*. From this, we obtain that (y^*, w^*, μ) is a feasible solution for (NIOD). Using Theorem 5.2 or Theorem 5.3, we have

$$\mathbf{T}(y^*) \oplus {y^*}^\top w^* \not\prec \mathbf{T}(v) \oplus v^\top w \oplus \sum_{j\in \mathbf{J}} \mu_j g_j(v). \tag{5.19}$$

Using by contradiction, (y^*, w^*, μ) is not a strictly efficient solution of order m to the problem (NIOD). It implies that for all $\alpha \in \mathbb{R}_+$, there exist $(v, w, \mu) \in \mathcal{M}_D$, such that

$$\mathbf{T}(y^*) \oplus {y^*}^\top w^* \oplus \sum_{j\in \mathbf{J}} \mu_j g_j(y^*) \oplus \alpha \|v - y^*\|^m \prec \mathbf{T}(v) \oplus v^\top w \oplus \sum_{j\in \mathbf{J}} \mu_j g_j(v)$$
$$\Longrightarrow \mathbf{T}(y^*) \oplus {y^*}^\top w^* \oplus \alpha \|v - y^*\|^m \prec \mathbf{T}(v) \oplus v^\top w \oplus \sum_{j\in \mathbf{J}} \mu_j g_j(v)$$
$$\Longrightarrow \mathbf{T}(y^*) \oplus {y^*}^\top w^* \prec \mathbf{T}(v) \oplus v^\top w \oplus \sum_{j\in \mathbf{J}} \mu_j g_j(v),$$

which results in the contradicts of (5.19). Hence, our assumption is incorrect. Then,

(y^*, w^*, μ) is strictly efficient with order m to the problem (NIOD).

Observe that Strong Duality 5.4 may not be satisfied unless we find $\mathbf{T}(y) \ \oplus y^\top w \oplus \sum_{j\in\mathbf{J}} \mu_j g_j(y)$ to be gH-type II strongly pseudoconvex. To illustrate this fact, we consider the following example:

Example 5.6.3 *Consider the following NIOP:*

$$\min\left\{\boldsymbol{T}(y) \oplus \boldsymbol{\sigma}_D(y) : g_j(y) \leq 0, j = 1, 2\right\} \tag{5.20}$$

where $D = [-1, 1]$, and $\boldsymbol{T} : [-3, 3] \to I(\mathbb{R})$ and support function $\boldsymbol{\sigma}_D : [-1, 1] \to \mathbb{R}$ by

$$\boldsymbol{T}(y) = \begin{cases} y^3 \odot [2, 3], & y > 0 \\ y^2 \odot [1, 2], & y \leq 0, \end{cases} \quad \textit{and} \quad \boldsymbol{\sigma}_D(y) = |y|, \ \textit{respectively.}$$

Clearly, $\boldsymbol{T}$ is gH-locally Lipschitz on $[-3, 3]$ and $\boldsymbol{\sigma}_D$ is locally Lipschitz on $[-1, 1]$. We see that

$$(\boldsymbol{T} \oplus \boldsymbol{\sigma}_D)(y) = \begin{cases} [2y^2 + y, 3y^2 + y], & y > 0 \\ [y^2 - y, 2y^2 - y], & y \leq 0. \end{cases}$$

Assume, further, that the real-valued constraints $g_j : [-3, 3] \to \mathbb{R}, j{=}1, 2$ are defined by

$$g_1(y) = \begin{cases} -e^y, & y > 0 \\ y^3, & y \leq 0 \end{cases} \quad \textit{and} \quad g_2(y) = \begin{cases} -y^2, & y > 0 \\ y^3, & y \leq 0. \end{cases}$$

Clearly, $g_j, j = 1, 2$ are locally Lipschitz on $[-3, 3]$. At $y^* = 0$

$$\boldsymbol{T}(y^*) \oplus \boldsymbol{\sigma}_D(y^*) \oplus \|y - y^*\|^2 \preceq \boldsymbol{T}(y) \oplus \boldsymbol{\sigma}_D(y),$$

Then, 0 is a strict efficient solution of (NIOP) for $\alpha = 1$ with order 2.

The set of feasible solution of (5.20) is $\mathcal{M} = \{y^* \in [-3, 3] : -3 \leq y^* \leq 3\}$. Let us check $y^\star = 0$ is a feasible point to the dual (NIOD) of (5.20). Since $g_j(y^*) = 0, j = 1, 2,$ we can select $\mu_j = 1, j = 1, 2$. For all $h \neq 0$ in $[-3, 3]$, we see that

$$\limsup_{\substack{y\to 0\\ \beta\to 0^+}} \frac{1}{\beta} \odot \{(\boldsymbol{T} \oplus \boldsymbol{\sigma}_D \oplus \sum_{i=1}^{2} g_j)(y + \beta h) \ominus_{gH} (\boldsymbol{T} \oplus \boldsymbol{\sigma}_D \oplus \sum_{i=1}^{2} g_j)(y)\} = \begin{cases} h \odot \boldsymbol{0}, & y \geq 0 \\ h \odot -\boldsymbol{1}, & y < 0. \end{cases}$$

At $y^* = 0$,

$$\partial^c(\boldsymbol{T}(y^*) \oplus \boldsymbol{\sigma}_D(y^*) \oplus \sum_{j=1}^{2} g_j(y^*)) = \left\{\boldsymbol{K} : -\boldsymbol{1} \preceq \boldsymbol{K} \preceq \boldsymbol{0}\right\}.$$

Since $\boldsymbol{K} = \boldsymbol{0} \in \partial^c(\boldsymbol{T}(y^*) \oplus \boldsymbol{\sigma}_D(y^*) \oplus \sum_{j=1}^{2} g_j(y^*))$, $y^* = 0 \in \mathcal{M}_D$. *Here, index set* $\boldsymbol{T} = \{1, 2\}$ *is divided into* $\boldsymbol{L} = \{1\}$ *and* $\boldsymbol{J} = \{2\}$.

Notice that $(\boldsymbol{T} \oplus \boldsymbol{\sigma}_D \oplus g_2)$ *is not type II strongly pseudoconvex of any order* m, *as for* $y = -2.5, y^* = 0$, *we have* $\boldsymbol{0} \preceq \alpha\|y\|^m \oplus (y - y^*)^\top \odot \boldsymbol{K}'$, *where* $\boldsymbol{K}' = \boldsymbol{0} \in \partial^c(\boldsymbol{T} \oplus \boldsymbol{\sigma}_D \oplus g_2)(y^*)$; *but,* $(\boldsymbol{T} \oplus \boldsymbol{\sigma}_D \oplus g_2)(0) \preceq (\boldsymbol{T} \oplus \boldsymbol{\sigma}_D \oplus g_2)(y)$ *is not valid for any* $\alpha \geq 0$ *and any* m. *It is evident that the function* g_1 *is strongly quasiconvex type II of order* 1 *with* $\alpha = 1$. *However,* $(y^*, w^*, \mu) = (0, 1, 1) \in \mathcal{M}_D$ *is not strictly efficient of order* 1 *for* (*NIOD*) *because for* $(v, w, \mu) = (1.5, 1, 1) \in \mathcal{M}_D$, *we observe that*

$$\boldsymbol{T}(0) \oplus \boldsymbol{\sigma}_D(0) \oplus g_2(0) \oplus 1.5\alpha \preceq \boldsymbol{T}(1.5) \oplus \boldsymbol{\sigma}_D(1.5) \oplus g_2(1.5) \text{ with } \alpha = 0.67 \ .$$

Hence, $(y^*, w^*, \mu) = (0, 1, 1)$ is not strictly efficient with order 1 for (NIOD) as $\mathbf{T}(y) \oplus y^\top w \oplus \sum_{j=1}^{2} \mu_j g_j(y)$ is not gH-type II strongly pseudoconvex. ☐

5.7 Conclusion

This chapter has explored two generalizations of strongly convexity of order m for gH-locally lipschitz IVFs with the support of gH-Clarke subdifferential. We extended the concepts of pseudoconvex (Definitions 5.6, 5.7) and quasiconvex (Definitions 5.9, 5.10) from real-valued functions to interval-valued functions. Under these extensions, sufficient optimality conditions which involves support function (Theorem 5.1) for interval-valued optimization problems have been established alongside with illustrative examples. Furthermore, we have applied the obtained sufficient condition to study weak duality (Theorems 5.2, 5.3) and strong duality for (Theorem 5.4) a nonsmooth IOPs, using the generalizations of strong convex IVF of order m and also the strict efficiency notion of order m.

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