Chapter 4

A Non-uniform Approximation for Fractional Sturm-Liouville Problem with Generalized Fractional Derivatives

In this chapter, a FSLP in terms of Caputo generalized fractional derivatives is studied. Section 4.1 consists of the introduction of the problem. Section 4.2.1 focuses on the well-posedness of the FSLP. A numerical algorithm based on non-uniform approximation is discussed in Section 4.3. In Section 4.4, the error approximation is presented. In Section 4.5, we provide illustrative examples to validate the proposed numerical approach for FSLP by considering the different scale and weight functions. Finally, we outline the main conclusions of the proposed work.

4.1 Introduction

In a recent work, Agrawal [17] proposed generalized fractional integrals and derivatives. The advantage of these integrals and derivatives is that many integrals and derivatives, such as R-L, Caputo, Riesz, Hadamard, etc., are the specific cases of the generalized fractional derivatives. Recently, Pandey et al. [73] discussed the Sturm's theorems for generalized derivative and generalized Sturm-Liouville problem. Motivated by this generalization, we study the FSLP in terms of Caputo generalized fractional derivatives and introduce a numerical algorithm based on non-uniform approximation to solve this problem.

The objective of this research is to convert the FSLP into an algebraic linear equation system using non-uniform node points. Furthermore, a numerical method is proposed to determine the eigenvalues and eigenfunctions.

Now, we define FSLP in terms of Caputo generalized fractional derivatives with mixed BCs.

4.2 Fractional Sturm-Liouville Problem with Generalized Fractional Derivatives

We consider a FSLP as follows

$$ {}_{b-}^{\;C}D^{\alpha}_{[\xi,w]}\left[p(x)\ {}_{a+}^{\;C}D^{\alpha}_{[\xi,w]}y(x)\right] + q(x)y(x) = \lambda r(x)y(x), \tag{4.1} $$

subjected to mixed boundary conditions,

$$ y(a) = 0, \qquad p(x)\ {}_{a+}^{\;C}D^{\alpha}_{[\xi,w]}y(x)|_{x=b} = 0, \tag{4.2} $$

where $\alpha \in (0,1)$ and $p(x), q(x),\ r(x) > 0$ are continuous functions on the interval $[a,b]$. Now, we will prove the well-posedness of the considered FSLP given by Eqs. (4.1)-(4.2).

4.2.1 Well-Posedness

Let $I = [a,b], \alpha \in (0,1)$ and

$$D(\cdot) = \xi'(x)^{C}_{b-}D^{\alpha}_{[\xi,w]}\left[p(x)\ {}^{C}_{a+}D^{\alpha}_{[\xi,w]}(\cdot)\right] + q(x)\xi'(x), \tag{4.3}$$

where $p(x)$ is nonvanishing and continuous in I. We define the bilinear form for some $v(x)$

$$\begin{aligned} L(u,v) &= (Du,v)_I, \\ L(u,v) &= \left(\xi'(x)^{C}_{b-}D^{\alpha}_{[\xi,w]}[p(x)\ {}^{C}_{a+}D^{\alpha}_{[\xi,w]}u(x)] + q(x)\xi'(x)u(x), v(x)\right)_I, \\ L(u,v) &= \int_a^b \xi'(x)v(x)^{C}_{b-}D^{\alpha}_{[\xi,w]}\left(p(x)\ {}^{C}_{a+}D^{\alpha}_{[\xi,w]}u(x)\right)dx + \int_a^b v(x)q(x)\xi'(x)u(x)dx. \end{aligned} \tag{4.4}$$

Now, using Eq. (1.24) we have

$$\begin{aligned} L(u,v) = \int_a^b \xi'(x)p(x)\ {}^{C}_{a+}D^{\alpha}_{[\xi,w]}u(x)^{RL}_{a+}D^{\alpha}_{[\xi,w]}v(x)dx - \left(p(x)\ {}^{C}_{a+}D^{\alpha}_{[\xi,w]}u(x)^{RL}_{a+}D^{\alpha-1}_{[\xi,w]}v(x)\right)\Big|_a^b \\ + \int_a^b v(x)q(x)\xi'(x))u(x)dx. \end{aligned}$$

Let $v(a) = 0; \quad {}^{C}_{a+}D^{\alpha}_{[\xi,w]}u(x)|_{x=b} = 0.$

This implies

$$L(u,v) = \left(\xi'(x)p(x)\ {}^{C}_{a+}D^{\alpha}_{[\xi,w]}u(x), {}^{RL}_{a+}D^{\alpha}_{[\xi,w]}v(x)\right)_I + (q(x)\xi'(x)u(x), v(x))_I. \tag{4.5}$$

Now, let

$$U_1 = \{u \in \mathbb{C}(I) |\ ||^{RL}_{a+}D^{\alpha}_{[\xi,w]}u(x)||_I < \infty, and\ u(a) = p(x)\ ^{RL}_{a+}D^{\alpha}_{[\xi,w]}u(x)^{RL}_{a+}D^{\alpha-1}_{[\xi,w]}u(x) = 0\}, \tag{4.6}$$

and

$$V_1 = \{v \in \mathbb{C}(I) |\ ||^{RL}_{a+}D^{\alpha}_{[\xi,w]}v(x)|| < \infty, and\ v(a) = 0\}. \tag{4.7}$$

As $U_1 \subset V_1$ therefore, we use Galerkin method and choose $U_1 = U_1 \cap V_1$ for the trial and test function u and v respectively. Hence, taking $Lu = \lambda\xi'(x)u$. We observe that U_1 is a Hilbert space and bilinear form $L(u, v)$ is linear and continuous. Thus, Further following the conclusion by Lemma 2.4 [66], the bilinear form $L(u, v)$ is coercive and by Lax-Milgram lemma, the considered FSLP is well-posed.

4.3 Numerical Algorithm

We are interested to determine the numerical solution of the FSLP given by Eqs. (4.1)-(4.2) using the matrix technique [67] and the finite difference scheme.
To find numerical solution, we have divided the given interval $[a, b]$, into subintervals with $a = x_0 < x_1 < x_2 < ... < x_N = b$, where n is an integer and $x_0, x_1, x_2, ...x_N$ are non-uniform node points. Let the step is denoted by $\tau_j = x_j - x_{j-1}, 1 \leq j \leq N$. The non-uniform node points are defined as

$$\tau_j = (N + 1 - j)\mu, \quad 1 \leq j \leq N, \tag{4.8}$$

where $\mu = \frac{2(b-a)}{N(N+1)}$. Now, we aim to approximate left/ right Caputo generalized fractional derivatives for non-uniform node points.

The value of left CGFD ${}_{a+}^{C}D_{[\xi,w]}^{\alpha}y$ at nodes $x_j, j = 0, 1, 2, ...N$ is approximated as follows:

$$
\begin{aligned}
{}_{a+}^{C}D_{[\xi,w]}^{\alpha}y(x)|_{x=x_0} &= 0,\\
{}_{a+}^{C}D_{[\xi,w]}^{\alpha}y(x)|_{x=x_j} &= \frac{[w(x_j)]^{-1}}{\Gamma(1-\alpha)}\int_a^{x_j}\frac{(w(t)y(t))'}{(\xi(x_j)-\xi(t))^{\alpha}}dt,
\end{aligned}
$$

$$
\begin{aligned}
{}_{a+}^{C}D_{[\xi,w]}^{\alpha}y(x)|_{x=x_j} &= \frac{[w(x_j)]^{-1}}{\Gamma(1-\alpha)}\sum_{k=0}^{j-1}\int_{x_k}^{x_{k+1}}\frac{1}{(\xi(x_j)-\xi(t))^{\alpha}}\frac{d}{dt}[w(t)y(t)]dt,\\
{}_{a+}^{C}D_{[\xi,w]}^{\alpha}y(x)|_{x=x_j} &\approx \frac{[w(x_j)]^{-1}}{\Gamma(1-\alpha)}\sum_{k=0}^{j-1}\frac{w_{k+1}y_{k+1}-w_k y_k}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}\frac{1}{(\xi(x_j)-\xi(t))^{\alpha}}dt. \quad (4.9)
\end{aligned}
$$

Let

$$
\begin{aligned}
V &= (\xi(x_j)-\xi(t)),\\
dV &= -\xi'(t)dt,\\
dV &\approx -\frac{(\xi(x_{k+1})-\xi(x_k))}{x_{k+1}-x_k}dt. \quad (4.10)
\end{aligned}
$$

Thus Eq. (4.9) implies

$$
\begin{aligned}
{}_{a+}^{C}D_{[\xi,w]}^{\alpha}y(x)|_{x=x_j} &\approx \frac{[w(x_j)]^{-1}}{\Gamma(2-\alpha)}\sum_{k=0}^{j-1}\frac{w_{k+1}y_{k+1}-w_k y_k}{(\xi(x_{k+1})-\xi(x_k))}\\
&\qquad\left[(\xi(x_j)-\xi(x_k))^{1-\alpha}-(\xi(x_j)-\xi(x_{k+1}))^{1-\alpha}\right]\\
&= \frac{[w(x_j)]^{-1}}{\Gamma(2-\alpha)}\sum_{k=0}^{j} l_{j,k}w_k y_k, \quad (4.11)
\end{aligned}
$$

where

$$l_{j,k} = \begin{cases} 0 & \text{for } j = 0, \\ \frac{[(\xi(x_j)-\xi(x_1))^{1-\alpha}-(\xi(x_j)-\xi(x_0))^{1-\alpha}]}{(\xi(x_1)-\xi(x_0))} & \text{for } j > 0,\ k = 0, \\ \frac{[(\xi(x_j)-\xi(x_{k-1}))^{1-\alpha}-(\xi(x_j)-\xi(x_k))^{1-\alpha}]}{(\xi(x_k)-\xi(x_{k-1}))} - & \\ \qquad \frac{[(\xi(x_j)-\xi(x_k))^{1-\alpha}-(\xi(x_j)-\xi(x_{k+1}))^{1-\alpha}]}{(\xi(x_{k+1})-\xi(x_k)} & \text{for } j > 0, k = 1, 2, ...j-1, \\ (\xi(x_k) - \xi(x_{k-1}))^{-\alpha} & \text{for } i > 0,\ j = k. \end{cases} \tag{4.12}$$

Thus the value of ${}^{C}_{a+}D^{\alpha}_{[\xi,w]}y(x)$ at nodes x_j can be written as a system of linear algebraic equations in the following matrix form :

$$F^{\alpha}_{L} = M^{\alpha}_{L} F, \tag{4.13}$$

where,

$$M^{\alpha}_{L} = \begin{bmatrix} l_{0,0} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ l_{1,0} & l_{1,1} & 0 & 0 & \cdots & 0 & 0 & 0 \\ l_{2,0} & l_{2,1} & l_{2,2} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ l_{i,0} & l_{i,1} & l_{i,2} & l_{i,3} & \cdots & l_{i,j} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ l_{N,0} & l_{N,1} & l_{N,2} & l_{N,3} & \cdots & l_{N,j} & \cdots & l_{N,N} \end{bmatrix},$$

$$F_L^\alpha = \begin{bmatrix} {}^C_{a+}D^\alpha_{[\xi,w]}y(x)|_{x=x_0} \\ {}^C_{a+}D^\alpha_{[\xi,w]}y(x)|_{x=x_1} \\ {}^C_{a+}D^\alpha_{[\xi,w]}y(x)|_{x=x_2} \\ \vdots \\ {}^C_{a+}D^\alpha_{[\xi,w]}y(x)|_{x=x_i} \\ \vdots \\ {}^C_{a+}D^\alpha_{[\xi,w]}y(x)|_{x=x_N} \end{bmatrix}, F = \begin{bmatrix} w_0y_0 \\ w_1y_1 \\ w_2y_2 \\ \vdots \\ w_iy_i \\ \vdots \\ w_Ny_N \end{bmatrix},$$

where M_L^α is a lower triangular matrix of order $(N+1)\times(N+1)$. In a similar way, the value of right CGFD ${}^C_{b-}D^\alpha_{[\xi,w]}y$ at nodes $x_j, j = 0, 1, 2, ...N$, is approximated as follows:

$$\begin{aligned} {}^C_{b-}D^\alpha_{[\xi,w]}y(x)|_{x=x_j} &= -\frac{[w(x_j)]}{\Gamma(1-\alpha)}\int_{x_k}^{x_N}\frac{((w(t))^{-1}y(t))'}{(\xi(t)-\xi(x_j))^\alpha}dt \\ &= -\frac{[w(x_j)]}{\Gamma(1-\alpha)}\sum_{k=j}^{N-1}\frac{w_{k+1}^{-1}y_{k+1}-w_k^{-1}y_k}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}\frac{1}{(\xi(t)-\xi(x_j))^\alpha}dt. \end{aligned} \tag{4.14}$$

Using Eq.(4.10), we have,

$$\begin{aligned} {}^C_{b-}D^\alpha_{[\xi,w]}y(x)|_{x=x_j} \approx &\frac{[w(x_j)]}{\Gamma(2-\alpha)}\sum_{k=j}^{N-1}\frac{w_{k+1}^{-1}y_{k+1}-w_k^{-1}y_k}{(\xi(x_{k+1})-\xi(x_k))} \\ &\Big[(\xi(x_{k+1})-\xi(x_j))^{1-\alpha}-(\xi(x_k)-\xi(x_j))^{1-\alpha}\Big] \\ = &\frac{[w(x_j)]}{\Gamma(2-\alpha)}\sum_{k=j}^{N}u_{j,k}w_k^{-1}y_k, \end{aligned} \tag{4.15}$$

where

$$u_{j,k} = \begin{cases} 0 & \text{for } j = N, \\ \frac{(\xi(x_{N-1})-\xi(x_j))^{1-\alpha}-(\xi(x_N)-\xi(x_j))^{1-\alpha}}{(\xi(x_N)-\xi(x_{N-1}))} & \text{for } j > 0,\ k = N, \\ \frac{(\xi(x_{k+1})-\xi(x_j))^{1-\alpha}-(\xi(x_k)-\xi(x_j))^{1-\alpha}}{(\xi(x_{k+1})-\xi(x_k))} + \frac{(\xi(x_{k-1})-\xi(x_j)^{1-\alpha}}{(\xi(x_k)-\xi(x_{k-1}))} & \text{for } j > 0,\ k = j+1, \\ & \qquad j+2, ...N-1, \\ (\xi(x_{k+1})-\xi(x_k))^{-\alpha} & \text{for } i > 0,\ j = k. \end{cases} \tag{4.16}$$

Thus, the value of ${}_{b-}^{\ C}D^{\alpha}_{[\xi,w]}y(x)$ at nodes x_j can be written as a system of linear algebraic equations in the following matrix form :

$$F^{\alpha}_U = M^{\alpha}_U F. \tag{4.17}$$

where M^{α}_U is a upper triangular matrix of order $(N+1) \times (N+1)$ and

$$M^{\alpha}_U = \begin{bmatrix} u_{0,0} & u_{0,1} & u_{0,2} & u_{0,3} & \cdots & u_{0,j} & \cdots & u_{0,N} \\ 0 & u_{1,1} & u_{1,2} & u_{1,3} & \cdots & d_{1,j} & \cdots & d_{1,N} \\ 0 & 0 & u_{2,2} & u_{2,3} & \cdots & u_{2,j} & \cdots & u_{2,N} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & u_{i,j} & \cdots & u_{i,N} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & u_{N,N} \end{bmatrix},$$

$$F_U^{\alpha} = \begin{bmatrix} {}_{b-}^{C}D_{[\xi,w]}^{\alpha}y(x)|_{x=x_0} \\ {}_{b-}^{C}D_{[\xi,w]}^{\alpha}y(x)|_{x=x_1} \\ {}_{b-}^{C}D_{[\xi,w]}^{\alpha}y(x)|_{x=x_2} \\ \vdots \\ {}_{b-}^{C}D_{[\xi,w]}^{\alpha}y(x)|_{x=x_i} \\ \vdots \\ {}_{b-}^{C}D_{[\xi,w]}^{\alpha}y(x)|_{x=x_N} \end{bmatrix}, F = \begin{bmatrix} w_0^{-1}y_0 \\ w_1^{-1}y_1 \\ w_2^{-1}y_2 \\ \vdots \\ w_i^{-1}y_i \\ \vdots \\ w_N^{-1}y_N, \end{bmatrix}.$$

We introduce three diagonal matrices P, S and R as $P = diag(p(x_0), p(x_1), p(x_2), ...p(x_N))$, $Q = diag(q(x_0), q(x_1), q(x_2), ...q(x_N)), R = diag(r_0, r_1, r_2, ...r_N)$,

where,

$$r_k = \frac{\int_{x_k}^{x_{k+1}} r(x)dx}{x_{k+1} - x_k}, \ k < N. \tag{4.18}$$

Now, using Eq. (4.13) and Eq. (4.17), Eq. (4.1) takes the matrix form

$$M_U^{\alpha} P M_L^{\alpha} F + QF = \lambda RF. \tag{4.19}$$

Eq. (4.19) can be rewritten as

$$R^{-1}AF = \lambda F, \tag{4.20}$$

where $A = M_U^{\alpha} P M_L^{\alpha} + Q$ and $F = [y_0, y_1, y_2....., y_N]^T$.

4.3.1 Orthogonality of Eigenfunctions

After the implementation of boundary conditions, the approximation of left and right CGFDs takes the following form

$$ {}_{a+}^{C}D_{[\xi,w]}^{\alpha} y(x)|_{x=x_j} \approx \frac{[w(x_j)]^{-1}}{\Gamma(2-\alpha)} \sum_{k=1}^{j} l_{j,k} w_k y_k. \tag{4.21}$$

$$ {}_{b-}^{C}D_{[\xi,w]}^{\alpha} y(x)|_{x=x_j} \approx \frac{[w(x_j)]}{\Gamma(2-\alpha)} \sum_{k=j}^{N-1} u_{j,k} w_k^{-1} y_k. \tag{4.22}$$

Now, the elements of matrix A are expressed as,

$$A_{j,k} = \sum_{i=max(j,k)}^{N-1} l_{j,i} p_i u_{i,k} + q_j \delta_{j,k}, \quad j,k = 1,2,3,...,N-1. \tag{4.23}$$

Since $l_{j,i} = u_{i,j}, \quad j,i = 1,2,3,...,N-1$. Thus, A is a symmetric matrix.

Now, we show that eigenfunctions corresponding to distinct eigenvalues are orthogonal in the respective $N-$dimensional vector space with the inner product defined as follows:

$$\langle Y, X\rangle_R = \sum_{i=0}^{N-1} Y_i r_i X_i = \langle X, Y\rangle_R. \tag{4.24}$$

Let λ_j and λ_k be eigenvalues of $R^{-1}A$ and Y_j , Y_k are corresponding eigenvectors, then

$$\langle Y_j, R^{-1}AY_k\rangle_R = \lambda_k \langle Y_j, Y_k\rangle_R, \tag{4.25}$$

$$\langle Y_k, R^{-1}AY_j\rangle_R = \lambda_j \langle Y_k, Y_j\rangle_R. \tag{4.26}$$

Subtracting Eq. (4.26) from Eq. (4.25), we get,

$$\langle Y_j, R^{-1}AY_k\rangle_R - \langle Y_k, R^{-1}AY_j\rangle_R = (\lambda_k - \lambda_j)\langle Y_j, Y_k\rangle_R. \tag{4.27}$$

Using the symmetricity of matrix A and the inner product definition given in Eq. (4.24), we get,

$$\langle Y_j, R^{-1}AY_k\rangle_R - \langle Y_k, R^{-1}AY_j\rangle_R = 0.$$

This proves that eigenfunctions derived from the proposed scheme are orthogonal, i.e.

$$\langle Y_k, Y_j\rangle_R = 0, \quad k \neq j. \tag{4.28}$$

Now, we find bound for the solution using variational form.

4.4 Bound for the Solution

This section contains the bound for the solution $y(x)$. To find this, we have used the variational form of the problem as follows:
Find $\lambda \in \mathbb{R}$ and $u \in U$ such that

$$B(u, v) = (Lu, v)_\Omega = \lambda(u, v)_{\Omega, w(x)} \ \ \forall u \in U. \tag{4.29}$$

For a given non-uniform partition, let u_τ be the solution of the approximated problem:

$$B(u, v) = \lambda_\tau(u_\tau, v)_{\Omega, w(x)} \ \ \forall u \in U_\tau, \tag{4.30}$$

where U_τ consists of linear functions.

Now,

$$T^k(y,\tau,\alpha) = |{}^C_{a+}D^\alpha_{[\xi,w]}y(x) - {}^C_{a+}D^\alpha_{[\xi,w]}y_\tau(x)|, \tag{4.31}$$

where $T^k(y,\tau,\alpha)$ is the truncation error in approximation of ${}^C_{a+}D^\alpha_{[\xi,w]}$ and $y_\tau(x)$ is approximation of $y(x)$ using linear interpolating function between points (x_j, y_j) and (x_{j+1}, y_{j+1}), defined as,

$$y_\tau(x) = \frac{(x - x_{j+1})}{(x_j - x_{j+1})}y_j + \frac{(x - x_j)}{(x_{j+1} - x_j)}y_{j+1}, \tag{4.32}$$

so we have,

$$y'_\tau(x) = \frac{y_{j+1} - y_j}{(x_{j+1} - x_j)}, \tag{4.33}$$

and

$$y(x) - y_\tau(x) = \frac{y''(\zeta_j)}{2}(x - x_j)(x - x_{j+1}), \tag{4.34}$$

where $\zeta_j \in (x_j, x_{j+1})$.

Since,

$$|{}^C_{a+}D^\alpha_{[\xi,w]}y(x) - {}^C_{a+}D^\alpha_{[\xi,w]}y_\tau(x)| = \left|\frac{w(x)^{-1}}{\Gamma(1-\alpha)}\int_a^x \frac{1}{(\xi(x) - \xi(s))^\alpha}(w(s)y(s) - w(s)y_\tau(s))'ds\right|,$$

Let $w(s)y(s) = g(s)$ and $w(s)y_\tau(s) = P_g(s)$

$$
\begin{aligned}
&= \left|\frac{w(x)^{-1}}{\Gamma(1-\alpha)}\int_a^x \frac{(g(s)-P_g(s))}{(\xi(x)-\xi(s))^\alpha}ds\right|,\\
&= \frac{w(x_k)^{-1}}{\Gamma(1-\alpha)}\left|\sum_{j=0}^{k-1}\int_{x_j}^{x_{j+1}} \frac{1}{(\xi(x_k)-\xi(s))^\alpha}\frac{g''(\xi_j)}{2}(s-x_j)(s-x_{j+1})ds\right|,\\
&= \frac{w(x_k)^{-1}}{\Gamma(1-\alpha)}\left|\sum_{j=0}^{k-1}\frac{g''(\xi_j)}{2}\int_{x_j}^{x_{j+1}} \frac{d}{ds}(\xi(x_k)-\xi(s))^{-\alpha}ds\right|,\\
&\leq \frac{w(x_k)^{-1}}{\Gamma(1-\alpha)}\left|\sum_{j=0}^{k-1}\frac{\tau_j^2 g''(\xi_j)}{4}[(\xi(x_k)-\xi(x_{j+1}))^{-\alpha}-(\xi(x_k)-\xi(x_j))^{-\alpha}]\right|\\
&\leq \frac{k-1}{4\Gamma(1-\alpha)}\max_{0\leq x_k\leq x_N}|[w(x_k)^{-1}]|\tau^2\max_{0\leq x_j\leq x_N}|g''(x_j)||(\xi(x_k)-\xi(x_0))|^{-\alpha},\\
&\leq \frac{cL^{-\alpha}\tau^{2-\alpha}(k-1)}{4\Gamma(1-\alpha)},
\end{aligned}
\tag{4.35}
$$

where $\max_{0\leq x_k\leq x_N}|[w(x_k)^{-1}]|\max_{0\leq x_j\leq x_N}|g''(x_j)| = c$ and L is Lipschitz constant. Therefore,

$$
T^k(y,\tau,\alpha) = |{}^C_{a+}D^\alpha_{[\xi,w]}(y(x)-y_\tau(x))| \leq \frac{cL^{-\alpha}\tau^{2-\alpha}(k-1)}{4\Gamma(1-\alpha)}, \tag{4.36}
$$

and

$$
\|{}^C_{a+}D^\alpha_{[\xi,w]}y(x) - {}^C_{a+}D^\alpha_{[\xi,w]}y_\tau(x)\|_{L^2} = \left(\sum_{i=1}^N\int_{x_{i-1}}^{x_i}|{}^C_{a+}D^\alpha_{[\xi,w]}(y(x)-y_\tau(x))|^2dx\right)^{1/2}, \tag{4.37}
$$

using Eq.(4.36), we have

$$
\begin{aligned}
\|{}^C_{a+}D^\alpha_{[\xi,w]}y(x) - {}^C_{a+}D^\alpha_{[\xi,w]}y_\tau(x)\|_{L^2} &\leq \left(\sum_{i=1}^N\int_{x_{i-1}}^{x_i}\left(\frac{cL^{-\alpha}\tau^{2-\alpha}(k-1)}{4\Gamma(1-\alpha)}\right)^2dx\right)^{1/2}\\
&\leq \frac{cL^{-\alpha}\tau^{2-\alpha}(k-1)}{4\Gamma(1-\alpha)}.
\end{aligned}
\tag{4.38}
$$

Now, using Eq.(1.41), the bound for the solution $y(x)$ is given as

$$\|y(x) - y_\tau(x)\|_{L^2} \leq C\|{}_{a+}^{C}D^{\alpha}_{[\xi,w]}y(x) - {}_{a+}^{C}D^{\alpha}_{[\xi,w]}y_\tau(x)\|_{L^2} \tag{4.39}$$

Thus, Eq. (4.38) implies

$$\|y(x) - y_\tau(x)\|_{L^2} \leq \frac{C_3 L^{-\alpha}\tau^{2-\alpha}(k-1)}{4\Gamma(1-\alpha)}. \tag{4.40}$$

4.5 Evaluation of Eigenvalues and Eigenfunctions

Here, we will discuss three examples and compute their eigenvalues and eigenfunctions using the proposed numerical method.

Example 1. Consider the following FSLP:

$${}_{b-}^{C}D^{\alpha}_{[\xi,w]}\,{}_{a+}^{C}D^{\alpha}_{[\xi,w]}y(x) = \lambda y(x), \qquad x \in [0,1], \tag{4.41}$$

subject to $y(0) = 0$, ${}_{a+}^{C}D^{\alpha}_{[\xi,w]}y(x)\big|_{x=1} = 0$. This is well known fractional oscillator equation. In this example, the proposed numerical algorithm is performed for FSLP given by Eq. (4.41). We have calculated eigenvalues and eigenfunctions for varying $w(x)$ and $\xi(x)$. Table 4.1 and Table 4.2 represent the first 10 eigenvalues for $w(x) = \{1, e^x\}, \xi(x) = \{x, x^2\}$ and $\alpha \in \{0.3, 0.5, 0.6, 0.7, 0.8, 0.9, 0.99\}$. In Table 4.1, It is observed that as α approaches to 1, eigenvalues approach to classical oscillator equation. Figure 4.1 represents the first four eigenfunctions of example 1 for $\alpha \in \{0.3, 0.6, 0.8, 0.99\}$ with $w(x) = 1$ and $\xi(x) = x$.

TABLE 4.1: First 10 eigenvalues of example 1 for different values of $\xi(x)$ and fixed $w(x)$

	$\xi(x) = x$ and $w(x) = 1$				$\xi(x) = x^2$ and $w(x) = 1$			
λ	α =0.6	α =0.8	α =0.9	α =0.99	α =0.6	α =0.8	α =0.9	α =0.99
λ_1	1.6460	2.0258	2.2636	2.5068	1.6247	2.0067	2.2513	2.5055
λ_2	6.7132	12.5075	16.9328	22.0680	6.6094	12.3290	16.7278	21.8636
λ_3	12.4261	28.4184	42.5213	60.4994	12.2221	27.9764	41.9410	59.8276
λ_4	18.9003	49.2038	78.2665	117.3825	18.5772	48.4127	77.1626	116.0299
λ_5	25.8282	74.0583	123.2560	192.1533	25.3765	72.8628	121.5332	189.9970
λ_6	33.2779	102.7917	177.0521	284.1860	32.6897	101.1637	174.6789	281.2303
λ_7	41.0915	134.9166	238.9571	392.6828	40.3679	132.8656	235.9836	389.0968
λ_8	49.3019	170.2482	308.4861	516.7408	48.4484	167.8257	305.0627	512.8989
λ_9	57.7967	208.3768	384.9354	655.2846	56.8277	205.6834	381.3287	651.8101
λ_{10}	66.5853	249.0785	467.7127	807.1104	65.5231	246.2742	464.3327	804.9266

TABLE 4.2: First 10 eigenvalues of example 1 for different values of $\xi(x)$ and fixed $w(x) = e^x$

	$\xi(x) = x, w(x) = e^x$				$\xi(x) = x^2, w(x) = e^x$			
λ	α =0.3	α =0.5	α =0.7	α =0.9	α =0.3	α =0.5	α =0.7	α =0.9
λ_1	1.6023	2.2119	3.0792	4.3040	1.5147	2.0226	2.7398	3.7553
λ_2	2.6509	5.0878	9.7405	18.4018	2.6418	5.0653	9.7043	18.3863
λ_3	3.5531	8.2783	19.1513	43.3364	3.5354	8.2307	19.0860	43.4399
λ_4	4.3615	11.6242	30.6032	78.1479	4.3332	11.5341	30.4330	78.1750
λ_5	5.0992	15.0516	43.6813	122.0572	5.0585	14.9063	43.3547	121.8921
λ_6	5.7959	18.5764	58.2317	174.5188	5.7440	18.3760	57.7366	174.1481
λ_7	6.4575	22.1679	74.0469	234.9206	6.3944	21.9112	73.3810	234.4101
λ_8	7.0962	25.8341	91.0232	302.7179	7.0231	25.5258	90.2104	302.2458
λ_9	7.7129	29.5509	108.9978	377.2700	7.6306	29.1961	108.0773	377.1287
λ_{10}	8.3138	33.3189	127.8669	457.9469	8.2237	32.9276	126.9060	458.5763

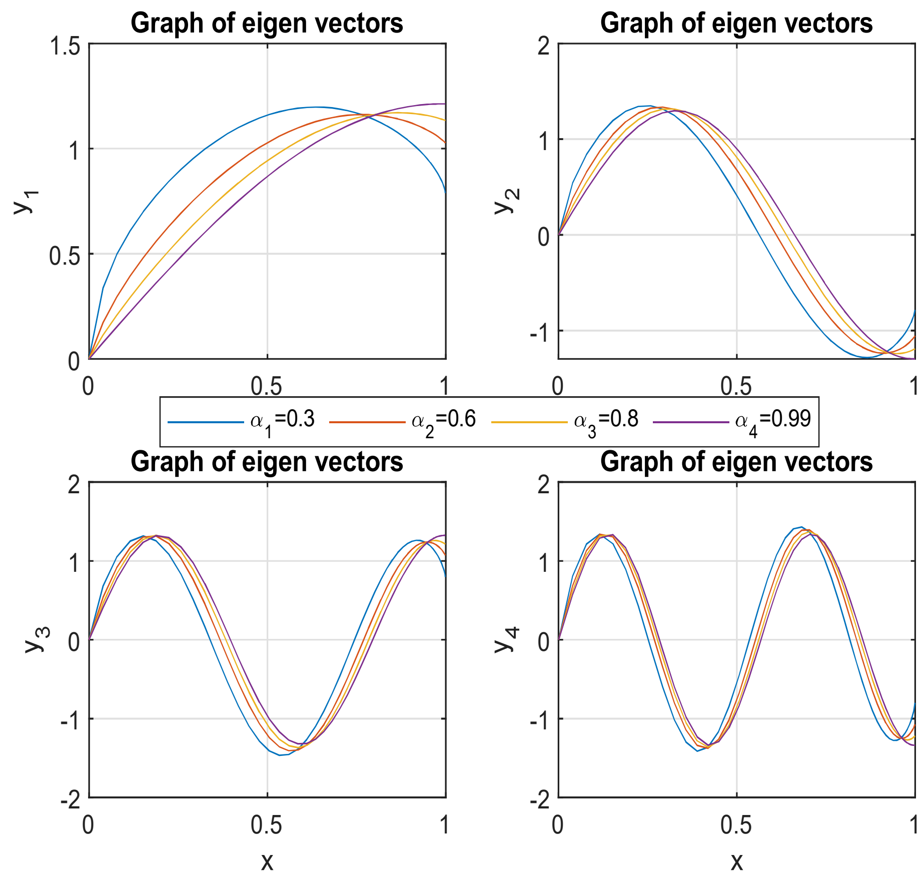

FIGURE 4.1: First four eigenfunctions $y_i,\ i = 1, 2, 3, 4$ of example 1 for different values of $\alpha \in \{0.3, 0.6, 0.8, 0.99\}$ with $\xi(x) = x$ and $w(x) = 1$.

Example 2. Consider the following FSLP:

$$(1 + x)^{1/2}\ {}^{C}_{b-}D^{\alpha}_{[\xi,w]}\ {}^{C}_{a+}D^{\alpha}_{[\xi,w]}y(x) = \lambda(1 + 100x^2)y(x) \tag{4.42}$$

subject to $\quad y(0) = 0, \quad {}_{a+}D^{\alpha}_{[\xi,w]}y(x)\big|_{x=1} = 0.$

Example 3. Consider the following FSLP:

$$(1 + 2x^2)\ {}^{C}_{b-}D^{\alpha}_{[\xi,w]}\ {}^{C}_{a+}D^{\alpha}_{[\xi,w]}y(x) + 5sin(\pi x)y(x) = \lambda 2e^{x}y(x) \tag{4.43}$$

TABLE 4.3: First 10 eigenvalues of example 2 for different values of $\xi(x)$ and fixed $w(x) = 1$

	$\xi(x) = x, w(x) = 1$				$\xi(x) = x^2, w(x) = 1$			
λ	α =0.4	α =0.6	α =0.8	α =0.99	α =0.4	α =0.6	α =0.8	α =0.99
λ_1	0.0351	0.0403	0.0463	0.0539	0.0277	0.0317	0.0371	0.0444
λ_2	0.1240	0.2635	0.5221	0.9452	0.0888	0.1740	0.3282	0.5787
λ_3	0.1868	0.4950	1.2389	2.8132	0.1342	0.3273	0.7717	1.6771
λ_4	0.2525	0.7719	2.2074	5.6362	0.1807	0.5072	1.3641	3.3275
λ_5	0.3115	1.0579	3.3539	9.3826	0.2234	0.6963	2.0702	5.5122
λ_6	0.3730	1.3753	4.7011	14.0289	0.2672	0.9036	2.8946	8.2149
λ_7	0.4307	1.7007	6.1988	19.5379	0.3091	1.1186	3.8142	11.4124
λ_8	0.4906	2.0507	7.8657	25.8768	0.3519	1.3474	4.8320	15.0808
λ_9	0.5478	2.4071	9.6604	32.9999	0.3933	1.5821	5.9288	19.1899
λ_{10}	0.6068	2.7831	11.5971	40.8643	0.4354	1.8272	7.1052	23.7082

TABLE 4.4: First 10 eigenvalues of example 3 for different values of $w(x)$ and fixed $\xi(x) = x$

	$\xi(x) = x, w(x) = 1$				$\xi(x) = x, w(x) = exp(x)$			
λ	α =0.3	α =0.5	α =0.7	α =0.9	α =0.2	α =0.4	α =0.65	α =0.8
λ_1	1.3646	1.4518	1.4898	1.5443	1.3329	1.7114	2.0228	2.2379
λ_2	2.2762	3.3378	5.5617	9.2191	1.8068	2.7770	4.9313	7.2821
λ_3	2.7050	4.9853	10.1325	21.5636	1.9919	3.6227	8.4561	14.8179
λ_4	3.1193	6.6512	15.8582	38.9711	2.2533	4.4561	12.6111	24.6818
λ_5	3.4808	8.3249	22.3093	60.8445	2.3822	5.2549	17.2163	36.5103
λ_6	3.8250	10.0724	29.5496	87.0298	2.5177	6.0429	22.2525	50.1697
λ_7	4.1476	11.8336	37.3712	117.1224	2.6446	6.8172	27.6328	65.4455
λ_8	4.4615	13.6494	45.8060	150.9194	2.7554	7.5875	33.3437	82.2231
λ_9	4.7627	15.4770	54.7023	188.0288	2.8633	8.3492	39.3189	100.3160
λ_{10}	5.0579	17.3417	64.0638	228.1780	2.9634	9.1075	45.5387	119.5955

subject to $\quad y(0) = 0, \quad {}_{a+}D^{\alpha}_{[\xi,w]}y(x)\big|_{x=1} = 0.$

Table 4.3 and Table 4.4 represents first 10 eigenvalues for example 2 and example 3 respectively. Eigenvalues are increasing as we increase the value of α.

4.5.1 Numerical Analysis of the Rate of Convergence

Here, we determine the convergence rate of the presented numerical method. When the analytic solution is not known, we can find the convergence order using the following formulas

$$erc_\lambda(N,k,\alpha) = \log_2 \frac{\lambda_k^{(N,\alpha)} - \lambda_k^{(N/2,\alpha)}}{\lambda_k^{(2N,\alpha)} - \lambda_k^{(N,\alpha)}}, \tag{4.44}$$

$$erc_y(N,k,\alpha) = \log_2 \frac{\|y_{k,N} - y_{k,N/2}\|_{L^2_w}}{\|y_{k,2N} - y_{k,N}\|_{L^2_w}} \,. \tag{4.45}$$

In Table 4.5, we present the convergence order for $p(x) = 1,\ q(x) = 0,\ r(x) = 1, \alpha = 0.9, N \in \{50, 100, 200, 400, 800\}, w(x) = 1$ and $\xi(x) \in \{x, x^2, x^3\}$. It is observed that convergence order does not depend on functions, it tends to 1 for all cases.

4.6 Application to Fractional Diffusion Equation with Generalized Fractional Derivatives (FDE)

We will examine the solution of FDE here:

$$ {}^{C}_{a+}D^{\beta}_{[\xi,w]} z(t,x) = -p(x)\ {}^{C}_{b-}D^{\alpha}_{[\xi,w]}\ {}^{C}_{a+}D^{\alpha}_{[\xi,w]} z(t,x), \tag{4.46}$$

$$z(0,x) = {}_{a+} I^{\alpha}_{[\xi,w]}\ {}_{b-}I^{\alpha}_{[\xi,w]} \frac{1}{p(x)} h(x), \quad h \in L^2_{1/p}, \tag{4.47}$$

$$z(t,0) = 0,\ {}^{C}_{a+}D^{\alpha}_{[\xi,w]} z(t,x)|_{x=b} = 0. \tag{4.48}$$

where $\beta \in (0,1)$ and $\alpha \in (0,1)$.

We now describe approximate weak solution and evaluate the error bound.

TABLE 4.5: Eigenvalues and experimental rates of convergence for $\alpha = 0.9,\ p(x)=1,\ q(x)=0,\ r(x)=1$ and for different $\xi(x)$

k	N	$\xi(x)=x, w(x)=1$			$\xi(x)=x^2, w(x)=1$			$\xi(x)=x^3, w(x)=1$		
		λ_k	erc_λ	erc_y	λ_k	erc_λ	erc_y	λ_k	erc_λ	erc_y
1	50	2.2500	-	-	2.6932	-	-	2.0899	-	-
	100	2.2203	0.98	0.98	2.1125	1.02	1.00	2.1178	1.08	1.00
	200	2.2053	0.98	0.98	2.1220	1.04	1.00	2.1310	1.09	1.00
	400	2.1976	0.97	0.98	2.1937	1.05	1.00	2.1373	1.08	1.00
	800	2.1937	-	-	2.1918	-	-	2.1402	-	-
2	50	16.8497	-	-	16.1984	-	-	16.0859	-	-
	100	16.5939	0.96	0.99	16.2850	1.22	1.02	16.2408	1.21	1.01
	200	16.4627	0.98	0.98	16.3223	1.15	1.00	16.3026	1.17	1.00
	400	16.3960	0.98	0.98	16.3390	1.11	1.00	16.3287	1.13	1.00
	800	16.3453	-	-	16.3468	-	-	16.3403	-	-
3	50	42.3165	-	-	40.4169	-	-	40.3461	-	-
	100	41.5923	0.92	0.99	40.7437		1.03	40.6765	1.48	1.04
	200	41.2108	0.96	0.98	40.7680		1.02	40.7564	1.39	1.02
	400	41.0156	0.99	0.98	40.7702		1.00	40.7732	1.29	1.00
	800	40.9170	-	-	40.7690	-	-	40.7758	-	-
4	50	77.9313	-	-	70.9342	-	-	74.3050	-	-
	100	76.4876	0.88	0.98	75.0946		1.07	74.9788	1.88	1.09
	200	75.6964	0.94	0.98	75.0345		1.03	75.0303	1.94	1.05
	400	75.2876	0.98	0.98	74.9697		1.01	74.9809	1.97	1.02
	800	75.0777	-	-	74.9301	-	-	74.9382	-	-

Theorem 4.1. *Consider the FDE given in Eqs. (4.46)-(4.48) and $\alpha \in (0,1)$. This equation has a unique weak solution of the form :*

$$z(t,x) = \sum_{k=1}^{\infty} \frac{1}{\Lambda_k} \langle h, \phi_k \rangle_{1/p} E_{\alpha,1}\Big(-\Lambda(\xi(x)-\xi(a))^{\alpha} \Big) \phi_k(x), \tag{4.49}$$

where the coefficient $\langle h, \phi_k \rangle_{1/p}$ is given by

$$\langle h, \phi_k \rangle_{1/p} = \int_a^b \frac{1}{p(x)} h(x) \phi_k(x) dx. \tag{4.50}$$

Proof. We prove this theorem following the same method as done in [51, 69]. The solution is expended in the following form:

$$z(t,x) = \sum_{k=1}^{\infty} b_k(t)\phi_k(x). \tag{4.51}$$

where $\phi_k(x)$ denotes the eigenfunctions of the operator, $L_x = p(x)\ {}^{C}_{b-}D^{\alpha}_{[\xi,w]}\ {}^{C}_{a+}D^{\alpha}_{[\xi,w]}$ and Λ_k be the associated eigenvalues.

Using the orthonormality of eigenfunctions, we get

$${}^{C}_{a+}D^{\beta}_{[\xi,w]} b_k(t) = -\Lambda_k b_k(t),\ \ k \in \mathbb{N}. \tag{4.52}$$

Solving Eq. (4.52) using weighted Laplace transform [74] , we get

$$b_k(t) = c_k E_{\alpha,1}\Big(- \Lambda(\xi(x) - \xi(a))^{\alpha}\Big), \tag{4.53}$$

where constants c_k are given as, $\phi_k,\ \ k \in \mathbb{N}$,

$$c_k = \frac{1}{\Lambda_k}\langle h, \phi_k\rangle_{1/p}\ . \tag{4.54}$$

Thus, the solution of Eqs. (4.46)-(4.48) will be of the form,

$$z(t,x) = \sum_{k=1}^{\infty} \frac{1}{\Lambda_k}\langle h, \phi_k\rangle_{1/p} E_{\alpha,1}\Big(- \Lambda(\xi(x) - \xi(a))^{\alpha}\Big)\phi_k(x). \tag{4.55}$$

Now, in order to prove that the resulting series given in Eq. (4.55) is continuous in $[0,\infty) \times [0,b]$. It is sufficient to demonstrate that the series of $z(t,x)$ is uniformly and absolutely convergent on $[0,\infty) \times [0,b]$, i.e.

$$\sum_{k=1}^{\infty} \left| \frac{1}{\Lambda_k}\langle h, \phi_k\rangle_{1/p} E_{\alpha,1}\Big(- \Lambda(\xi(x) - \xi(a))^{\alpha}\Big)\phi_k(x)\right| < \infty, \tag{4.56}$$

for this, we noticed that

$$\begin{aligned}
&\sum_{k=1}^{\infty}\left|\frac{1}{\Lambda_k}\langle h,\phi_k\rangle_{1/p}E_{\alpha,1}\Big(-\Lambda(\xi(x)-\xi(a))^{\alpha}\Big)\phi_k(x)\right| \\
&\leq \sum_{k=1}^{\infty}\left|\langle h,\phi_k\rangle_{1/p}\|E_{\alpha,1}(-\Lambda(\xi(x)-\xi(a))^{\alpha}\|\frac{\phi_k(x)}{\Lambda_k}\right| \\
&\leq M\sum_{k=1}^{\infty}|\langle h,\phi_k\rangle_{1/p}|\frac{C}{1+\Lambda(\xi(x)-\xi(a))^{\alpha}} \\
&< \infty. \qquad (4.57)
\end{aligned}$$

Thus, Weierstrass-Majorant theorem implies that the series of z is uniformly and absolutely convergent on $[0,\infty]\times[0,b]$.

Now, we will approximate the exact solution as follows

$$z(t,x) = z_M(t,x) + R_M(t,x), \qquad (4.58)$$

where, $z_M(t,x)$, the theoretical approximation is defined as,

$$z_M(t,x) = \sum_{k=1}^{M-1}\frac{1}{\Lambda_k}\langle h,\phi_k\rangle_{1/p}E_{\alpha,1}\Big(-\Lambda(\xi(x)-\xi(a))^{\alpha}\Big)\phi_k(x), \qquad (4.59)$$

and $R_M(t,x)$, error of this approximation is given as

$$R_M(t,x) = \sum_{k=M}^{\infty}\frac{1}{\Lambda_k}\langle h,\phi_k\rangle_{1/p}E_{\alpha,1}\Big(-\Lambda(\xi(x)-\xi(a))^{\alpha}\Big)\phi_k(x). \qquad (4.60)$$

The error of this approximation $R_M(t,x)$ is bounded as

$$\|R_M(t,x)\|^2_{1/p} = \sum_{k=M}^{\infty} \frac{1}{\Lambda_k^2} |\langle h, \phi_k\rangle_{1/p}|^2 E_{\alpha,1}\Big(- \Lambda(\xi(x)-\xi(a))^{\alpha}\Big)^2,$$
$$\leq \sum_{k=M}^{\infty} \frac{1}{\Lambda_k^2} |\langle h, \phi_k\rangle_{1/p}|^2 \frac{C_1^2}{(1+\Lambda(\xi(x)-\xi(a)))^2}.$$

Now, Schwarz-Bunyakovsky inequality implies

$$\|R_M(t,x)\| \leq \frac{C_2}{\Lambda_M}. \tag{4.61}$$

□

4.6.1 Numerical Approximation and Error Bound

We approximate the exact solution as

$$\tilde{z}_{M,N}(t,x) = \sum_{k=1}^{M-1} \frac{1}{\lambda_k} \langle h, \phi_{k,N}\rangle_{1/p} E_{\alpha,1}\Big(- \lambda(\xi(x)-\xi(a))^{\alpha}\Big)\phi_{k,N}(x). \tag{4.62}$$

where $N > M$.

Now, we calculate the error

$$\|z_M(t,\cdot)-\tilde{z}_{M,N}(t,\cdot)\|_{1/p} = \Bigg\| \sum_{k=1}^{M-1} \frac{1}{\Lambda_k} \langle h, \phi_k\rangle_{1/p} E_{\alpha,1}\Big(- \Lambda(\xi(x)-\xi(a))^{\alpha}\Big)\phi_k(x)$$
$$- \sum_{k=1}^{M-1} \frac{1}{\lambda_k} \langle h, \phi_{k,N}\rangle_{1/p} E_{\alpha,1}\Big(- \lambda(\xi(x)-\xi(a))^{\alpha}\Big)\phi_{k,N}(x)\Bigg\|_{1/p}$$
$$\leq \Bigg\| \sum_{k=1}^{M-1} \left(\frac{1}{\Lambda_k} - \frac{1}{\lambda_k}\right) \langle h, \phi_{k,N}\rangle_{1/p} E_{\alpha,1}\Big(- \lambda(\xi(x)-\xi(a))^{\alpha}\Big)\phi_{k,N}(x)\Bigg\|_{1/p}$$

$$+\left\| \sum_{k=1}^{M-1} \frac{1}{\Lambda_k} \langle h, \phi_{k,N} \rangle_{1/p} E_{\alpha,1}\Big(-\Lambda(\xi(x)-\xi(a))^{\alpha}\Big) - E_{\alpha,1}\Big(-\lambda(\xi(x)-\xi(a))^{\alpha}\Big)\phi_{j,N}(x)\right\|_{1/p}$$
$$+\left\| \sum_{k=1}^{M-1} \frac{1}{\Lambda_k} \langle h, \phi_k - \phi_{k,N} \rangle_{1/p} E_{\alpha,1}\Big(-\Lambda(\xi(x)-\xi(a))^{\alpha}\Big)\phi_j(x)\right\|_{1/p}$$
$$+\left\| \sum_{k=1}^{M-1} \frac{1}{\xi_k} \langle h, \phi_{k,N} \rangle_{1/p} E_{\alpha,1}\Big(-\Lambda(\xi(x)-\xi(a))^{\alpha}\Big)(\phi_k(x)-\phi_{k,N}(x))\right\|_{1/p}, \tag{4.63}$$

and now define $\Delta\phi_{k,N}$ and $\Delta\xi_{k,N}$

$$\Delta\phi_{k,N} = |\phi_k(x) - \phi_{k,N}(x)|,\ \Delta\xi_{k,N} = |\Lambda_k - \lambda_k|. \tag{4.64}$$

Since,

$$\begin{aligned}\left\| \sum_{k=1}^{M-1} \left(\frac{1}{\Lambda_k} - \frac{1}{\lambda_k}\right) \langle h, \phi_{k,N} \rangle_{1/p} E_{\alpha,1}(-\lambda(\xi(x)-\xi(a))^{\alpha})\phi_{k,N}(x)\right\|_{1/p} \\ \le C\frac{\max_{k<M}(\Delta w_{k,N})}{\xi_1\lambda_1}\left\| \sum_{k=1}^{M-1} \langle h, \phi_{k,N} \rangle_{1/p}\right\|_{1/p} \\ \le C\frac{\max_{k<M}(\Delta w_{k,N})}{\xi_1\lambda_1}\|h\|_{1/p}.\end{aligned} \tag{4.65}$$

Similarly,

$$\begin{aligned}\left\| \sum_{k=1}^{M-1} \frac{1}{\Lambda_k} \langle h, \phi_{k,N} \rangle_{1/p} E_{\alpha,1}\Big(-\Lambda(\xi(x)-\xi(a))^{\alpha}\Big) - E_{\alpha,1}\Big(-\lambda(\xi(x)-\xi(a))^{\alpha}\Big)\phi_{k,N}(x)\right\|_{1/p} \\ \le C_1\frac{\max_{k<M}(\Delta w_{k,N})}{\xi_1}\|h\|_{1/p},\end{aligned} \tag{4.66}$$

$$\begin{aligned}\left\| \sum_{k=1}^{M-1} \frac{1}{\Lambda_k} \langle h, \phi_k - \phi_{k,N} \rangle_{1/p} E_{\alpha,1}\Big(-\Lambda(\xi(x)-\xi(a))^{\alpha}\Big)\phi_k(x)\right\|_{1/p} \\ \le C \max_{k<M} \|\Delta\phi_{k,N}\|_{1/p}\|h\|_{1/p}\left(\sum_{k=1}^{\infty}\frac{1}{\xi_k^2}\right)^{\frac{1}{2}},\end{aligned} \tag{4.67}$$

$$\Big\| \sum_{k=1}^{M-1} \frac{1}{\xi_k} \langle h, \phi_{k,N} \rangle_{1/p} E_{\alpha,1}\Big(-\Lambda(\xi(x)-\xi(a))^{\alpha}\Big)(\phi_k(x)-\phi_{k,N}(x)) \Big\|_{1/p} \leq \frac{C \max_{k<M} \|\Delta\phi_{k,N}\|_{1/p}}{\xi_1} \|h\|_{1/p}. \tag{4.68}$$

Thus, we have

$$\begin{aligned} \|z_M(t,\cdot) - \tilde{z}_{M,N}(t,\cdot)\|_{1/p} &\leq \Big(\frac{C}{\xi_1\lambda_1} + \frac{C_1}{\xi_1}\Big) \max_{k<M}(\Delta w_{k,N}) \|h\|_{1/p} \\ &+ \Big(C \Big(\sum_{j=1}^{\infty} \frac{1}{\xi_k^2} \Big)^{\frac{1}{2}} + \frac{C}{\xi_1} \Big) \max_{k<M} \|\Delta\phi_{k,N}\|_{1/p} \xi_1 \|h\|_{1/p}. \end{aligned} \tag{4.69}$$

Therefore,

$$\begin{aligned} \|z(t,\cdot) - \tilde{z}_{M,N}(t,\cdot)\|_{1/p} &\leq \|z(t,\cdot) - z_M(t,\cdot)\|_{1/p} + \|z_M(t,\cdot) - \tilde{z}_{M,N}(t,\cdot)\|_{1/p} \\ &\leq \frac{C}{\xi_1} + \Big(\frac{C}{\xi_1\lambda_1} + \frac{C_1}{\xi_1}\Big) \max_{k<M}(\Delta w_{k,N}) \|h\|_{1/p} \\ &+ \Big(C \Big(\sum_{k=1}^{\infty} \frac{1}{\xi_k^2} \Big)^{\frac{1}{2}} + \frac{C}{\xi_1} \Big) \max_{k<M} \|\Delta\phi_{k,N}\|_{1/p} \xi_1 \|h\|_{1/p}. \end{aligned} \tag{4.70}$$

As a result, the error estimation of the approximation is affected by both M and N.

4.7 Conclusions

We investigated a numerical algorithm to solve the FSLP given by Eqs. (4.1)-(4.2) in this chapter. First, we demonstrated that the considered FSLP is well-posed. Following that, a numerical approach for solving the suggested FSLP is provided. We utilized the finite difference method and non-uniform node points to discretize CGFD. A system of linear equations is then formed, and we determine the approximated eigenvalues and eigenfunctions. The numerical results obtained for the test examples are provided in Tables 4.1-4.4, and Table 4.5 shows the convergence order is close to one. By changing the weight and scale functions, the suggested approach can easily be expanded to find solutions to different FSLPs. We derived the solution to the FDE provided by Eqs. (4.46)-(4.48) in order to build an application.

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