Chapter 2

Riemann Problem and Elementary Wave Interaction

2.1 Introduction

Dusty gas is a mixture of gas and small solid particles. The solid particles have a small volume fraction but its density may be large relative to the gas, such types of flow have broad range of application from industrial processes to geophysical flows. There are many processes in which mass fraction of solid particles is very small in comparison to gas such as volcanic jets, solid particle motion in rocket exhaust etc Pai (1977), Miura and Glass (1983), Pai, Menon and Fan (1980), Laibe and Price (2014), Pelanti and Leveque (2006), Probstein and Fassio (1969). So the study of

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Riemann problem for such type of flow plays a major role in the field of mathematics as well as in engineering and physics.

The governing equations describing a planar isentropic dusty gas flow are given by Chadha and Jena (2014)

$$\left.\begin{aligned} \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 \\ \rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}) + \frac{\partial p}{\partial x} = 0 \end{aligned}\right\}, \tag{2.1.1}$$

where $\rho \geq 0$ is the density of dusty gas, u is the flow velocity, p is the pressure and $t > 0$, $x \in \mathbb{R}$. The specific heat of dusty gas at constant pressure is given by $c_{pd} = k_p c_{sp} + (1 - k_p)\, c_p$, where c_p and c_{sp} stand for specific heat of gas and specific heat of solid particle respectively and $k_p = m_{sp}/m_{gd}$ is the mass fraction of solid particle in total mass of dusty gas, where m_{sp} is mass of solid particles and m_{gd} is mass of dusty gas. If c_{vd} denotes the specific heat of dusty gas at constant volume, the ratio of specific heats for dusty gas is given by Pai (1977) and Marble(1970)

$\Gamma = \frac{c_{pd}}{c_{vd}} = \frac{\gamma(1+\delta\beta)}{1+\delta\beta}$, where $\delta = k_p/(1 - k_p)$, $\beta = c_{sp}/c_p$, $\gamma = c_p/c_v$, with c_v as specific heat of gas at constant volume. For isentropic dusty gas flow, equation of state is given by Gupta et al. (2015), $p = k(\rho/(1 - Z))^{\Gamma}$, where k is a positive constant and the quantity Z denotes the volume fraction of solid particle in total volume of dusty gas. The relation between the entities Z and k_p is $k_p = Z\rho_{sp}/\rho$, where ρ_{sp} stands for density of solid particles in dusty gas. Since mass fraction of solid particle must be constant in the equilibrium flow therefore $Z/\rho =$ constant (say θ).

A detailed analytical and numerical investigation of Riemann problem in different material media has been carried out in Chadha and Jena (2014), Chorin (1976), Smoller (1993), Chen (1994), Toro (1989), Toro (1995), Smoller and Temple (1993), Chen (1995), LeVeque (2002), Courant and Friedrichs (1999) and Lax (1957). Also the problem of interaction of elementary waves for unsteady one dimensional Euler equations was studied by Smoller (1994) and Chang and Hsiao (1989). Liu (2012) discussed the interaction of elementary waves for nonlinear degenerate wave equations. The Riemann problem for magnetogasdynamics and their wave interactions was studied by Sekhar and Sharma (2010), Liu and Sun (2013, 2016).

In the present chapter, we consider Riemann problem for (2.1.1) with initial data given by

$$U(x,0) = \begin{cases} U_l = (\rho_l, u_l) & \text{if } x \leq 0 \\ U_r = (\rho_r, u_r) & \text{if } x > 0 \end{cases}, \tag{2.1.2}$$

where (ρ_l, u_l) and (ρ_r, u_r) are constant states.

In the next sections we will discuss the effect of dust particles on the shock and rarefaction wave and its properties by using the method of characteristics. The existence and uniqueness of the solution under certain conditions will be obtained constructively. The numerical solution of Riemann problem of dusty gas flow for different initial data will be determined. We also discuss all possible interactions of elementary waves.

2.2 Shock and Rarefaction Waves

Equation (2.1.1) can be written in conservative form as follows

$$\left.\begin{aligned}\frac{\partial}{\partial t}(\rho)+\frac{\partial}{\partial x}(\rho u)=0\\ \frac{\partial(\rho u)}{\partial t}+\frac{\partial}{\partial x}(p+\rho u^2)=0\end{aligned}\right\}. \tag{2.2.1}$$

Equations (2.2.1) may be written in the matrix form as

$$U_t+AU_x=0, \tag{2.2.2}$$

where $U=(\rho,\rho u)^{tr}$, tr denotes transposition and the A is a 2x2 matrix may be written as

$$A=\begin{bmatrix}0 & 1\\ -\frac{(\rho u)^2}{\rho^2}+\sqrt{p'(\rho)} & \frac{2(\rho u)}{\rho}\end{bmatrix}.$$

Here, prime denotes the differentiation with respect to ρ. The eigenvalues of A are

$$\lambda_1=u-(p'(\rho))^{1/2}, \qquad \lambda_2=u+(p'(\rho))^{1/2},$$

where $p'(\rho)=\left(\Gamma p/\left(\rho\left(1-Z\right)\right)\right)^{1/2}$, so that the system (2.2.1) is strictly hyperbolic. The eigenvector of A corresponding to eigenvalue λ_i is $r_i=(1\ \ \lambda_i)^{tr}$.

Also,

$$\begin{aligned}\nabla\lambda_i\bullet r_i&=\nabla\left(\frac{(\rho u)}{\rho}+(-1)^i\sqrt{p'(\rho)}\right)\begin{pmatrix}1\\ \frac{\rho u}{\rho}+(-1)^i\sqrt{p'(\rho)}\end{pmatrix}\\ &=(-1)^i\left(\frac{p''(\rho)}{2\sqrt{p'(\rho)}}+\frac{\sqrt{p'(\rho)}}{\rho}\right),\end{aligned}$$

which is not equal to zero for $p''(\rho)\geq 0$ therefore both the characteristic fields are genuinely nonlinear when $p''(\rho)\geq 0$. Hence the waves associated with r_i, $i=1,2$ characteristic fields are either shocks or rarefaction waves.

2.2.1 Shock waves

Let ρ_l, $u_l = u(\rho_l)$, $p_l = p(\rho_l)$ and $\rho, u = u(\rho)$, $p = p(\rho)$ denote respectively the left and right hand states of either a shock or a rarefaction wave. Then the Rankine-Hugoniot (R-H) jump conditions for the system (2.2.1) are

$$s\left[u - u_l\right] = \left[\rho u - \rho_l u_l\right], \tag{2.2.3}$$

$$s\left[\rho u - \rho_l u_l\right] = \left[p + \rho u^2 - p_l - \rho_l u_l^2\right], \tag{2.2.4}$$

where s is the shock speed.

Lemma 2.1 Let S_1 and S_2 respectively denotes 1-and 2-shock associated with λ_1 and λ_2 characteristic fields. Also let the states U_l and U satisfy the R-H jump conditions (2.2.3) and (2.2.4). Then the shock curves satisfy

$$u = u_l - \varphi(\rho_{l,}\rho), \tag{2.2.5}$$

where $\varphi(\rho_l, \rho) = \sqrt{(p - p_l)\left(\frac{\rho - \rho_l}{\rho\rho_l}\right)}$, such that for $1 < \Gamma < 2$, we have for $\rho > \rho_l$, $u' < 0$ and $u'' > 0$ on S_1. Whilst for $\rho < \rho_l$ we have $u' > 0$ and $u'' < 0$ on S_2.

Proof. Putting the value of s from (2.2.3) in (2.2.4), we get (2.2.5). Let $\tau(\rho) = \varphi^2(\rho_l, \rho)$, then $u' = -\frac{\tau'(\rho)}{2\sqrt{\tau(\rho)}}$, which is negative for $\rho > \rho_l$. It is clear that τ and τ' are positive for $\rho > \rho_l$ and $\tau(\rho_l) = \tau'(\rho_l) = 0$. Let $\psi(\rho) = (\tau'(\rho))^2 - 2\tau(\rho)\tau''(\rho)$, then $\psi(\rho_l) = 0$, and $\psi'(\rho) = -2\tau(\rho)\tau'''(\rho)$. Since, for $1 < \Gamma < 2$, $\tau'' > 0$, and $\tau''' < 0$, therefore $\psi'(\rho) > 0$, which implies that $\psi(\rho) > \psi(\rho_l) = 0$, for $\rho > \rho_l$. Thus, for $1 < \Gamma < 2$, we have $u'' = \frac{(\tau'(\rho))^2 - 2\tau(\rho)\tau''(\rho)}{4\tau(\rho)^{3/2}} > 0$, on S_1. Similarly for $\rho < \rho_l$ and $1 < \Gamma < 2$, it can be shown that $u' > 0$ and $u'' < 0$ on S_2.

Lemma 2.2 If $p' > 0$ and $p'' \geq 0$, then the Lax conditions holds i.e. 1-shock satisfies

$s < \lambda_1(U_l)$ and $\lambda_1(U) < s < \lambda_2(U)$.

While 2-shock satisfies

$\lambda_1(U_l) < s < \lambda_2(U_l)$ and $\lambda_2(U) < s$.

Proof. First, let us consider 1-shock and prove that $s < \lambda_1(U_l)$. On 1-shock, $\rho_l < \rho$, and for the considered problem $p' > 0$ and $p'' \geq 0$, so, by Lagrange mean value theorem, there exists a $\eta_1 \in (\rho_l, \rho)$, such that $p'(\eta_1) = \frac{p(\rho)-p(\rho_l)}{\rho-\rho_l}$. Further, since $p'' \geq 0$, we have $p'(\eta_1) > p'(\rho_l)$, which implies that $p'(\rho_l) < \frac{(p-p_l)\rho}{\rho_l(\rho-\rho_l)}$. Hence, $p'(\rho_l) < \frac{(p-p_l)\rho}{\rho_l(\rho-\rho_l)} \Rightarrow \frac{-\rho\sqrt{(p-p_l)(\rho-\rho_l)/\rho\rho_l}}{(\rho-\rho_l)} < -(p'(\rho_l))^{1/2}$.

From equation (2.2.3)and above inequality, we have

$$s < \lambda_1(U_l) = u_l - (p'(\rho_l))^{1/2}.$$

Now since $p' \geq 0$ and $\rho_l < \rho$ on 1-shock, we have $p'(\eta_2) = \frac{p(\rho)-p(\rho_l)}{\rho-\rho_l} < p'(\rho)$, for some $\eta_2 \in (\rho_l, \rho)$, and hence

$$p'(\rho) > \frac{(p-p_l)\rho_l}{\rho(\rho-\rho_l)} \Rightarrow \frac{-\rho_l\sqrt{(p-p_l)(\rho-\rho_l)/\rho\rho_l}}{(\rho-\rho_l)} > -(p'(\rho))^{1/2}. \tag{2.2.6}$$

Also, equations (2.2.3) and (2.2.5) imply that $u - (p'(\rho))^{1/2} < \frac{\rho u-\rho_l u_l}{\rho-\rho_l} = s$, and hence $\lambda_1(U) < s$. Now we show that $s < \lambda_2(U)$. Equation (2.2.6) implies that $-\sqrt{\frac{(p-p_l)\rho_l}{(\rho-\rho_l)\rho}} < p'(\rho)$. For first shock curve, we use (2.2.3) and get $\frac{(u-u_l)\rho}{\rho-\rho_l} < p'(\rho)$, hence 1-shock satisfies Lax conditions. Similarly we can prove for 2-shock curve.

Theorem 2.1 The 1-shock and 2-shock are starlike with respect to (ρ_l, u_l) when $p = k\left(\frac{\rho}{1-Z}\right)^{\Gamma}$, for $1 \leq \Gamma \leq 2$.

Proof. We shall prove that any ray through the point (ρ_l, u_l) intersects 1-shock curve in at most one point, for this, it is adequate to prove that for any two rays through (ρ_l, u_l) and two other different points on 1-shock curve (ρ_1, u_1) and (ρ_2, u_2)

having slopes $\frac{u_1-u_l}{\rho_1-\rho_l}$ and $\frac{u_2-u_l}{\rho_2-\rho_l}$ are different. For 1-shock curve, equation (2.2.5) implies that

$$\left(\frac{u-u_l}{\rho-\rho_l}\right)^2 = \frac{p-p_l}{\rho\rho_l(\rho-\rho_l)}.$$

Let $f(\rho) = \frac{p-p_l}{\rho\rho_l(\rho-\rho_l)}$, then $f'(\rho) = \frac{\rho\rho_l(\rho-\rho_l)p'(\rho)-(p-p_l)(2\rho\rho_l-\rho_l^2)}{\rho^2\rho_l^2(\rho-\rho_l)^2}$. Let $g(\rho) = \rho\rho_l(\rho-\rho_l)p'(\rho)-(p-p_l)(2\rho\rho_l-\rho_l^2)$, then $g(\rho_l)=0$ and $g'(\rho) = \rho\rho_l(\rho-\rho_l)p''(\rho)-2(p-p_l)\rho_l$, so we have $g'(\rho_l)=0$.

Further, $g''(\rho) = \rho\rho_l(\rho-\rho_l)p'''(\rho)+(2\rho\rho_l-\rho_l^2)p''(\rho)-2p'(\rho)\rho_l$.

Putting the value of $p'(\rho)$, $p''(\rho)$ and $p'''(\rho)$ in above equation, we get

$$g''(\rho) = \frac{k\rho_l\rho^{\Gamma-2}}{(1-Z)^{\Gamma+3}}\left[4\left(\Gamma+1\right)Z\rho-\left(\Gamma+1\right)\left(2-\Gamma\right)\rho-\left(\Gamma-1\right)^2\rho_l-\left(5\Gamma-3\right)Z\rho_l-4\rho_l Z^2\right].$$

For $1 \leq \Gamma \leq 2$ and using the fact that the volume fraction of solid particles is less than 5% of the total volume of the dusty gas, we have $g''(\rho) < 0$. Therefore $g'(\rho)$ is monotonically decreasing function of ρ, which implies that for $\rho > \rho_l$, we have $g'(\rho) < g'(\rho_l)$. But $g'(\rho_l) = 0$ implies that $g(\rho)$ is monotonically decreasing function of ρ, which implies that for $\rho > \rho_l$, we have $g'(\rho) < g'(\rho_l)$. But $g(\rho_l) = 0$ implies that $f(\rho)$ is monotonically decreasing function of ρ. Thus $\frac{u-u_l}{\rho-\rho_l}$ is monotonically decreasing function of ρ and hence 1-shock curve is starlike with respect to (ρ_l, u_l). Similarly we can show that 2-shock curve is starlike.

2.2.2 Rarefaction wave

For rarefaction wave, we have smooth solution of the form $U\left(x/t\right)$ such that

$$U(x,t)=\begin{cases} U_l, & \text{for } x\leq\lambda_i(U_l)t,\\ U\left(x/t\right), & \text{for } \lambda_i(U_l)t\leq x\leq\lambda_i(U_r)t,\\ U_r, & \text{for } x\geq\lambda_i(U_r)t, \end{cases} \tag{2.2.7}$$

where $i=1,2$. Setting $\xi=x/t$ then $U\left(x/t\right)$ is defined as

$$\frac{dU}{d\xi}=\begin{bmatrix}\frac{d\rho}{d\xi}\\ \frac{d(\rho u)}{d\xi}\end{bmatrix}=\frac{1}{(-1)^i\left(\frac{p''(\rho)}{2\sqrt{p'(\rho)}}+\frac{\sqrt{p'(\rho)}}{\rho}\right)}\begin{bmatrix}1\\ \lambda_i\end{bmatrix}.$$

So the expressions of wave curve, in terms of conserved variables, is given by $\frac{d(\rho u)}{d\rho}=\lambda_i$, which implies that

$$R_i\equiv u+(-1)^{i+1}\int\frac{(p'(\rho))^{1/2}}{\rho}d\rho=Constant,\qquad for\quad i=1,2. \tag{2.2.8}$$

Here, R_i is the i^{th} Riemann invariant and Eq.(2.2.8) represents i^{th} rarefaction wave curve, i.e.

$$R_1\equiv u+\int\frac{(p'(\rho))^{1/2}}{\rho}d\rho=Constant, \tag{2.2.9}$$

and

$$R_2\equiv u-\int\frac{(p'(\rho))^{1/2}}{\rho}d\rho=Constant. \tag{2.2.10}$$

Theorem 2.2 On i^{th} rarefaction wave curve, the corresponding Riemann invariant R_i is constant for $i=1,2$.

Proof. Let U be a i^{th}-rarefaction wave of the form (2.2.7), and let R be i^{th}-Riemann invariant. Since U is continuous and R is assumed to be smooth, the function $R : (x,t) \rightarrow R(U)$ is continuous for $t > 0$.

So, $R(U)$ is constant for $x \leq \lambda_i(U_l)t$ and $x \geq \lambda_i(U_r)t$.

Further, since $\xi = x/t$, we have

$$\frac{dR(U)}{d\xi} = \nabla R(U).U'(\xi). \tag{2.2.11}$$

As U is an integral curve of $\overrightarrow{r_i}$, the right hand side of (2.2.11) is zero, and this proves the theorem.

Lemma 2.3 Characteristic speed increases from left to right hand state across 1-and 2-rarefaction waves iff across 1-rarefaction waves (respectively 2-rarefaction waves) $\rho \leq \rho_l$ and $u_l \leq u$ (respectively $\rho \geq \rho_l$ and $u \geq u_l$).

Proof. Let us consider 1-rarefaction wave with $\rho \leq \rho_l$ and $u_l \leq u$.

Let $\omega = (p'(\rho))^{1/2}$, then $\frac{d\omega}{d\rho} = \frac{p''(\rho)}{2\omega} > 0$, therefore ω is increasing function of ρ, so

$\omega(\rho) \leq \omega(\rho_l)$ or $-\omega \geq -\omega_l$, this inequality together with $u_l \leq u$ implies that $u_l - \omega_l \leq u - \omega$ i.e. $\lambda_1(U_l) \leq \lambda_1(U)$.

Similarly we can prove that $\lambda_2(U_l) \leq \lambda_2(U)$ for 2-rarefaction waves.

Conversely, for 1- rarefaction wave $\lambda_1(U_l) \leq \lambda_1(U)$ implies that

$$\omega - \omega_l \leq u - u_l. \tag{2.2.12}$$

Further, since in first rarefaction wave region R_1 is constant, we have $u - u_l = \int_0^{\rho_l} \frac{(p'(h))^{1/2}}{h} dh - \int_0^{\rho} \frac{(p'(h))^{1/2}}{h} dh$. In view of (2.2.12) $\omega - \omega_l \leq \int_0^{\rho_l} \frac{(p'(h))^{1/2}}{h} dh - \int_0^{\rho} \frac{(p'(h))^{1/2}}{h} dh$

implies that $\rho \leq \rho_l$ and $u - u_l = \int_0^{\rho_l} \frac{(p'(h))^{1/2}}{h} d\rho - \int_0^{\rho} \frac{(p'(h))^{1/2}}{h} dh \geq 0$. Hence $\rho \leq \rho_l$ and $u_l \leq u$.

Similarly we can prove for 2-rarefaction wave $\rho \leq \rho_l$.

Theorem 2.3 The 1-rarefaction curve is convex and monotonic decreasing while 2-rarefaction curve is concave and monotonic increasing.

Proof. Let us consider 1-rarefaction wave

$$u - u_l = \int_{\rho}^{\rho_l} \frac{(p'(\rho))^{1/2}}{\rho} d\rho, \qquad if \qquad \rho \leq \rho_l, \tag{2.2.13}$$

$$\frac{du}{d\rho} = -\frac{(p'(\rho))^{1/2}}{\rho} < 0,$$

and

$$\frac{d^2u}{d\rho^2} = \frac{2(p'(\rho)) - \rho p''(\rho)}{2(p'(\rho))^{1/2}\rho^2}. \tag{2.2.14}$$

Since $p = k\left(\frac{\rho}{1-Z}\right)^{\Gamma}$, therefore equation (2.2.14) becomes

$\frac{d^2u}{d\rho^2} = k\Gamma \frac{\rho^{\Gamma-1}}{(1-Z)^{\Gamma+2}}\left[(3-\Gamma) - 4Z\right]$, which is positive for $1 \leq \Gamma \leq 2$ and volume fraction of solid particles is less than 5% of the total volume of dusty gas. Hence u is convex with respect to ρ for 1-rarefaction waves. In similar way we can prove for 2-rarefaction wave.

Lemma 2.4 With $p'(\rho) > 0$ and $p''(\rho) > 0$, the inequalities $0 < \frac{dR_1}{dR_2} < 1$ and $0 < \frac{dR_2}{dR_1} < 1$ holds along i^{th} shock with

$$R_i \equiv u + (-1)^{i+1} \int \frac{(p'(h))^{1/2}}{h} dh. \tag{2.2.15}$$

Proof. From (2.2.15), we have $\frac{dR_i}{d\rho} = \frac{du}{d\rho} + (-1)^{i+1}\frac{(p'(\rho))^{1/2}}{\rho}$. As we know that along 1-shock curve $\frac{du}{d\rho} < 0$, also along 1-shock curve $\left|\frac{dR_1}{d\rho}\right| < \left|\frac{dR_2}{d\rho}\right|$, we have $\left|\frac{dR_1}{dR_2}\right| < 1$.

In order to show that $0 < \frac{dR_1}{dR_2} < 1$, it is sufficient to show that $\frac{du}{d\rho} < 0$. For 1-shock curve, equation (2.2.5) implies that

$$\begin{aligned}\frac{du}{d\rho} + \frac{(p'(\rho))^{1/2}}{\rho} &= \frac{(p'(\rho))^{1/2}}{\rho} - \frac{\frac{(p-p_l)}{\rho^2} + p'(\frac{\rho-\rho_l}{\rho\rho_l})}{2\sqrt{(p-p_l)(\frac{\rho-\rho_l}{\rho\rho_l})}},\\ &= -\frac{\left[\sqrt{\frac{(p-p_l)}{\rho^2}} - p'(\frac{\rho-\rho_l}{\rho\rho_l})\right]^2}{2\sqrt{(p-p_l)(\frac{\rho-\rho_l}{\rho\rho_l})}}.\end{aligned}$$

This implies that $\frac{dR_1}{d\rho} < 0$. Similarly we can show that $0 < \frac{dR_1}{dR_2} < 1$, along 2-shock curve.

2.3 The Riemann Problem

We now solve the Riemann problem described by the system (2.1.1) and initial condition given by (2.1.2) for a class of functions which consists of constant states, separated by either shock waves or rarefaction waves. The solution obtained here consists of, at most three constant states (including (ρ_l, u_l) and (ρ_r, u_r)) which are separated by either a shock or a rarefaction wave.

Lemma 2.5 The mapping $(\rho, u) \to (R_1, R_2)$ is one to one and the Jacobian of this mapping is non zero if $\rho > 0$.

Proof. Since $R_1 = u + \int^{\rho}\frac{(p'(h))^{1/2}}{h}dh$, and $R_2 = u - \int^{\rho}\frac{(p'(h))^{1/2}}{h}dh$ we have $\frac{\partial R_1}{\partial\rho} = \frac{(p'(h))^{1/2}}{h}$, $\frac{\partial R_1}{\partial u} = 1$, $\frac{\partial R_2}{\partial\rho} = -\frac{(p'(h))^{1/2}}{h}$, and $\frac{\partial R_2}{\partial u} = 1$. Thus the Jacobian of the mapping $(\rho, u) \to (R_1, R_2)$ is equal to $\frac{2(p'(\rho))^{1/2}}{\rho}$ which is non zero for $\rho > 0$. Hence the mapping is bijective.

Thus $(R_1(\rho_l, u_l), R_2(\rho_l, u_l))$ and $(R_1(\rho_r, u_r), R_2(\rho_r, u_r))$ are uniquely determined for given (ρ_l, u_l) and (ρ_r, u_r). When (ρ_r, u_r) and (ρ_l, u_l) are sufficiently closed then the existence and uniqueness of the solution of Riemann problem for system (2.1.1) in the class of elementary waves follows from the general theorem of Lax, which applies to any system of conservation laws that is strictly hyperbolic and genuinely nonlinear in each characteristic field (see Pai, Menon and Fan (1980), Liu and Sun(2016)). We discuss the existence and uniqueness of the solution of the Riemann problem for system (2.1.1) for arbitrary initial data.

The shock and rarefaction curve is plotted by taking the physical variables ρ and u as coordinate axis. The curves divide the whole plane into four disjoint open regions namely I, II, III and IV as shown in Fig.2.1. The numerical value of k is taken to be 1 throughout the present chapter.

For given (ρ_l, u_l), allow (ρ_r, u_r) to vary in $\rho - u$ plane, then (ρ_r, u_r) lies on either any one of above curves or in any one open regions.

If (ρ_r, u_r) lies on any one of the above wave curves then we already know how to solve Riemann problem. Let U_r belongs in one of the above four open regions I, II, III and IV.

For $U \in R^+ \times R$, we define

$$S_j(U) = \{(\rho, u) : (\rho, u) \in S_j\},$$

$$R_j(U) = \{(\rho, u) : (\rho, u) \in R_j\},$$

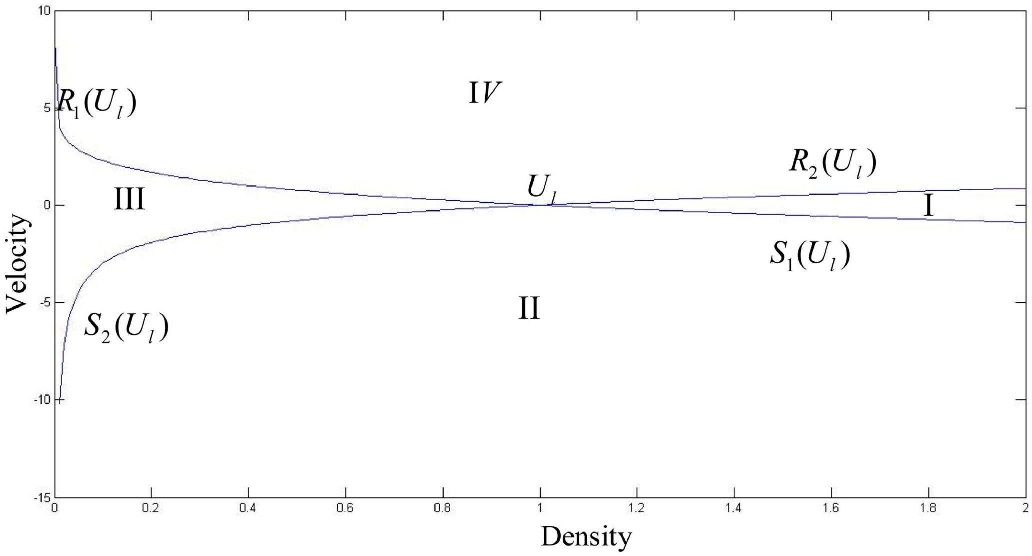

FIGURE 2.1: Wave curves in $\rho - u$ plane for $Z = 0.04, K_p = 0.3, \beta = 1.0$ and $\gamma = 1.4$

and

$$T_j(U) = S_j(U) \cup R_j(U), \quad j = 1, 2.$$

For fixed $U_l \in R^+ \times R$, we consider $S = \{T_2(U) : U \in T_1(U_l)\}$. As $\rho - u$ plane is covered univalently by the family of curves S, i.e. through each point U_r, there possesses exactly one curve $T_2(U)$ of S. To solve the Riemann problem, we connect U to U_l on right by a backward (shock or rarefaction) wave and U_r to U on the right by a forward (shock or rarefaction) wave. Particular type of wave pattern depends on the location of U_r.

Theorem 2.4 If $U_l, U_r \in R^+ \times R$ with U_l fixed and U_r allowed to vary then Riemann problem have unique solution.

Proof. Let $U_r \in I$, draw a line $\rho = \rho_r$, it meets the curves $R_2(U_l)$ and $S_1(U_l)$ at points P and Q respectively as shown in fig.2.2. We observe that the subfamily of curves in S consisting of the set $\{T_2(U) \equiv T_2(\rho, u) : \rho_l \leq \rho \leq \rho_r\}$, includes a continuous mapping $g \to \varphi(g)$ from the arc U_lA to the line segment PQ, see (Smoller

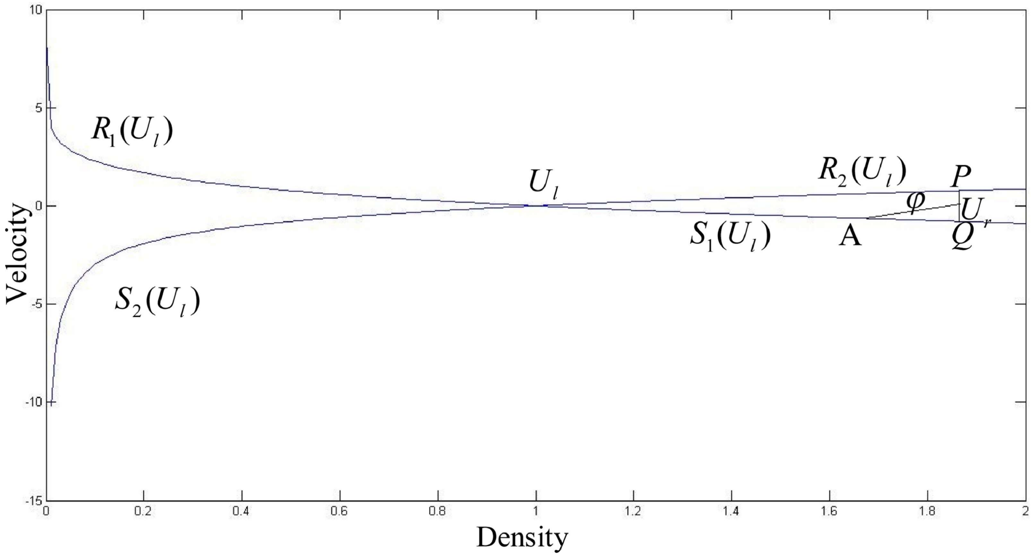

FIGURE 2.2: Wave curves in $\rho - u$ plane for $Z = 0.04, K_p = 0.3, \beta = 1.0$ and $\gamma = 1.4$

(1994)), let $(\rho_m, u_m) \to U_r$, then

$$u = u_l + \int_{\rho}^{\rho_r} \frac{p'(h)}{h} dh - \sqrt{(p - p_l)(\rho - \rho_l)/(\rho\rho_l)},$$

which gives, $\frac{du}{d\rho}_{at\ \rho=\rho_m} < 0$, this implies that (ρ_m, u_m) is unique. Similar line of proof is followed if U_r lies in regions II, III and IV. Since the set S covers the regions I, II, III and IV in a 1-1 fashion therefore solution of Riemann problem exists for any arbitrary data U_r lying in the regions I, II, III and IV.

Theorem 2.5 If $R_1(\rho_l, u_l) - R_2(\rho_r, u_r) \leq 0$, then the vacuum exists.

Proof. Since across i^{th}-rarefaction wave, i^{th}-Riemann invariant is constant therefore $R_1(\rho_l, u_l) = R_1(\rho_m, u_m)$, $R_2(\rho_m, u_m) = R_2(\rho_r, u_r)$.

So, $R_1(\rho_m, u_m) - R_2(\rho_m, u_m) = R_1(\rho_l, u_l) - R_2(\rho_r, u_r) \leq 0$,

but $R_1(\rho_m, u_m) - R_2(\rho_m, u_m) = 2\int^{\rho_m} \frac{(p'(h))^{1/2}}{h} dh > 0$, this implies that $\rho_m = 0$, hence vacuum exist.

2.4 Numerical Solution

For a given left state U_l and a right state U_r one can find the unknown state U_m in the (x,t)-plane, for this, we proceed as:

Case A. For $\rho_l < \rho_m$ and $\rho_r > \rho_m$, eliminating u_m from (2.2.5) and (2.2.10), we get

$$u_r - u_l + \varphi(\rho_l, \rho_m) + \int_{\rho_r}^{\rho_m} \frac{(p'(h))^{1/2}}{h} dh = 0.$$

Case B. When $\rho_l > \rho_m$ and $\rho_r > \rho_m$, eliminating u_m from (2.2.9) and (2.2.10), we get

$$u_r - u_l + \int_{\rho_l}^{\rho_m} \frac{(p'(h))^{1/2}}{h} dh + \int_{\rho_r}^{\rho_m} \frac{(p'(h))^{1/2}}{h} dh = 0.$$

Case C. When $\rho_l > \rho_m$ and $\rho_r < \rho_m$, eliminating u_m from (2.2.9) and (2.2.5), we get

$$u_r - u_l + \int_{\rho_l}^{\rho_m} \frac{(p'(h))^{1/2}}{h} dh + \varphi(\rho_m, \rho_r) = 0.$$

Case D. For $\rho_l < \rho_m$ and $\rho_r < \rho_m$, then from (2.2.5) we get

$$u_r - u_l + \varphi(\rho_l, \rho_m) + \varphi(\rho_m, \rho_r) = 0.$$

All the above four cases are combined as

$$f_r\left(\rho_m, U_r\right) + f_l\left(\rho_m, U_l\right) + u_r - u_l = 0,$$

where

$$f_r\left(\rho_m, U_r\right) = \begin{cases} \varphi\left(\rho_r, \rho_m\right) & \text{if } \rho_m > \rho_r \\ \frac{(k\Gamma)^{1/2}(\Gamma-1)}{4(\Gamma-2)}\left[\left(\frac{\rho_r}{1-\theta\rho_r}\right)^{2(\Gamma-2)} - \left(\frac{\rho_m}{1-\theta\rho_m}\right)^{2(\Gamma-2)}\right] & \text{if } \rho_m < \rho_r \end{cases},$$

and

$$f_l\left(\rho_m, U_l\right) = \begin{cases} \varphi\left(\rho_l, \rho_m\right) & \text{if } \rho_m > \rho_l \\ \frac{(k\Gamma)^{1/2}(\Gamma-1)}{4(\Gamma-2)}\left[\left(\frac{\rho_l}{1-\theta\rho_l}\right)^{2(\Gamma-2)} - \left(\frac{\rho_m}{1-\theta\rho_m}\right)^{2(\Gamma-2)}\right] & \text{if } \rho_m < \rho_l \end{cases},$$

In the case of rarefaction waves, we have derived the solution inside the rarefaction wave. For 1-rarefaction wave, the slope of characteristics joining the points $(0,0)$ and (x,t) is

$$\frac{dx}{dt} = \frac{x}{t} = u - \sqrt{p'(\rho)}, \tag{2.4.1}$$

where u is function of ρ.

Since R_1 is constant along 1-rarefaction wave region, we have

$$u = u_l + \int_h^{h_l} \frac{(p'(h))^{1/2}}{h} dh.$$

From (2.4.1), we have

$$u_l + \int_\rho^{\rho_l} \frac{(p'(h))^{1/2}}{h} dh - \frac{x}{t} - \sqrt{p'(\rho)} = 0.$$

Above equation can be solved numerically by using Newton Raphson method.

The value of θ is chosen in such a way so that the volume fraction of solid particles does not exceed 5% level on left and right state in a dusty gas. Since Z/ρ is constant

Table 2.1: Initial data for Riemann Problem

Test	ρ_l	u_l	ρ_r	u_r
1	1.0	0.0	0.125	0.0
2	1.0	1.08	1.8	1.11
3	1.0	-2.0	1.0	2.0
4	5.99924	19.5975	5.99242	-6.19633

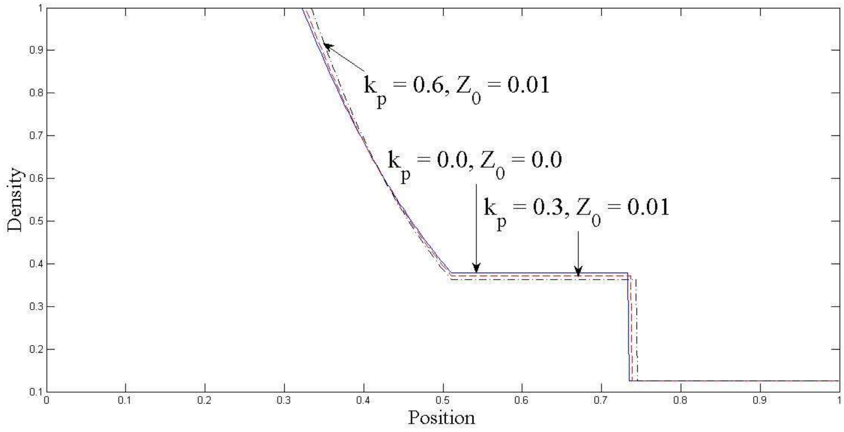

Figure 2.3: Density profiles for test 1 with varying k_p.

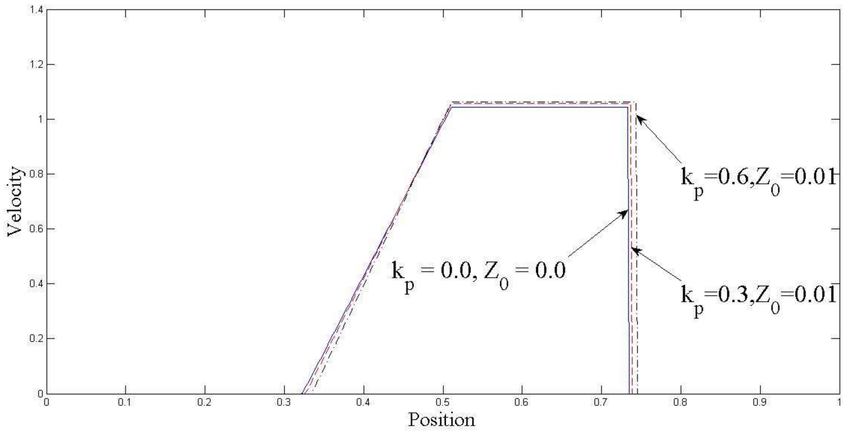

Figure 2.4: Velocity profiles for test 1 with varying k_p

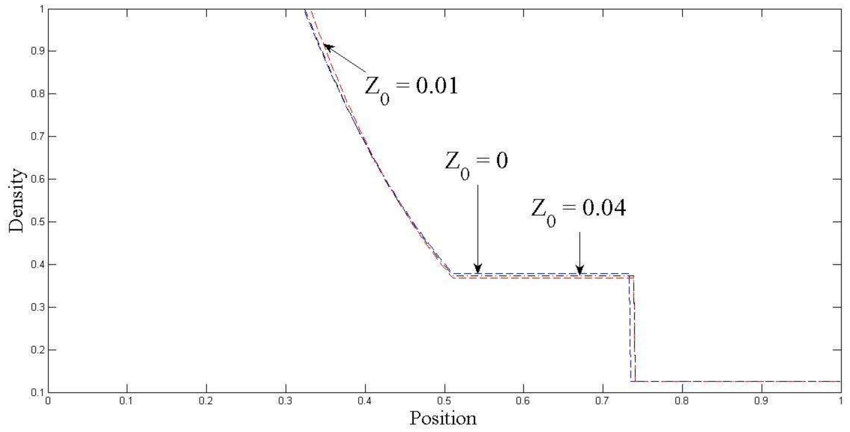

FIGURE 2.5: Density profiles for test 1 with varying Z_0

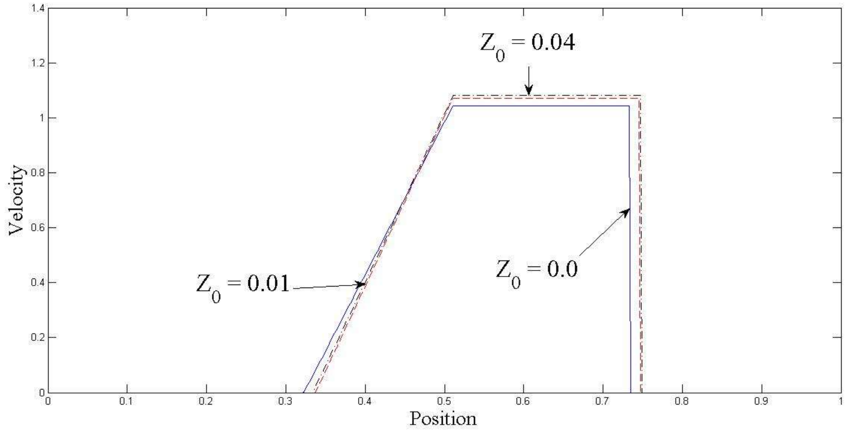

FIGURE 2.6: Velocity profiles for test 1 with varying Z_0

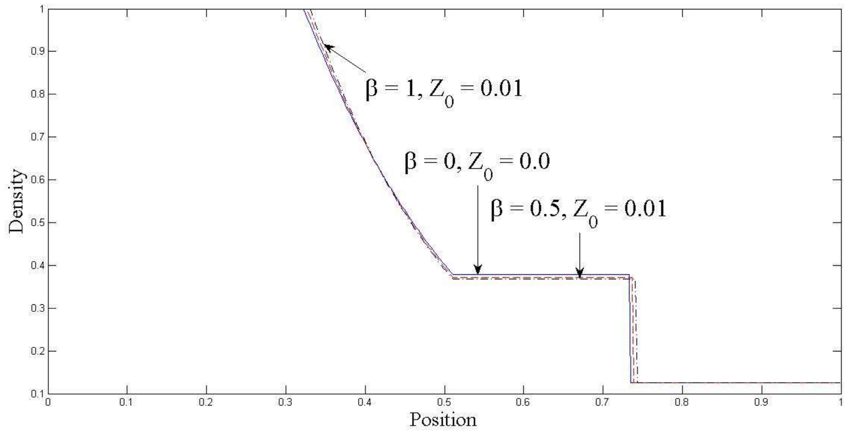

FIGURE 2.7: Density profiles for test 1 with varying β

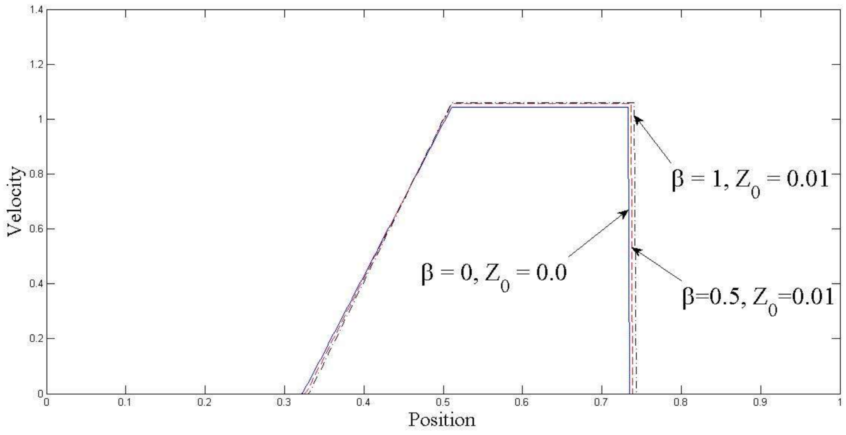

FIGURE 2.8: Velocity profiles for test 1 with varying β

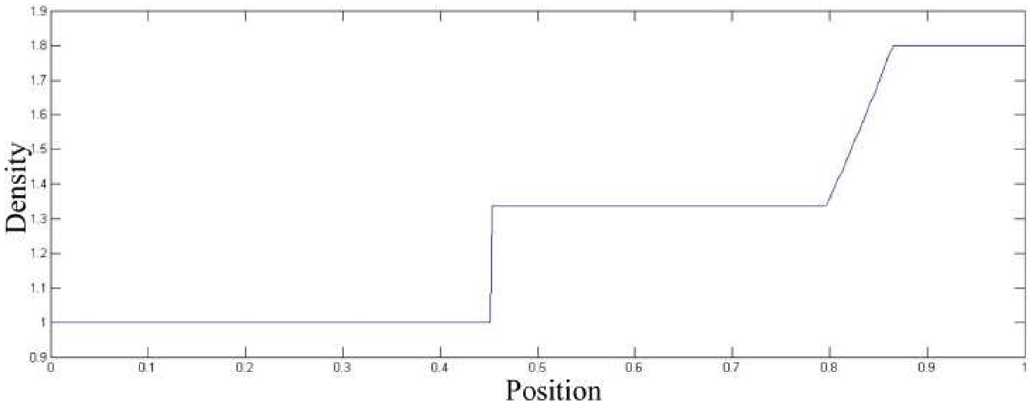

FIGURE 2.9: Density profiles for test 2

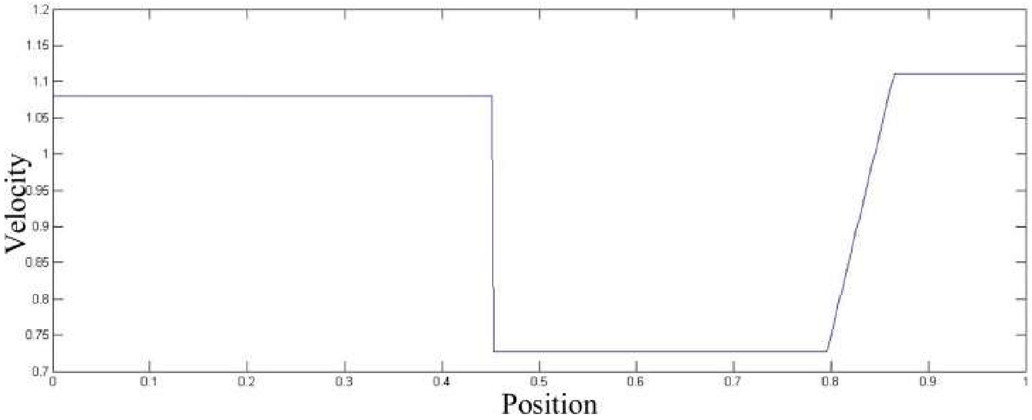

FIGURE 2.10: Velocity profiles for test 2

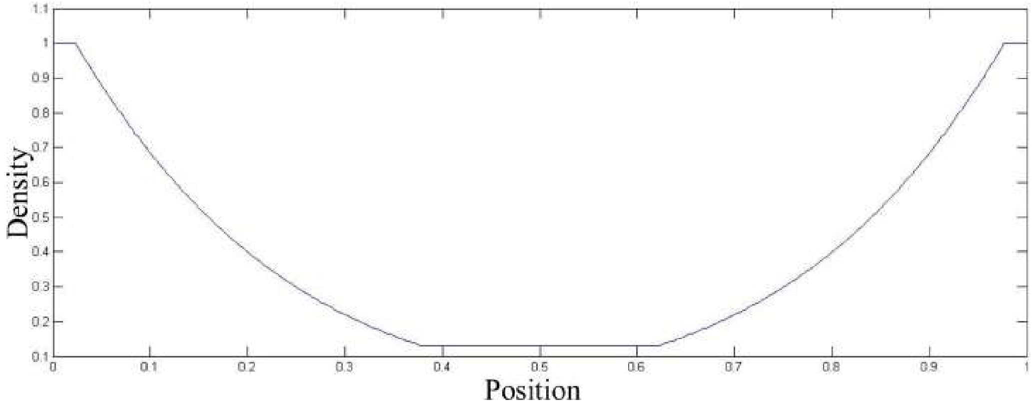

FIGURE 2.11: Density profiles for test 3

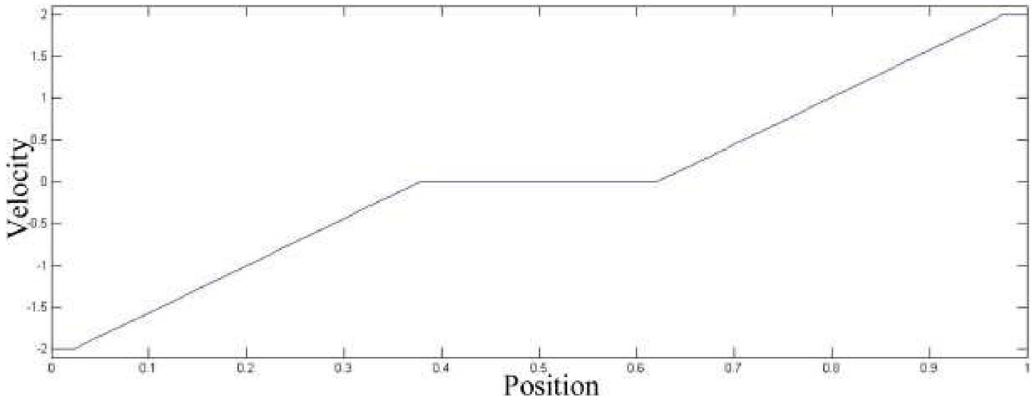

FIGURE 2.12: Velocity profiles for test 3

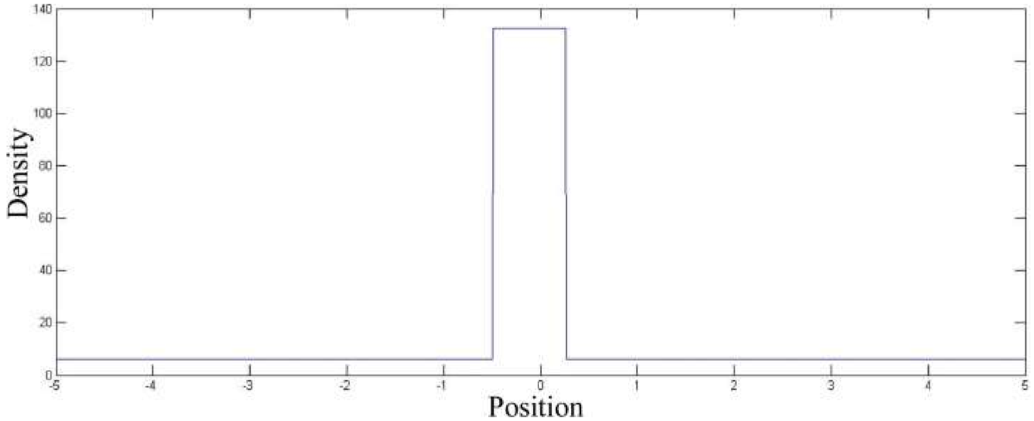

FIGURE 2.13: Density profiles for test 4

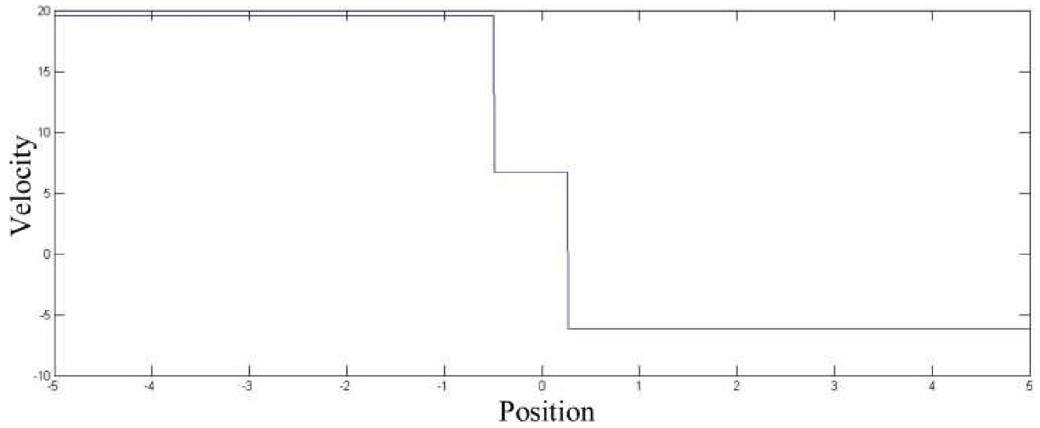

FIGURE 2.14: Velocity profiles for test 4

so an obvious choice is $\theta = Z/\rho_0$ where $\rho_0 = Max\,(\rho_l, \rho_r)$.

For case A, the solution profiles of the Riemann problem consist of 1-rarefaction wave and 2-shock wave and the corresponding solution profiles for Test 1 at $t = 0.15$ are shown in Figs. 2.3 – 2.8.

In Figs. 2.3 – 2.4, the solution profiles correspond to constant $\beta = 0.5$ with varying mass fraction at $k_p = 0.0$ (which corresponds to ordinary gas), $k_p = 0.3$, 0.6. It is observed that the increasing values of mass concentration of solid particles in gas causes the head of rarefaction wave move faster and shock to become weaker as compared to ordinary gas case.

Figs. 2.5 and 2.6 correspond to contribution of constant specific heat and mass fraction of solid particles having numerical values, $\beta = 0.8, k_p = 0.3, \delta = 0.4286$, with varying volume fraction of solid particles, $Z = 0.0$ (corresponds to ordinary gas), $Z = 0.01$, 0.04. Here, the increasing values of the volume fraction of solid particles in dusty gas leads to head of the rarefaction wave move slower and shock wave to become stronger.

Figs. 2.7 and 2.8 correspond to solution profiles at constant $Z = 0.01$ and $k_p = 0.3$ with varying value of $\beta = 0.0$ (which corresponds to ordinary gas), $\beta = 0.5$, 1.0.

Here, it is observed that an increase in the value of specific heat of solid particles in dusty gas causes the head of rarefaction wave move faster and shock becomes weaker.

For case B, the solution profiles, shown in Fig 2.9 and 2.10, consist of 1-shock wave and 2-rarefaction wave of a dusty gas for Test 2 at $t = 0.15s$. The value of the constants appearing in the computations are taken as $\beta = 0.8, \theta = 0.01, \gamma = 1.4, \delta = 0.4286$. Since the effect of dust particles for this case is similar to the case A, so details are omitted.

For case C, the solution profiles, shown in Fig 2.11 and 2.12, consist of 1-rarefaction wave and 2-rarefaction wave of a dusty gas for Test 3 at $t = 0.15s$. The value of the constants appearing in the computations are taken as $\beta = 0.8, \theta = 0.01, \gamma = 1.4, \delta = 0.111111$. Since the effect of dust particles for this case also is similar to the case A, so details are omitted.

For case D, the solution profiles, shown in Fig 2.13 and 2.14, consists of 1-shock wave and 2-shock wave of a dusty gas for Test 4, at $t = 0.035s$ The value of the constants appearing in the computations are taken as $\beta = 0.8, \theta = 0.001667, \gamma = 1.4, \delta = 0.111111$. Since effect of dust particles for this case is similar to the case A, so details are omitted.

2.5 Interaction of Elementary Waves

In this section, the interaction of elementary waves obtained from the Riemann problem described by (2.1.1) is discussed. We consider the following initial condition, with two jump discontinuities at x_1 and x_2, as:

$$U(x,0) = \begin{cases} U_l = (\rho_l, u_l) & \text{if } -\infty < x < x_1, \\ U_* = (\rho_*, u_*) & \text{if } x_1 < x < x_2, \\ U_r = (\rho_r, u_r) & \text{if } x_2 < x < \infty. \end{cases}$$

For the above initial data we have two Riemann problems. For appropriate selection of (ρ_*, u_*) and (ρ_r, u_r) in terms of (ρ_l, u_l) and arbitrary x_1 and $x_2 \in R$, the elementary wave of one dimensional Riemann problem may interact with elementary wave of another Riemann problem which results in a new Riemann problem at the time of interaction of elementary waves. So, by knowing the process of solving Riemann problem we will actually be able to solve the certain type of interaction problem. For one dimensional Euler equations in ideal gas and magnetogasdynamics, analytical solution and interaction of elementary waves may be found in Smoller (1994), Chang and Hsiao (1989), Liu (2012), Sekhar and Sharma (2010), Liu and Sun (2013, 2016). It may be noted that the analytical solution for the problem of interaction of elementary waves for one dimensional Euler equations in a dusty gas has not been attempted till now. Here, we use the notation $R_2S_1 \to S_1R_2$, which means that a 2-rarefaction wave, R_2, of first Riemann problem (connecting (u_l, ρ_l) to (u_*, ρ_*)) interacts with 1-shock, S_1, of second Riemann problem (connecting (u_*, ρ_*) to (u_r, ρ_r)) and after interaction we have a new Riemann problem connecting (u_l, ρ_l) to (u_r, ρ_r) via (u_*, ρ_*) and its solution consists of 1-shock and 2-rarefaction. To solve this problem we need to determine that (u_r, ρ_r) lies in which region with respect to (u_l, ρ_l) see Fig.2.1. The discussion of interaction of elementary waves is divided into two subsections. In first section, interactions of elementary waves belonging to different families (S_2S_1, S_2R_1, R_2R_1, R_2S_1) is discussed and second, to same family (S_2S_2, S_1S_1, S_1R_1, R_1S_1, S_2R_2, R_2S_2) is discussed.

2.5.1 Interaction of elementary waves from different families

2.5.1.1 Collision of two shocks (S_2S_1)

Let $U_* \in S_2(U_l)$ and $U_r \in S_1(U_*)$, i.e. for a given (ρ_l, u_l) we choose (ρ_*, u_*) and (ρ_r, u_r) such that $\rho_* < \rho_l$, $u_* = u_l - \varphi(\rho_{l,}\rho_*)$ and $\rho_* < \rho_r$, $u_r = u_* - \varphi(\rho_r, \rho_*)$. Since s_2, speed of S_2 is positive and s_1, speed of S_1 is negative. So S_1 collides with S_2 after some time. To prove that for any arbitrary state (ρ_l, u_l), the state (ρ_r, u_r) lies in the region IV (see Fig.2.1), it is enough to show that $\varphi(\rho_*, \rho) - \varphi(\rho_l, \rho) + \varphi(\rho_l, \rho_*) > 0$, for $\rho_* < \rho_l$ and $\rho_* < \rho$.

Let $\varphi(\rho_*, \rho) - \varphi(\rho_l, \rho) + \varphi(\rho_l, \rho_*) \leq 0$, then

$$\varphi^2(\rho_*, \rho) + \varphi^2(\rho_l, \rho_*) + 2\varphi(\rho_*, \rho)\varphi(\rho_l, \rho_*) \leq \varphi^2(\rho_l, \rho).$$

This gives

$$(p_* - p)\left(\frac{\rho_* - \rho_l}{\rho_*\rho_l}\right) + (p_* - p_l)\left(\frac{\rho_* - \rho}{\rho_*\rho}\right) + 2\varphi(\rho_l, \rho_*)\varphi(\rho_*, \rho) \leq 0. \tag{2.5.1}$$

But left hand side of (2.5.1) is strictly positive, which contradicts to our assumption. Therefore, the curve S_1 (ρ_*, u_*) lies below the curve $S_1(\rho_l, u_l)$, hence (u_r, ρ_r) lies in region IV. Results are shown in fig. 2.15.

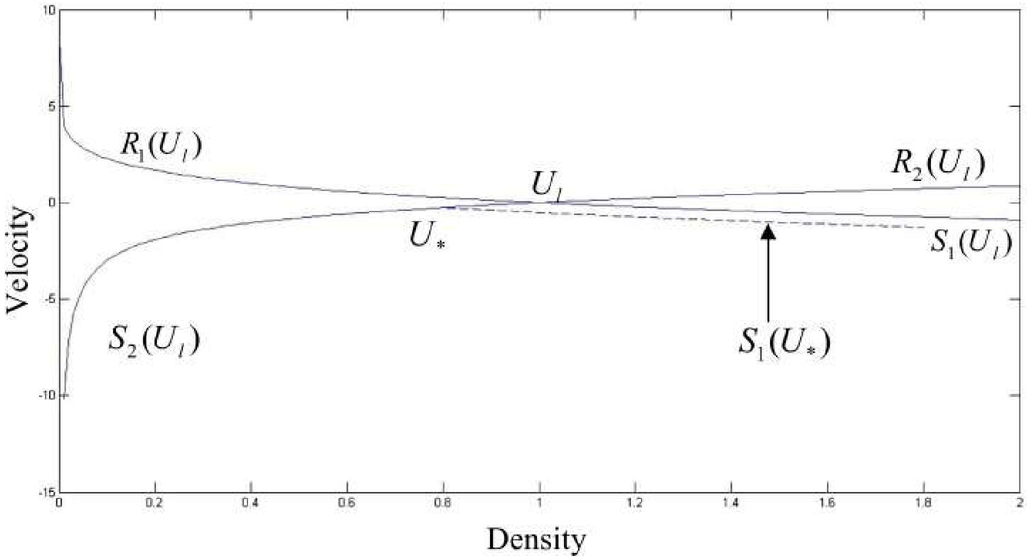

FIGURE 2.15: Collision of S_2S_1

2.5.1.2 Collision of a shock and rarefaction (S_2R_1)

Let $U_* \in S_2(U_l)$ and $U_r \in R_1(U_*)$. Since velocity of S_2 is positive and velocity of R_1 is negative. So, R_1 interacts with S_2 after some time $t > 0$. For given U_l and $\rho < \rho_* < \rho_l$ we have $\int_{\rho}^{\rho_l} \frac{p'(h)}{h} dh - \int_{\rho}^{\rho_*} \frac{p'(h)}{h} dh + \varphi(\rho_l, \rho_*) > 0$, it shows that the curve $R_1(U_*)$ lies below the curve $R_1(U_l)$, therefore U_r lies in III quadrant hence $S_2R_1 \to R_1S_2$. Results are shown in Fig.2.16.

2.5.1.3 Collision of two rarefaction (R_2R_1)

Let $U_* \in R_2(U_l)$ and $U_r \in R_1(U_*)$. Since the 2-rarefaction has positive velocity and 1-rarefaction has negative velocity therefore they interact after some time $t > 0$. For $\rho < \rho_* < \rho_l$ and given U_l, we have $\int_{\rho}^{\rho_*} \frac{(p'(h))^{1/2}}{h} dh - \int_{\rho}^{\rho_l} \frac{(p'(h))^{1/2}}{h} dh + \int_{\rho_l}^{\rho_*} \frac{(p'(h))^{1/2}}{h} dh > 0$, which implies that the curve $R_1(U_*)$ lies above the curve $R_1(U_l)$, so U_r lies in III quadrant, hence $R_2R_1 \to R_1R_2$. Results are shown in Fig 2.17.

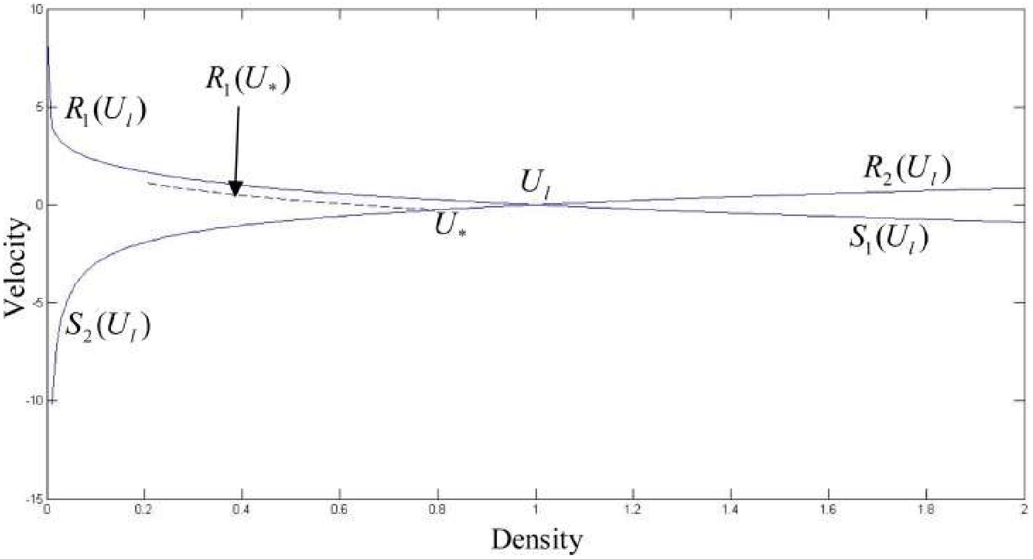

FIGURE 2.16: Collision of S_2R_1

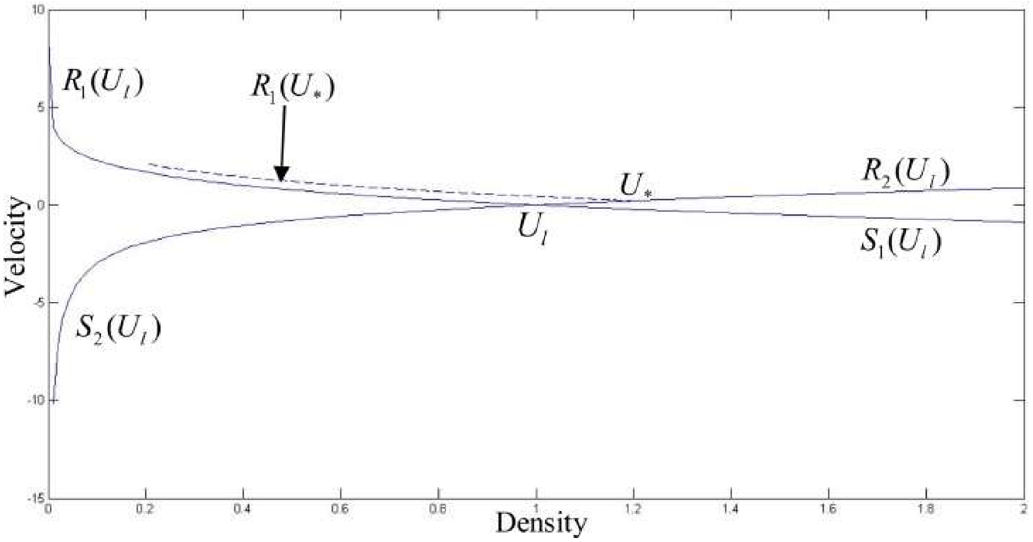

FIGURE 2.17: Collision of S_2R_1

2.5.1.4 Collision of two rarefaction (R_2S_1)

Let $U_* \in S_2(U_l)$ and $U_r \in R_1(U_*)$. Since velocity of S_2 is positive and velocity of R_1 is negative. So, R_1 interacts with S_2. For given U_l and $\rho < \rho_* < \rho_l$, we have $\int_{\rho}^{\rho_l} \frac{p'(h)}{h}dh - \int_{\rho}^{\rho_*} \frac{p'(h)}{h}dh + \varphi(\rho_l, \rho_*) > 0$, which implies that the curve $R_1(U_*)$ lies below the curve $R_1(U_l)$, therefore U_r lies in III quadrant hence $R_2S_1 \to S_1R_2$. Results are shown in Fig 2.18.

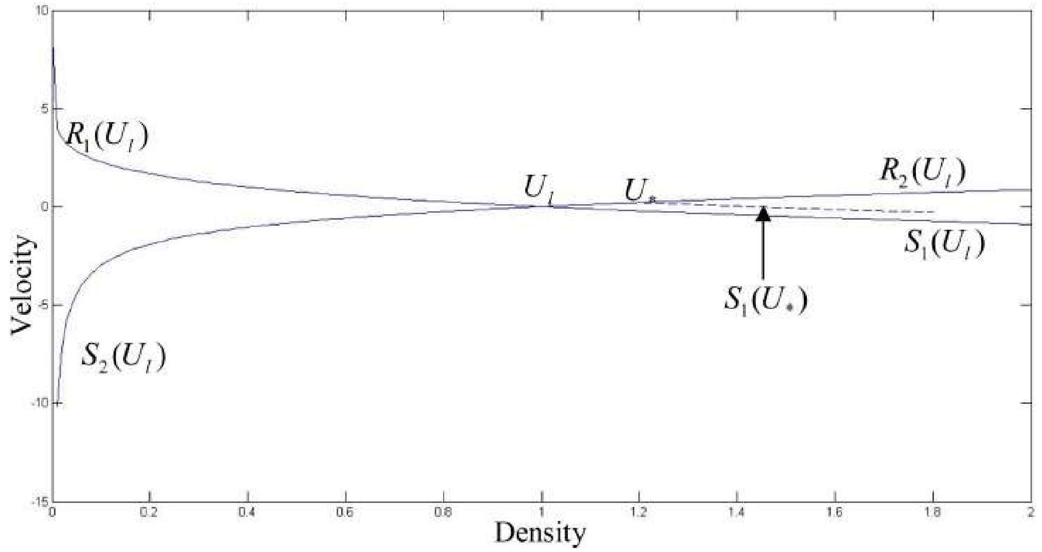

FIGURE 2.18: Collision of R_2S_1

2.5.2 Interaction of elementary waves from same families

2.5.2.1 2-shock wave overtaking another 2-shock wave (S_2S_2)

Let (ρ_l, u_l) be connected to (ρ_*, u_*) by 2-shock, S_2 with speed s_2 of first Riemann problem and (ρ_*, u_*) be connected to (ρ_r, u_r) by 2-shock S_2 with speed s_2 of second Riemann problem. By Lax stability condition, we have

$$\lambda_1(U_l) < s_2(U_l, U_*) < \lambda_2(U_l), \qquad \lambda_2(U_*) < s_2(U_l, U_*),$$

$$\lambda_1(U_*) < s_2(U_*, U_r) < \lambda_2(U_*), \qquad \lambda_2(U_r) < s_2(U_*, U_r).$$

From the above inequalities, it is clear that $s_2(U_*, U_r) < s_1(U_l, U_*)$. So after some time $t > 0$, interaction between elementary waves occurs. We now show that for given (ρ_l, u_l), (ρ_r, u_r) lies in the III-quadrant. For this, it is enough to show that

$$\varphi(\rho_l, \rho) - \varphi(\rho_*, \rho) - \varphi(\rho_l, \rho_*) > 0, \qquad \rho < \rho_* < \rho_l.$$

Let $\varphi(\rho_l,\rho)-\varphi(\rho_*,\rho)-\varphi(\rho_l,\rho_*)\leq 0$, $\rho<\rho_*<\rho_l$, then we have $\varphi^2(\rho_l,\rho)+\varphi^2(\rho_*,\rho)-2\varphi(\rho_l,\rho)\varphi(\rho_*,\rho)\leq\varphi^2(\rho_l,\rho_*)$, which is equivalent to

$\left(\frac{1}{\rho_l}-\frac{1}{\rho_*}\right)(p-p_l)+\left(\frac{1}{\rho_l}-\frac{1}{\rho}\right)(p_*-p_l)\leq 2\varphi(\rho_l,\rho)\varphi(\rho_*,\rho)$, which after simplification yields

$$\left[\left(\frac{1}{\rho_l}-\frac{1}{\rho_*}\right)(p-p_l)+\left(\frac{1}{\rho_l}-\frac{1}{\rho}\right)(p_*-p_l)\right]^2\leq 0.$$

But LHS of the above inequality is strictly positive, so our assumption is wrong. Therefore, $S_2S_2\rightarrow R_1S_2$. Results are shown in Fig.2.19.

2.5.2.2 1-shock wave overtaking another 1-shock wave (S_1S_1)

The Proof of (ρ_r,u_r) lies in I quadrant is parallel to the previous case, so details are omitted. Result is $S_1S_1\rightarrow S_1R_1$.

2.5.2.3 1-rarefaction wave overtaking 1-shock wave (R_1S_1)

Let (ρ_l,u_l) be connected to (ρ_*,u_*) by 1-rarefaction of the first Riemann problem and (ρ_*,u_*) connected to (ρ_r,u_r) by 1-shock of second Riemann problem. First, we show that $S_1(U_*)$ lies below the curve $R_1(U_l)$, for $\rho_*<\rho\leq\rho_l$, to prove this, we show that for a given (ρ_l,u_l) and $\rho_*<\rho\leq\rho_l$, we have $\varphi(\rho_*,\rho)+\int_{\rho}^{\rho_l}\frac{(p'(h))^{1/2}}{h}dh-\int_{\rho_*}^{\rho_l}\frac{(p'(h))^{1/2}}{h}dh>0$. Let $f_1(\rho)=\varphi(\rho_*,\rho)+\int_{\rho}^{\rho_l}\frac{(p'(h))^{1/2}}{h}dh-\int_{\rho_*}^{\rho_l}\frac{(p'(h))^{1/2}}{h}dh$, then $f_1(\rho_*)=0$ and $f_1'(\rho)>0$, which implies that $0=f_1(\rho_*)<f_1(\rho)$. Thus $S_1(U_*)$ lies below the curve $R_1(U_l)$, for $\rho_*<\rho\leq\rho_l$.

Next we show that $S_1(U_l)$ lies above the curve $S_1(U_*)$, for $\rho_l\leq\rho$. For this, we show that

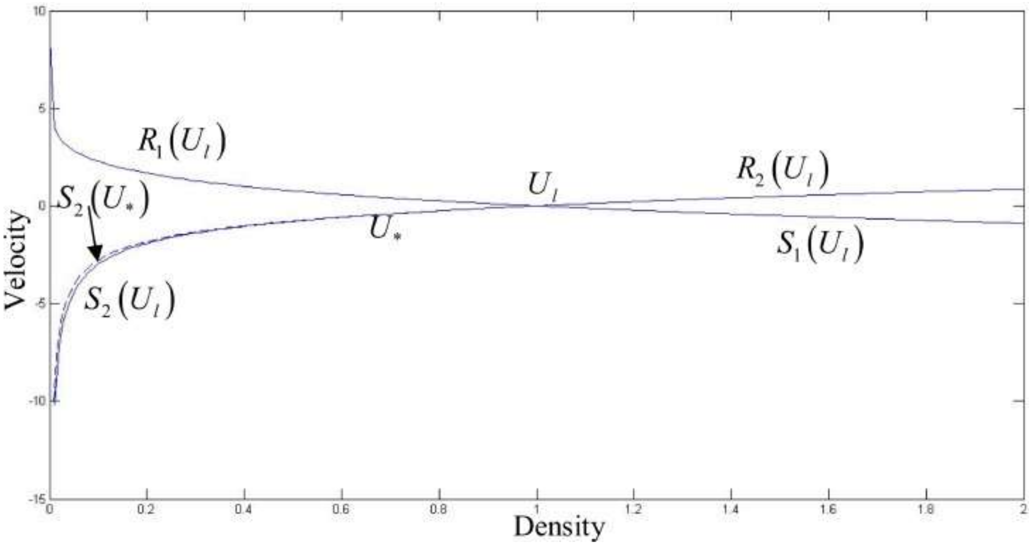

FIGURE 2.19: S_2 overtakes S_2

$$\varphi(\rho_*,\rho)-\varphi(\rho_l,\rho)-\int_{\rho_*}^{\rho_l}\frac{(p'(h))^{1/2}}{h}dh>0,\quad forall \rho_l\le\rho.$$

Let $f_2(\rho)=\varphi(\rho_*,\rho)-\varphi(\rho_l,\rho)-\int_{\rho_*}^{\rho_l}\frac{(p'(h))^{1/2}}{h}dh$ then $f_2(\rho_l)=f_1(\rho_l)>0$.

Let $\varphi(\rho_*,\rho)-\varphi(\rho_*,\rho_l)\le\varphi(\rho_l,\rho)$ for $\rho_*<\rho<\rho_l$ then

$$\varphi^2(\rho_*,\rho)+\varphi^2(\rho_*,\rho_l)-2\varphi(\rho_*,\rho)\varphi(\rho_*,\rho_l)\le\varphi^2(\rho_l,\rho),$$

which gives

$$\left[(p_l-p_*)\,(\frac{1}{\rho_*}-\frac{1}{\rho})-(p-p_*)\,(\frac{1}{\rho_*}-\frac{1}{\rho_l})\right]\le 0,$$

which contradicts our assumption, so $\varphi(\rho_*,\rho)-\varphi(\rho_l,\rho)>\varphi(\rho_*,\rho_l)$, for $\rho_*<\rho<\rho_l$, which implies that $\varphi(\rho_*,\rho)-\varphi(\rho_l,\rho)-\int_{\rho_*}^{\rho_l}\frac{(p'(h))^{1/2}}{h}dh>\varphi(\rho_*,\rho_l)-\int_{\rho_*}^{\rho_l}\frac{(p'(h))^{1/2}}{h}dh=f_2(\rho_l)>0$.

Next we show that $S_1(U_*)$ and $S_2(U_l)$ intersect at some point (ρ_1,u_1).

Let $f_3(\rho)=\varphi(\rho_*,\rho)-\varphi(\rho_l,\rho)-\int_{\rho_*}^{\rho_l}\frac{(p'(h))^{1/2}}{h}dh$, for $\rho_*\le\rho\le\rho_l$, then $f_3(\rho_l)>0$ and $f_3(\rho_*)<0$. So by IMVT there exist a ρ_1 between ρ_* and ρ_l such that $f_3(\rho_1)=0$,

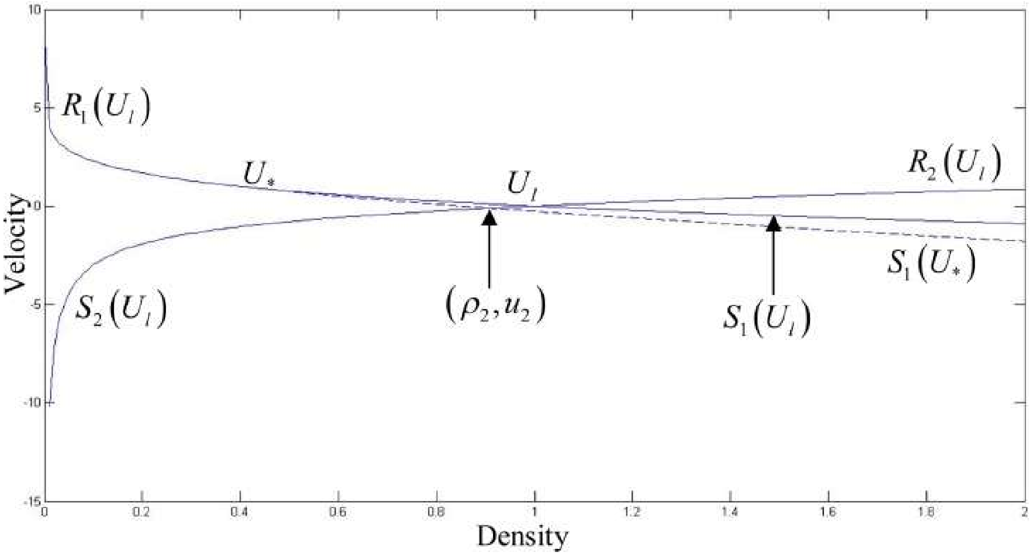

FIGURE 2.20: R_1 overtakes S_1

thus intersection of $S_1(U_*)$ and $S_2(U_l)$ are uniquely determined by ρ_r, see Fig.2.20. We separate the above conversation into three cases as:

(a) if $\rho_r < \rho_1$, then U_r lies in III-quadrant and interaction result is $R_1S_1 \to R_1S_2$.

(b) If $\rho_r = \rho_1$ then U_r lies on $S_2(U_l)$ and the result is $R_1S_1 \to S_2$ indeed when two wave of first family interact then they destroy to each other and gives a wave of second kind.

(c) If $\rho_r > \rho_1$ then U_r lies in IV-quadrant and interaction result is $R_1S_1 \to S_1S_2$.

2.5.2.4 1-shock wave overtaking 1-rarefaction wave (S_1R_1)

The result of overtaking 1-shock wave by 1-rarefaction wave are summarized as below, detailed proofs are omitted due to similarity with the previous case.

There exists a ρ_2 satisfying $\rho_2 < \rho_l < \rho_*$ such that

(a) If $\rho_r > \rho_2$, then U_r lies in II-quadrant and the interaction result is $S_1R_1 \to S_1S_2$.

(b) If $\rho_r = \rho_2$ then U_r lies on $S_2(U_l)$ and the result is $S_1R_1 \to S_2$ i.e. when two waves of first family interact then they destroy to each other and gives a wave of second kind.

(c) If $\rho_r < \rho_2$ then in U_r lies in III-quadrant and the interaction result is $S_1R_1 \to R_1S_2$.

2.5.2.5 2-shock wave overtaking 2-rarefaction wave (S_2R_2)

Let (ρ_l, u_l) be connected to (ρ_*, u_*) by 2-shock of the first Riemann problem and (ρ_*, u_*) connected to (ρ_r, u_r) by 2-rarefaction of second Riemann problem. First, we show that $R_2(U_*)$ lies below the curve $S_2(U_l)$, for $\rho_* < \rho \leq \rho_l$. For this, we show that for a given (ρ_l, u_l) and $\rho_* < \rho \leq \rho_l$, we have $\varphi(\rho_l, \rho_*) + \varphi(\rho_l, \rho) - \int_{\rho_*}^{\rho} \frac{(p'(h))^{1/2}}{h} dh > 0$.

Let $g_1(\rho) = \varphi(\rho_l, \rho_*) + \varphi(\rho_l, \rho) - \int_{\rho_*}^{\rho} \frac{(p'(h))^{1/2}}{h} dh$ then $g_1(\rho_*) = 0$ and $g_1'(\rho) > 0$, which implies that $g_1(\rho_*) < g_1(\rho)$ i.e. $g_1(\rho) > 0$. Thus $R_2(U_*)$ lies below the curve $S_2(U_l)$, for $\rho_* < \rho \leq \rho_l$.

Next we show that $R_2(U_l)$ lies above the curve $R_2(U_*)$ for $\rho_l \leq \rho$. For this, we show that

$$\varphi(\rho_l, \rho_*) + \int_{\rho_*}^{\rho} \frac{(p'(h))^{1/2}}{h} dh - \int_{\rho_l}^{\rho} \frac{(p'(h))^{1/2}}{h} dh > 0, \quad \forall \rho_* < \rho_l \leq \rho.$$

Since LHS of this inequality is $g_1(\rho_l)$ which has been previously shown to be positive. Hence $R_2(U_l)$ lies above the curve $R_2(U_*)$, for $\rho_l \leq \rho$.

Next we show that $R_2(U_*)$ and $S_1(U_l)$ intersect at a unique point say (ρ_3, u_3).

Let $g_2(\rho) = \varphi(\rho_l, \rho) - \varphi(\rho_l, \rho_*) + \int_{\rho_*}^{\rho} \frac{(p'(h))^{1/2}}{h} dh$, for $\rho_* \leq \rho_l \leq \rho$ then $g_2(\rho_l) < 0$ and we can choose a constant $k > 0$ such that $g_2(\rho) > 0$, for all $\rho > k$. Therefore, by

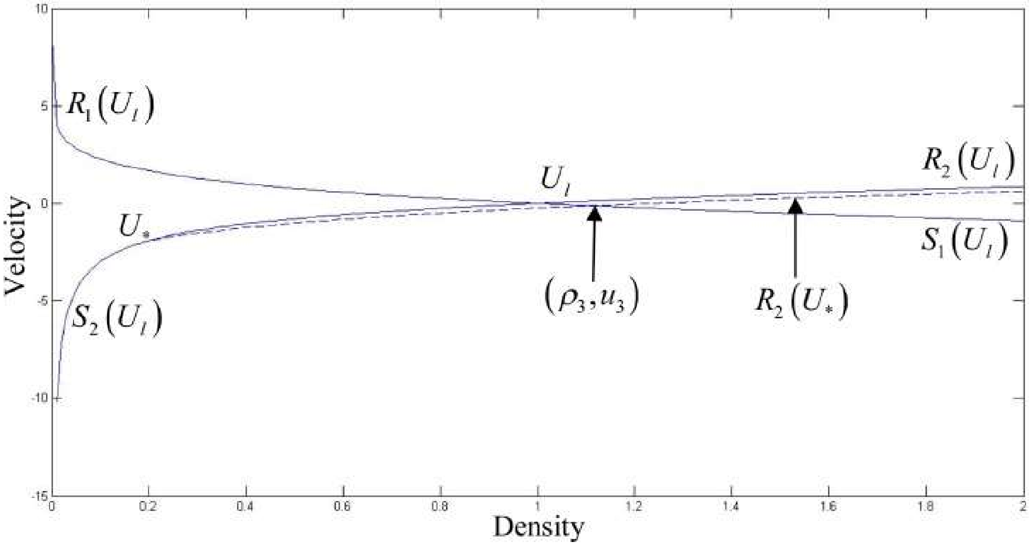

FIGURE 2.21: S_2 overtakes R_2

IMVT there exist a ρ_3 between ρ and ρ_l such that $g_2(\rho_3) = 0$, thus intersection of $R_2(U_*)$ and $S_1(U_l)$ are uniquely determined by ρ_r. Computed results are shown in Fig. 2.21, we can divide the above discussion into three cases as

(a) if $\rho_r < \rho_3$, then U_r lies in II-quadrant and interaction result is $S_2R_2 \to S_1S_2$.

(b) if $\rho_r = \rho_3$ then U_r lies on $S_1(U_l)$ and the result is $S_2R_2 \to S_1$ indeed when two waves of second family interact then they destroy to each other and gives a wave of first kind.

(c) if $\rho_r > \rho_3$ then U_r lies in I-quadrant and interaction result is $S_2R_2 \to S_1R_2$.

2.5.2.6 2-rarefaction wave overtaking 2-shock wave (R_2S_2)

The result of overtaking 2-rarefaction wave by 2-shock wave are precisely given as below, detailed proofs are omitted due to similarity with the previous case.

There exists a ρ_4 between ρ_* and ρ_l such that

(a) If $\rho_r > \rho_4$, then $U_r$lies in I-quadrant and interaction result is $R_2S_2 \to S_1R_2$.

(b) If $\rho_r = \rho_4$ then U_r lies on $S_1(U_l)$ and the result is $R_2S_2 \rightarrow S_1$, i.e. when two waves of first family interact then they destroy to each other and gives a wave of second kind.

(c) If $\rho_r < \rho_4$ then in U_r lies in II-quadrant and interaction result is $R_2S_2 \rightarrow S_1S_2$.

2.6 Conclusion

In the present chapter, the Riemann problem and elementary wave Interaction in a dusty gas is discussed and following conclusions may be drawn from the above discussion.

- Shock wave and its properties are investigated in section 2.2.1.
- Rarefaction wave and its properties are investigated in section 2.2.2.
- The Uniqueness and existence of the solution of considered Riemann problem is discussed.
- Solution of Riemann problem for considered problem has obtained.
- All possible interactions of elementary waves are discussed.

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