Chapter 3

An asymptotic expansion for a Lambert series associated to Siegel cusp forms of degree n

In this chapter, we extend the result of Chapter 2 to Siegel modular forms of arbitrary degree $n > 1$.

3.1 Introduction

The main aim of this chapter is to study certain Lambert series involving Fourier-Jacobi coefficients of Siegel cusp forms of arbitrary degree n when $n > 1$. More specifically, we study an asymptotic expansion of a Lambert series associated to Siegel cusp forms of degree n involving their Fourier-Jacobi coefficients with a twist by a Dirichlet character χ. Quite interestingly, we observe a connection with the non-trivial zeros of the Dirichlet L-function $L(s, \chi^2)$. More precisely, let χ be a Dirichlet character modulo N and $F, G \in S_k(\Gamma_n)$ with their Fourier-Jacobi expansions as follows:

$$F(Z) = \sum_{m=1}^{\infty} \phi_m(Z_1, w) e(mz), \qquad G(Z) = \sum_{m=1}^{\infty} \psi_m(Z_1, w) e(mz). \tag{3.1}$$

For $\alpha > 0$, we study the following Lambert series:

$$\sum_{m=1}^{\infty} \chi(m) \langle \phi_m, \psi_m \rangle \exp(-4\pi m \alpha). \tag{3.2}$$

Before stating the main result of this paper, we first introduce a few notations. First we define the Rankin-Selberg type Dirichlet series associated to Siegel cusp forms of degree n. Let $F, G \in S_k(\Gamma_n)$ with their Fourier-Jacobi expansions as given in (3.1) The Dirichlet series associated to F and G is defined as follows:

$$\mathcal{D}_{F,G}(s) := \sum_{m=1}^{\infty} \frac{\langle \phi_m, \psi_m \rangle}{m^s}. \tag{3.3}$$

The analytic properties of (3.3) have been studied by Krieg [46]. Next we define the twist of the series in (3.3) by a Dirichlet character. Let N be a positive integer and χ a Dirichlet character modulo N. The twist of the series $\mathcal{D}_{F,G}(s)$ by χ is defined as

$$\mathbb{D}_{F,G,\chi}(s) := \sum_{m=1}^{\infty} \chi(m) \frac{\langle \phi_m, \psi_m \rangle}{m^s}. \tag{3.4}$$

The analytic properties of $\mathbb{D}_{F,G,\chi}(s)$ were studied by Kohnen [42] for degree 2. Later, Kohnen, Krieg and Sengupta [43] studied the analytic properties of $\mathbb{D}_{F,G,\chi}(s)$ for arbitrary degree $n > 1$.

In view of the (1.22), we see that the series (3.3) and (3.4) are absolutely convergent for $\Re(s) > k$. We complete the Dirichlet series $\mathcal{D}^*_{F,G}(s)$ and $\mathbb{D}^*_{F,G,\chi}(s)$ associated to F and G as follows:

$$\mathcal{D}^*_{F,G}(s) := (2\pi)^{-2s}\Gamma(s)\Gamma(s+n-k)\zeta(2s+2n-2k)D_{F,G}(s), \tag{3.5}$$

$$\mathbb{D}^*_{F,G,\chi}(s) := \left(\frac{2\pi}{N}\right)^{-2s} \Gamma(s)\Gamma(s+n-k)L(2s+2n-2k,\chi^2)\mathbb{D}_{F,G,\chi}(s). \tag{3.6}$$

Next, for $\Re(s) > k$, we define an arithmetical function $a_{F,G}(m,\chi)$ that satisfies the following generating function:

$$\sum_{m=1}^{\infty} \frac{a_{F,G}(m,\chi)}{m^s} = \frac{L(2s-2k+2n,\bar{\chi}^2)\mathbb{D}_{F,G,\bar{\chi}}(s)}{L(2s-2k+1,\bar{\chi}^2)}, \tag{3.7}$$

where $\mathbb{D}_{F,G,\chi}(s)$ is the twisted Rankin-Selberg L-function defined in (3.4). Note that for $N = 1$, $\bar{\chi}(t) = \bar{\chi}^2(t) = 1, \mathbb{D}_{F,G,\bar{\chi}}(s) = \mathcal{D}_{F,G}(s)$ and $L(s, \bar{\chi}^2) = \zeta(s)$.

With the above notation and definitions, we are now ready to state the main result of this chapter.

3.2 Statement of results

Theorem 3.1. *Let F and G be Siegel cusp forms of weight k and degree n with their Fourier-Jacobi expansions as given in (3.1) and satisfying (1.22). Let α and β be positive real numbers such that $\alpha\beta = 1$. Let χ be a Dirichlet character modulo N such that χ^2 is primitive. Under the assumption of the grand simplicity hypothesis of the non-trivial zeros of $L(s, \chi^2)$, we have*

$$\begin{aligned}&\sum_{m=1}^{\infty} \chi(m)\langle \phi_m, \psi_m\rangle \exp(-4\pi m\alpha)\\ &= \left(\frac{\beta^{2k-n}N^{2n-2k+2}}{\pi^{n-\frac{1}{2}}g(\bar{\chi})^4 g(\chi^2)}\right)\sum_{m=1}^{\infty} a_{F,G}(m,\chi)\left(\frac{4\pi m\beta}{N^2}\right)^{\frac{n-1-k}{2}} W_{\frac{n+k}{2}-1,\frac{k-n}{2}}\left(\frac{4\pi m\beta}{N^2}\right)\exp\left(\frac{-2\pi m\beta}{N^2}\right)\\ &+\sum_{\rho} \frac{\mathbb{D}^*_{F,G,\chi}\left(\frac{\rho}{2}+k-n\right)(4\pi\alpha)^{n-k-\frac{\rho}{2}}}{(2\pi/N)^{2n-2k-\rho}\Gamma(\frac{\rho}{2})L'(\rho,\chi^2)} + \mathcal{R}_k,\end{aligned}$$

where $W_{a,b}$ is the Whittaker function, the term $\mathcal{R}_k$ is defined as

$$\mathcal{R}_k = \begin{cases} \frac{(-1)^{n+1}2(2n)!\langle F,G\rangle}{(4\pi)^n\alpha^k(n-1)!B_{2n}} & \text{if } N = 1,\\ 0 & \text{otherwise,}\end{cases} \tag{3.8}$$

$L'(s, \chi^2)$ denotes the derivative of $L(s, \chi^2)$ and the sum runs over non-trivial zeros ρ of $L(s, \chi^2)$ satisfying the bracketing condition, i.e., the terms corresponding to ρ_1

and ρ_2 are included in the same bracket if they satisfy

$$|\Im(\rho_1) - \Im(\rho_2)| < \exp\left(-\frac{C_0|\Im(\rho_1)|}{\log(|\Im(\rho_1)|+3)}\right) + \exp\left(-\frac{C_0|\Im(\rho_2)|}{\log(|\Im(\rho_2)|+3)}\right), \quad (3.9)$$

where C_0 is some positive constant.

In particular, if we consider χ to be the trivial character with modulus $N = 1$, then we have the following result.

Corollary 3.1. *Let F, G, α and β be as in Theorem 3.1. Under the assumption of the grand simplicity hypothesis of the non-trivial zeros of $\zeta(s)$, we have*

$$\begin{aligned}
&\sum_{m=1}^{\infty} \langle \phi_m, \psi_m \rangle \exp(-4\pi m\alpha) \\
&= \frac{\beta^{2k-n}}{\pi^{n-\frac{1}{2}}} \sum_{m=1}^{\infty} a_{F,G}(m)(4\pi m\beta)^{\frac{n-1-k}{2}} W_{\frac{n+k}{2}-1, \frac{k-n}{2}}(4\pi m\beta)\exp(-2\pi m\beta) \\
&+ \frac{(-1)^{n+1}2(2n)!\langle F, G\rangle}{(4\pi)^n \alpha^k (n-1)! B_{2n}} + \sum_{\rho} \frac{\mathcal{D}^*_{F,G}\left(\frac{\rho}{2}+k-n\right)(4\pi\alpha)^{n-k-\frac{\rho}{2}}}{(2\pi)^{2n-2k-\rho}\Gamma(\frac{\rho}{2})\zeta'(\rho)},
\end{aligned} \quad (3.10)$$

where the sum runs over non-trivial zeros ρ of $\zeta(s)$ satisfying the bracketing condition (3.9) and B_{2n} denotes the $2n$-th Bernoulli number.

As an application of the above corollary, we have the following asymptotic expansion of the Lambert series in (3.10).

Corollary 3.2. *Let F and G be as in Theorem 3.1. If $\langle F, G\rangle \neq 0$, then for $\alpha \to 0^+$, we have*

$$\alpha^k \sum_{m=1}^{\infty} \langle \phi_m, \psi_m \rangle \exp(-4\pi m\alpha) \sim \frac{(-1)^{n+1}2(2n)!\langle F, G\rangle}{(4\pi)^n (n-1)! B_{2n}}. \quad (3.11)$$

3.3 Preparatory results

In this section, we state a few results without proof which will play a crucial role in the proof of Theorem 3.1.

The first result is about the analytic properties of $\mathcal{D}^*_{F,G}(s)$ proved by Krieg [46].

Theorem 3.2. *[46, p. 249, Theorem 1] Let F and G be Siegel cusp forms of weight k and degree n. Then $\mathcal{D}^*_{F,G}(s)$ is a holomorphic function of $s \in \mathbb{C}$ except for possible simple poles at $s = k$ and $s = k - n$ with residues $\pi^{n-k}\langle F, G\rangle$ and $-\pi^{n-k}\langle F, G\rangle$ respectively, and satisfies the functional equation $\mathcal{D}^*_{F,G}(2k - n - s) = \mathcal{D}^*_{F,G}(s)$.*

The above result of Krieg [46] was extended by Kohnen, Krieg and Sengupta [43] where the authors studied the analytic properties of $\mathbb{D}^*_{F,G,\chi}(s)$. More precisely, they proved the following result.

Theorem 3.3. *([43, p. 495, Theorem 1],[43, p. 497 Theorem 2]) Let F and G be Siegel cusp forms of weight k and degree n, and χ be a Dirichlet character modulo $N > 1$. If χ is not principal, then $\mathbb{D}^*_{F,G,\chi}(s)$ extends to an entire function of s. Further, if χ^2 is primitive, then $\mathbb{D}^*_{F,G,\chi}(s)$ satisfies the following functional equation*

$$\mathbb{D}^*_{F,G,\chi}(2k - n - s) = \left(\frac{g(\chi)}{\sqrt{N}}\right)^4 \mathbb{D}^*_{F,G,\bar{\chi}}(s),$$

where $g(\chi) := \sum_{v \,(\mathrm{mod}\, N)} \chi(v) e^{\frac{2\pi i v}{N}}$ denotes the Gauss sum associated to the character χ.

Lemma 3.1. *([53, p. 8, Lemma 4.3], [15, p. 7, Lemma 4.6]) Assume that there exists a sequence of positive numbers T_n with arbitrary large absolute value satisfying $|T_n - \Im(\rho)| > \exp(-A|\Im(\rho)|)/\log(|\Im(\rho)|)$ for every non-trivial zero ρ of $L(s, \chi)$, where A is some suitable positive constant. Then for $0 < B < \frac{\pi}{4}$, we have*

$$\frac{1}{L(\sigma + iT_n, \chi)} < e^{BT}.$$

3.4 Proof of results

Proof of Theorem 3.1. Using the definition of $\Gamma(s)$ and $\mathbb{D}_{F,G,\chi}(s)$ and for $\Re(s) > k+1$, we see that

$$\Gamma(s)\mathbb{D}_{F,G,\chi}(s) = \int_0^\infty \sum_{m=1}^{\infty} \chi(m)\langle \phi_m, \psi_m \rangle \exp(-mx) x^{s-1} dx. \tag{3.12}$$

Thus, for any real number $c > k+1$, the inverse Mellin transform yields

$$\begin{aligned} \sum_{m=1}^{\infty} \chi(m)\langle \phi_m, \psi_m \rangle \exp(-4\pi m\alpha) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\mathbb{D}_{F,G,\chi}(s)(4\pi\alpha)^{-s} ds \\ &= \frac{1}{2\pi i}\int_{(c)} \frac{\mathbb{D}^*_{F,G,\chi}(s)(4\pi\alpha)^{-s}}{(2\pi/N)^{-2s}\Gamma(s+n-k)L(2s+2n-2k,\chi^2)} ds, \end{aligned} \tag{3.13}$$

where the symbol (c) denotes the line integral $c - i\infty$ to $c + i\infty$. In view of Theorem 3.3, it is easy to deduce that the trivial zeros of $L(2s + 2n - 2k, \chi^2)$ in (3.13) are neutralized by the poles of $\Gamma(s + n - k)$. Nevertheless, from generalized Riemann hypothesis and grand simplicity hypothesis the non-trivial zeros of $L(2s-2k+2n, \chi^2)$ will give us infinitely many simple poles of the integrand in the strip $k-n < \Re(s) < k - n + \frac{1}{2}$. Thus, we construct a closed rectangular contour $\mathcal{C}$ consisting of the end points $c - iT, c + iT, c_1 + iT, c_1 - iT$, where T is some large positive constant. Here we consider $c_1 \in (k-n-1, k-n)$ so that the non-trivial zeros of $L(2s-2k+2n, \chi^2)$ lie inside the contour $\mathcal{C}$.

When $N = 1$, the Dirichlet character χ becomes trivial and Dirichlet L-function reduces to the Riemann zeta function. In view of Theorem 3.2, one can check that the integrand function will have an extra pole at $s = k$. However, there would not be a pole at $s = k - n$.

Now apply the Cauchy's residue theorem to obtain

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{(2\pi/N)^{2s}\mathbb{D}^*_{F,G,\chi}(s)(4\pi\alpha)^{-s}}{\Gamma(s+n-k)L(2s-2k+2n,\chi^2)}ds = \sum_{|\Im(\rho)|<T} \mathcal{R}_\rho + \mathcal{R}_k, \tag{3.14}$$

where the term $\mathcal{R}_\rho$ denotes the residual term at the non-trivial zero ρ of $L(s,\chi^2)$. The residual term $\mathcal{R}_k$ is given by

$$\mathcal{R}_k = \begin{cases} \frac{(-1)^{n+1}2(2n)!\langle F,G\rangle}{(4\pi)^n\alpha^k(n-1)!B_{2n}} & \text{if } N=1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.15}$$

To evaluate the above residual term we have used (1.6). Under the assumption that the non-trivial zeros of Dirichlet L-functions are simple, we evaluate $\mathcal{R}_\rho$ as follows:

$$\mathcal{R}_\rho = \frac{\mathbb{D}^*_{F,G,\chi}\left(\frac{\rho}{2}+k-n\right)(4\pi\alpha)^{n-k-\frac{\rho}{2}}}{(2\pi/N)^{2n-2k-\rho}\Gamma\left(\frac{\rho}{2}\right)L'(\rho,\chi^2)}, \tag{3.16}$$

where L' denotes the derivative of the Dirichlet L-function. Employing Stirling's bound (1.5) and Lemma 3.1, horizontal integrals tend to zero as $T\to\infty$. Therefore, making use of (3.14) in (3.13), we obtain

$$\begin{aligned}&\sum_{m=1}^{\infty}\chi(m)\langle\phi_m,\psi_m\rangle\exp(-4\pi m\alpha)\\ &= \frac{1}{2\pi i}\int_{(c_1)} \frac{\mathbb{D}^*_{F,G,\chi}(s)(4\pi\alpha)^{-s}}{(2\pi/N)^{-2s}\Gamma(s+n-k)L(2s-2k+2n,\chi^2)}ds + \sum_{\rho}\mathcal{R}_\rho + \mathcal{R}_k. \end{aligned} \tag{3.17}$$

Note that the above sum runs over all the non-trivial zeros ρ of the Dirichlet L-function $L(s,\chi^2)$. The convergence of this kind of infinite sum is not known without bracketing condition (3.9). A similar infinite series appeared in the work of Hardy and Littlewood [27, p. 156]. For more details on these kind of sums, we refer the readers to [1, 24].

Next, we concentrate on the left vertical integral

$$\begin{aligned}\mathbb{V}_k(\alpha,\chi) :&= \frac{1}{2\pi i}\int_{(c_1)} \frac{\mathbb{D}^*_{F,G,\chi}(s)(4\pi\alpha)^{-s}}{(2\pi/N)^{-2s}\Gamma(s+n-k)L(2s-2k+2n,\chi^2)}ds\\ &= \frac{1}{2\pi i}\int_{(c_1)} \frac{(2\pi/N)^{2s}\mathbb{D}^*_{F,G,\bar{\chi}}(2k-n-s)(4\pi\alpha)^{-s}}{\left(\frac{g(\bar{\chi})}{\sqrt{N}}\right)^4\Gamma(s+n-k)L(2s-2k+2n,\chi^2)}ds. \end{aligned} \tag{3.18}$$

Here, we have utilized Theorem 3.3. Further, use (3.6) to obtain

$$\begin{aligned}&\mathbb{D}^*_{F,G,\bar{\chi}}(2k-n-s)\\ &=(2\pi/N)^{-2(2k-n-s)}\Gamma(2k-s-n)\Gamma(k-s)L(2k-2s,\bar{\chi}^2)\mathbb{D}_{F,G,\bar{\chi}}(2k-n-s).\end{aligned} \tag{3.19}$$

Employ (3.19) in (3.18) to deduce that

$$\mathbb{V}_k(\alpha,\chi) = \frac{1}{2\pi i}\int_{(c_1)} \frac{\Gamma(2k-n-s)\Gamma(k-s)L(2k-2s,\bar{\chi}^2)\mathbb{D}_{F,G,\bar{\chi}}(2k-n-s)(4\pi\alpha)^{-s}}{\left(\frac{2\pi}{N}\right)^{4k-4s-2n}\left(\frac{g(\bar{\chi})}{\sqrt{N}}\right)^4\Gamma\left(s-k+n\right)L(2s-2k+2n,\chi^2)}ds. \tag{3.20}$$

To simply further, change the variable by letting $2k-n-s=w$ in (3.20) to obtain

$$\begin{aligned}&\mathbb{V}_k(\alpha,\chi)\\ &=\left(\frac{\pi}{N^2\alpha}\right)^{2k-n}\frac{1}{2\pi i}\int_{(d_1)} \frac{\Gamma(w)\Gamma(w-k+n)L(2w+2n-2k,\bar{\chi}^2)\mathbb{D}_{F,G,\bar{\chi}}(w)}{\left(\frac{g(\bar{\chi})}{\sqrt{N}}\right)^4\Gamma\left(k-w\right)L(2k-2w,\chi^2)}\left(\frac{\alpha N^4}{4\pi^3}\right)^w dw,\end{aligned} \tag{3.21}$$

where $k<\Re(w)=d_1<k+1$. Now, we use the functional equation of the Dirichlet L-function (given in Theorem 1.4) to obtain

$$L(2k-2w,\chi^2) = \frac{g(\chi^2)}{\sqrt{N}}\frac{(N/\pi)^{\frac{1}{2}-2k+2w}\Gamma\left(w-k+\frac{1}{2}\right)L(2w-2k+1,\bar{\chi}^2)}{\Gamma(k-w)}. \tag{3.22}$$

Substituting (3.22) in (3.21), we obtain

$$\mathbb{V}_k(\alpha,\chi) = \left(\frac{\alpha^{n-2k}N^{2n-2k+2}}{\pi^{n-\frac{1}{2}}g(\bar{\chi})^4 g(\chi^2)}\right)$$
$$\times \frac{1}{2\pi i}\int_{(d_1)} \frac{\Gamma(w)\Gamma(w-k+n)L(2w+2n-2k,\bar{\chi}^2)\mathbb{D}_{F,G,\bar{\chi}}(w)}{\Gamma\left(w-k+\frac{1}{2}\right)L(2w-2k+1,\bar{\chi}^2)}\left(\frac{4\pi}{\alpha N^2}\right)^{-w} dw. \tag{3.23}$$

For $\Re(w) > k$, we know that the Dirichlet series $\mathbb{D}_{F,G,\bar{\chi}}(w)$, $L(2w+2n-2k,\bar{\chi}^2)$ and $1/L(2w-2k+1,\bar{\chi}^2)$ are absolutely and uniformly convergent. Thus, we can write

$$L(2w+2n-2k,\bar{\chi}^2)\frac{\mathbb{D}_{F,G,\bar{\chi}}(w)}{L(2w-2k+1,\bar{\chi}^2)} = \sum_{m=1}^{\infty}\frac{a_{F,G}(m,\chi)}{m^w}, \quad \Re(w) > k, \tag{3.24}$$

where $a_{F,G}(m,\chi)$ is defined in (3.7). Now substituting (3.24) in (3.23) and simplify for $\alpha\beta = 1$ to obtain

$$\mathbb{V}_k(\alpha,\chi) = \left(\frac{\beta^{2k-n}N^{2n-2k+2}}{\pi^{n-\frac{1}{2}}g(\bar{\chi})^4 g(\chi^2)}\right)\sum_{m=1}^{\infty} a_{F,G}(m,\chi)\mathbb{I}_k(m,\beta), \tag{3.25}$$

where

$$\mathbb{I}_k(m,\beta) := \frac{1}{2\pi i}\int_{(d_1)}\frac{\Gamma(w)\Gamma(w-k+n)}{\Gamma\left(w-k+\frac{1}{2}\right)}\left(\frac{4m\pi\beta}{N^2}\right)^{-w} dw.$$

Recalling the Definition 1.6, one can show that the above integral is nothing but

$$\mathbb{I}_k(m,\beta) = G_{1,2}^{2,0}\left(\begin{array}{c} \frac{1}{2}-k \\ 0, n-k \end{array}\middle| \frac{4\pi m\beta}{N^2}\right). \tag{3.26}$$

Using the relation between Meijer G-function and Whittaker function given in Lemma 1.5, we obtain

$$\mathbb{I}_k(m,\beta) = \left(\frac{4\pi m\beta}{N^2}\right)^{\frac{n-1-k}{2}} W_{\frac{n+k}{2}-1,\frac{k-n}{2}}\left(\frac{4\pi m\beta}{N^2}\right)\exp\left(\frac{-2\pi m\beta}{N^2}\right). \tag{3.27}$$

Finally, putting (3.27) in (3.25), the left vertical integral simplifies to

$$\mathbb{V}_k(\alpha,\chi) = \frac{\sqrt{\pi}\beta^{2k-n}N^{2n-2k+2}}{\pi^n g(\bar{\chi})^4 g(\chi^2)}\sum_{m=1}^{\infty} a_{F,G}(m,\chi)\left(\frac{4\pi m\beta}{N^2}\right)^{\frac{n-1-k}{2}} W_{\frac{n+k}{2}-1,\frac{k-n}{2}}\left(\frac{4\pi m\beta}{N^2}\right)\exp\left(\frac{-2\pi m\beta}{N^2}\right). \tag{3.28}$$

The convergence of the above series is an easy consequence of the fact that $a_{F,G}(m,\chi)$ has a polynomial growth and the Whittaker function decays exponentially (see Lemma 1.4). Finally, the proof of Theorem 3.1 follows by using the evaluation (obtained in (3.28)) of the left vertical integral $V_k(\alpha,\chi)$ in (3.17) together with the residual terms (3.15) and (3.16). □

Proof of Corollary 3.1. The proof is similar to the proof of Theorem 3.1 and follows by putting $N = 1$ in the above proof. □

Proof of Corollary 3.2. Multiply α^k on both sides of (3.10) and use $\alpha\beta = 1$ to see that

$$\begin{aligned}\alpha^k &\sum_{m=1}^{\infty}\langle\phi_m,\psi_m\rangle\exp(-4\pi m\alpha)\\ &= \frac{\beta^{k-n}}{\pi^{n-\frac{1}{2}}}\sum_{m=1}^{\infty} a_{F,G}(m)(4\pi m\beta)^{\frac{n-1-k}{2}} W_{\frac{n+k}{2}-1,\frac{k-n}{2}}(4\pi m\beta)\exp(-2\pi m\beta)\\ &+ \frac{(-1)^{n+1}2(2n)!\langle F,G\rangle}{(4\pi)^n(n-1)!B_{2n}} + \sum_{\rho}\frac{\mathcal{D}^*_{F,G}\left(\frac{\rho}{2}+k-n\right)(4\pi)^{n-k-\frac{\rho}{2}}\alpha^{n-\frac{\rho}{2}}}{(2\pi)^{2n-2k-\rho}\Gamma(\frac{\rho}{2})\zeta'(\rho)}.\end{aligned} \tag{3.29}$$

Now considering the first infinite series on the right side of (3.29) and using Lemma 1.4, for $\alpha \to 0^+$, we deduce that

$$\begin{aligned}
&\frac{\beta^{k-n}}{\pi^{n-\frac{1}{2}}}\sum_{m=1}^{\infty}\frac{a_{F,G}(m)}{(4\pi m\beta)^{\frac{k-n+1}{2}}}\exp(-2\pi m\beta)W_{\frac{n+k}{2}-1,\frac{k-n}{2}}(4\pi m\beta)\\
&\ll \beta^{k-n}\sum_{m=1}^{\infty}\frac{a_{F,G}(m)}{(4\pi m\beta)^{\frac{k-n+1}{2}}}\exp(-4\pi m\beta)(4\pi m\beta)^{\frac{n+k}{2}-1} \ll \frac{1}{\beta^K},
\end{aligned} \tag{3.30}$$

where $K > 0$ is a large number. Therefore, the above sum tends to zero. Next, we also observe that the infinite sum over ρ on the right side of (3.29) vanishes. As we know that the non-trivial zeros ρ of $\zeta(s)$ satisfy $0 < \Re(\rho) < 1$, we have $n - \frac{1}{2} < \Re\left(n - \frac{\rho}{2}\right) < n$ and hence the infinite sum over ρ vanishes as $\alpha \to 0^+$. This completes the proof of this corollary. □

3.5 Conclusion

In this chapter, we have generalised the result obtained in Chapter 2 by studying the asymptotic behaviour of similar Lambert series for Siegel cusp forms of different weights k_1, k_2 and degree $n > 1$, using the similar technique which we have used to prove the result in Chapter 2. Also, we have studied twist of this Lambert series by a Dirichlet character and established relation between the Fourier Jacobi coefficients and non-trivial zeros of Dirichlet L-function, using the analytic properties of Dirichlet series studied by Kohnen, Krieg and Sengupta, under the assumption of bracketing condition. Moreover, the result obtained involves Bernoulli numbers and Whittaker function.

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