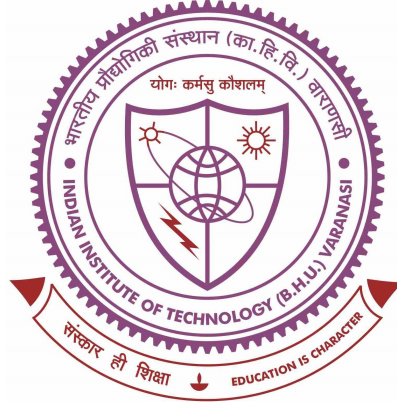


A Priori and A Posteriori Error Estimates for Singularly Perturbed Differential and Integro-Differential Equations



Thesis submitted in partial fulfillment

for the award of degree of

Doctor of Philosophy

by

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CERTIFICATE

It is certified that the work contained in this thesis titled “**A Priori and A Posteriori Error Estimates for Singularly Perturbed Differential and Integro-Differential Equations**” by **Shashikant Kumar** has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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It is certified that the above statement made by the student is correct to the best of my knowledge.


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Varanasi

Shashikant Kumar

Dedicated
to
My Family

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Symbols

\mathbb{R}	Set of real numbers
$\varepsilon, \varepsilon_1, \varepsilon_2$	Perturbation parameters
N	Discretization parameter
λ, λ^N	Control parameter
\mathcal{O}	Landau symbol
$\mathcal{T}_\lambda, \mathcal{L}, \mathcal{L}, \hat{\mathcal{L}}$	Continuous operator
$\mathcal{T}_{\lambda^N}^N, \mathcal{L}_N, \mathbf{L}^N, \hat{\mathbf{L}}^N, \mathcal{T}^N$	Discrete operators
$J, \bar{J}, J_0, \mathcal{G}, \Omega$	Domain of continuous problem
$\omega, \omega_{N_0}, \Omega^N$	Domain of discrete problem
ϕ_i	$\phi(x_i)$
$\boldsymbol{\phi}(x)$	$(\phi_1(x), \phi_2(x))^T$
$\boldsymbol{\phi}_i$	$(\phi_{1,i}, \phi_{2,i})^T$
$\ \phi\ _\infty$	$\sup_{x \in \bar{J}} \phi(x) $
$\ \phi\ _\omega$	$\max_{x_i \in \bar{\omega}} \phi(x_i) $
$\ \boldsymbol{\phi}\ _\infty$	$\max \left(\sup_{x \in \bar{J}} \phi_1(x) , \sup_{x \in \bar{J}} \phi_2(x) \right)$
$\ \boldsymbol{\phi}\ _{\bar{\omega}}$	$\max \left(\max_{x_i \in \bar{\omega}} \phi_{1,i} , \max_{x_i \in \bar{\omega}} \phi_{2,i} \right)$
C	Generic positive constant, independent of ε and N

