A Priori and A Posteriori Error Estimates for Singularly Perturbed Differential and Integro-Differential Equations

Thesis submitted in partial fulfillment

for the award of degree of

Doctor of Philosophy

by

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CERTIFICATE

It is certified that the work contained in this thesis titled **"A Priori and A Posteriori Error Estimates for Singularly Perturbed Differential and Integro-Differential Equations"** by **Shashikant Kumar** has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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DECLARATION BY THE CANDIDATE

I, **Shashikant Kumar**, certify that the work embodied in this thesis is my own bona fide work and carried out by me under the supervision of **Dr. Sunil Kumar** from **January 2020** to **September 2023** at the **Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi**. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not willfully copied any other's work, paragraphs, text, data, results, *etc.*, reported in journals, books, magazines, reports dissertations, theses, *etc.*, or available at websites and have not included them in this thesis and have not cited as my own work.

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It is certified that the above statement made by the student is correct to the best of my knowledge.

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Varanasi **Shashikant Kumar**

Dedicated

to

My Family

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Symbols

$\mathbb{R}$	Set of real numbers
$\varepsilon,\ \varepsilon_1,\ \varepsilon_2$	Perturbation parameters
N	Discretization parameter
λ, λ^N	Control parameter
$\mathcal{O}$	Landau symbol
$\mathcal{T}_\lambda,\ \mathcal{L},\ \mathscr{L},\ \hat{\boldsymbol{\mathcal{L}}}$	Continuous operator
$\mathcal{T}^N_{\lambda^N},\ \mathscr{L}_N,\ \boldsymbol{L}^N,\ \hat{\boldsymbol{L}}^N,\ \mathcal{T}^N$	Discrete operators
$J,\ \mathcal{I},\ \mathcal{I}_0,\ \mathcal{G},\ \Omega$	Domain of continuous problem
$\omega, \omega_{N_0},\ \Omega^N$	Domain of discrete problem
ϕ_i	$\phi(x_i)$
$\boldsymbol{\phi}(x)$	$(\phi_1(x), \phi_2(x))^T$
$\boldsymbol{\phi_i}$	$(\phi_{1,i}, \phi_{2,i})^T$
$\Vert\phi\Vert_\infty$	$\sup\limits_{x\in\bar{J}} \vert\phi(x)\vert$
$\Vert\phi\Vert_\omega$	$\max\limits_{x_i\in\bar{\omega}} \vert\phi(x_i)\vert$
$\Vert\boldsymbol{\phi}\Vert_\infty$	$\max\left(\sup\limits_{x\in\bar{J}} \vert\phi_1(x)\vert, \sup\limits_{x\in\bar{J}} \vert\phi_2(x)\vert\right)$
$\Vert\boldsymbol{\phi}\Vert_{\bar{\omega}}$	$\max\left(\max\limits_{x_i\in\bar{\omega}} \vert\phi_{1,i}\vert, \max\limits_{x_i\in\bar{\omega}} \vert\phi_{2,i}\vert\right)$
C	Generic positive constant, independent of ε and N