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Abbreviations

PDE	P artial D ifferential E quation
IDE	I ntegro D ifferential E quation
FPDE	F ractional P artial D ifferential E quation
LW	L egendre W avelets
RADE	R eaction- A dvection- D iffusion E quation
FRADE	F ractional-order R eaction- A dvection- D iffusion E quation
ADE	A dvection- D iffusion E quation
FRDE	F ractional-order R eaction- D iffusion E quation
SLCM	S hifted L egendre C ollocation M ethod
BFE	B urgers- F isher E quation