CHAPTER 5

ON GENERALIZED PORO-THERMOELASTICITY THEORY

5.1 Variational Principle and Continuous Dependence Results on the Generalized Poro-Thermoelasticity Theory with One Relaxation Parameter

5.1.1 Introduction[1]

This chapter also includes two subchapters, which is devoted to the study of generalized poro-thermoelasticity theory. It must be mentioned here that the poro-thermoelasticity is focused on characterizing the relationship between thermoelastic solid deformation and viscous fluid flow within a porous medium. One must recall that Biot (1956; 1962) firstly developed the theory of poroelasticity to explain the mechanical behavior of water-saturated soil and proposed the idea of describing the deformation of poroelastic material. Such material is made up of fluid and deformable solid, with the solid forming a porous skeleton with numerous microscopic spaces that are connected to one another and filled with the fluid. The theory of poro-thermoelasticity combines the poroelasticity equations with the heat conduction equation to describe the coupling

[1] The content of this sub chapter is published in ***Continuum Mechanics and Thermodynamics***, 34, 867-881 (2022)

between the temperature and displacement in porous medium. Coupled thermal and poromechanical processes play a significant role in a number of geomechanics-related issues, such as hydraulic fracturing in geothermal reservoirs or highly heated petroleum-bearing formations and borehole stability. Numerous applications exist for the topic of poro-thermoelasticity, particularly in examining the effect of employing waste materials on the disintegration of asphalt concrete mixture (ACM). This subject is therefore of utmost interest of researchers during several years. However, the thermo-mechanical coupling in the poroelastic material is more complicated than it is in the classical case since it involves mechanical and thermal interaction both inside solid phase and liquid phase material as well as coupling between the phases. The complete system of equations for porothermoelasticity and a study of the heat effects on wave propagation in liquid-filled porous medium were both accomplished by Pecker and Deresiewicz (1973). Further, McTigue (1986), Kurashige (1989), and Wang and Papamichos (1994) investigated the thermoelasticity of a fluid-saturated porous medium and discussed the fluid and heat flow in a poroelastic medium. In the framework of LS thermoelasticity theory, Youssef (2007) formulated the generalized theory of porothermoelasticity. The governing equations in this theory were derived by using the LS theory of thermoelasticity, and therefore, this theory also predicts the finite speed of thermal waves.

The objective of the present subchapter is to discuss some theoretical aspects (variational principle and continuous dependence results) in the context of the poro-thermoelasticity theory with one-relaxation parameter as proposed by Youssef (2007). As discussed in previous subchapter 4.1, a variational principle is an effective tool in deducing a number of approximate methods for the static and dynamic thermoelastic problems. Moreover, it is an alternative approach to describe the state of dynamics of a thermoelastic system by determining an extremum of a function or a functional. The variational principle has applications for both the unified development of the theory and the approximate solution of fully coupled initial-boundary value problems in thermoelasticity theory.

In addition, it provides a theoretical foundation for some numerical methods such as the finite element method and boundary element methods (see Nickell and Sackman (1968a), Aboustit et al. (1985), Gladysz (1986), Darrall and Dargush (2018)). Sherief and Dhaliwal (1980) and further Chandrasekharaiah (1987) presented the variational principles for generalized thermoelasticity theory and the micropolar thermoelasticity theory, respectively. Variational principles have been discussed in detail which are recorded in some articles and books (Youssef and Al-Lehaibi (2010), Lebon (1980), Marin et al. (2020a)). A continuous dependence result on the initial data and external supply terms in the thermoelasticity theory with one relaxation parameter was studied by Bem (1982). In this respect, some continuous dependence results can be found in the papers (Iesan (1998), Chirita (2003), El-Karamany and Ezzat (2013), Marin et al. (2020b)).

In order to establish the variational principle and continuous dependence results, we begin with introducing the fundamental equations in the context of generalized porothermoelasticity theory by taking into account an isotropic and homogeneous porous material. An alternative characterization of a mixed initial-boundary value problem is then presented by combining initial conditions with the field equations. Further, the convolution type variational principle is derived on the basis of this alternative characterization. This variational principle is not based on the Laplace transform technique and hence it is the more general type of variational principle with respect to a dynamical system of porothermoelasticity involving the general type of initial and boundary conditions. Lastly, the continuous dependence result depending on initial data and external supply terms (heat source and body force) is established.

5.1.2 Basic Equations and Problem Formulation

We consider an isotropic and homogeneous porothermoelastic material occupying a regular region B in three-dimensional Euclidean space. The closure and boundary of B are denoted by $\overline{B}$ and ∂B, respectively. Let n_i represents the components of unit outward normal to the boundary ∂B and it is considered that $\partial B_i, (i = 1, 2, 3, 4, 5, 6, 7, 8)$ are the subsets of ∂B such that

$$\partial B_1 \cup \partial B_2 = \partial B_3 \cup \partial B_4 = \partial B_5 \cup \partial B_6 = \partial B_7 \cup \partial B_8 = \partial B,$$
$$\partial B_1 \cap \partial B_2 = \partial B_3 \cap \partial B_4 = \partial B_5 \cap \partial B_6 = \partial B_7 \cap \partial B_8 = \phi.$$

In the present context of generalized poro-thermoelasticity theory proposed by Youssef (2007), the basic governing equations and the constitutive relations for an isotropic and homogeneous porous material can be expressed in the following way:

Equations of motion:

$$\sigma_{ij,j} + \rho^s F_i^s = \rho_{11}\ddot{u}_i + \rho_{12}\ddot{U}_i, \tag{5.1.1}$$
$$\sigma_{,i} + \rho^f F_i^f = \rho_{12}\ddot{u}_i + \rho_{22}\ddot{U}_i. \tag{5.1.2}$$

Equations of energy:

$$\rho H^s - \rho T_0 \dot{\eta}^s = q_{i,i}^s, \tag{5.1.3}$$
$$\rho H^f - \rho T_0 \dot{\eta}^f = q_{i,i}^f. \tag{5.1.4}$$

Constitutive relations:

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e_{kk}\delta_{ij} + \left(Q\epsilon - R_{11}\theta^s - R_{12}\theta^f\right)\delta_{ij}, \tag{5.1.5}$$
$$\sigma = R\epsilon + Qe_{kk} - R_{22}\theta^f - R_{21}\theta^s, \tag{5.1.6}$$
$$\rho\eta^s = R_{11}e_{kk} + R_{21}\epsilon + \frac{F_{11}}{T_0}\theta^s, \tag{5.1.7}$$
$$\rho\eta^f = R_{12}e_{kk} + R_{22}\epsilon + \frac{F_{22}}{T_0}\theta^f, \tag{5.1.8}$$

$$q_i^s + \tau_q^s \dot{q}_i^s = -K^s \theta_{,i}^s, \tag{5.1.9}$$

$$q_i^f + \tau_q^f \dot{q}_i^f = -K^f \theta_{,i}^f. \tag{5.1.10}$$

Geometrical relations:

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = u_{(i,j)}, \tag{5.1.11}$$

$$\epsilon = U_{i,i}, \tag{5.1.12}$$

where η^s is the entropy for the solid phase per unit mass of aggregate, η^f is the entropy for the fluid phase per unit mass of aggregate, $\rho_{11} = \rho^s - \rho_{12}$ is the mass coefficient for the solid phase, $\rho_{22} = \rho^f - \rho_{12}$ is the mass coefficient for the fluid phase, and ρ_{12} is the coefficient of dynamics coupling. $F_{11} = \rho C_E^s$, $F_{22} = \rho C_E^f$, $R_{11} = \alpha_t^s P + \alpha_t^{fs} Q$, $R_{22} = \alpha_t^f R + 3\alpha_t^{sf} Q$, $R_{12} = \alpha_t^f Q + \alpha_t^{sf} P$, $R_{21} = 3\alpha_t^s Q + \alpha_t^{fs} R$, $P = 3\lambda + 2\mu$, R, and Q are poroelastic coefficients. Further, using Eqs. (5.1.3), (5.1.4), (5.1.7)-(5.1.10), the heat transport equations are acquired in the following forms:

$$\left(1 + \tau_q^s \frac{\partial}{\partial t}\right)\left(R_{11}\dot{e}_{kk} + R_{21}\dot{\epsilon} + \frac{F_{11}\dot{\theta}^s}{T_0} - \frac{\rho H^s}{T_0}\right) = \frac{K^s \theta_{,ii}^s}{T_0}, \tag{5.1.13}$$

$$\left(1 + \tau_q^f \frac{\partial}{\partial t}\right)\left(R_{12}\dot{e}_{kk} + R_{22}\dot{\epsilon} + \frac{F_{22}\dot{\theta}^f}{T_0} - \frac{\rho H^f}{T_0}\right) = \frac{K^f \theta_{,ii}^f}{T_0}. \tag{5.1.14}$$

Further, the material parameters satisfying the following conditions are considered:

$$\begin{gathered}
\rho^s > 0,\ \ \rho^f > 0,\ \ C_E^s > 0,\ \ C_E^f > 0,\ \ K^s > 0,\ \ K^f > 0,\\
\tau_q^s > 0,\ \ \tau_q^f > 0,\ \ \mu > 0,\ \ 3\lambda + 2\mu > 0,\ \ T_0 > 0,\\
\alpha_t^s > 0,\ \ \alpha_t^{fs} > 0,\ \ P > 0,\ \ Q > 0,\ \ R > 0.
\end{gathered} \tag{5.1.15}$$

With the above governing equations, the following initial conditions are adjoined for

$x \in B$:

$$u_i(x,0) = u_{i0}, \quad \dot{u}_i(x,0) = \dot{u}_{i0}, \quad U_i(x,0) = U_{i0}, \quad \dot{U}_i(x,0) = \dot{U}_{i0},$$
$$q_i^s(x,0) = q_{i0}^s, \quad q_i^f(x,0) = q_{i0}^f, \quad \theta^s(x,0) = \theta_0^s, \quad \theta^f(x,0) = \theta_0^f, \;\; \text{for all} \;\; x \in B. \tag{5.1.16}$$

and the boundary conditions are considered as

$$\left.\begin{aligned}
u_i = \overline{u}_i \;\; &\text{on} \;\; \partial B_1 \times [0,\infty)\,, \\
U_i = \overline{U}_i \;\; &\text{on} \;\; \partial B_2 \times [0,\infty)\,, \\
d_i = \sigma_{ji} n_j = \overline{d}_i \;\; &\text{on} \;\; \partial B_3 \times [0,\infty)\,, \\
p_i = \sigma n_i = \overline{p}_i \;\; &\text{on} \;\; \partial B_4 \times [0,\infty)\,, \\
\theta^s = \overline{\theta}^s \;\; &\text{on} \;\; \partial B_5 \times [0,\infty)\,, \\
\theta^f = \overline{\theta}^f \;\; &\text{on} \;\; \partial B_6 \times [0,\infty)\,, \\
q^s = q_i^s n_i = \overline{q}^s \;\; &\text{on} \;\; \partial B_7 \times [0,\infty)\,, \\
q^f = q_i^f n_i = \overline{q}^f \;\; &\text{on} \;\; \partial B_8 \times [0,\infty)\,,
\end{aligned}\right\} \tag{5.1.17}$$

where the functions u_{i0}, U_{i0}, $\dot{u}_{i0}$, $\dot{U}_{i0}$, q_{i0}^s, q_{i0}^f, θ_0^s, θ_0^f are the prescribed initial solid phase displacement component, initial fluid phase displacement component, initial solid phase velocity component, initial fluid phase velocity component, initial solid heat flux component, initial fluid heat flux component, initial solid phase temperature, and initial fluid phase temperature, respectively. Also, $\overline{u}_i$, $\overline{U}_i$, $\overline{d}_i$, $\overline{p}_i$, $\overline{\theta}^s$, $\overline{\theta}^f$, $\overline{q}^s$, $\overline{q}^f$ denote the prescribed component of solid phase surface displacement, surface displacement component of the fluid phase, component of solid traction, component of fluid traction, temperature of the solid phase, temperature of the fluid phase, normal heat flow in the solid, and normal heat flow in the fluid, respectively.

Moreover, the following smoothness requirements and regularity conditions are assumed:

(1) u_{i0}, U_{i0}, $\dot{u}_{i0}$, $\dot{U}_{i0}$, q^s_{i0}, q^f_{i0}, θ^s_0, and θ^f_0 are continuous on $\overline{B}$.

(2) F^s_i, F^f_i, H^s, and H^f are continuously differentiable on $\overline{B} \times [0, \infty)$.

(3) $\overline{u}_i$, $\overline{U}_i$, $\overline{\theta}^s$, and $\overline{\theta}^f$ are continuous on $\partial B_1 \times [0, \infty)$, $\partial B_2 \times [0, \infty)$, $\partial B_5 \times [0, \infty)$ and on $\partial B_6 \times [0, \infty)$, respectively.

(4) $\overline{d}_i$, $\overline{p}_i$, $\overline{q}^s$, and $\overline{q}^f$ are piecewise continuous on $\partial B_3 \times [0, \infty)$, $\partial B_4 \times [0, \infty)$, $\partial B_7 \times [0, \infty)$ and on $\partial B_8 \times [0, \infty)$, respectively.

Now, an ordered array of functions u_i, U_i, q^s_i, q^f_i, e_{ij}, ϵ, η^s, η^f, σ_{ij}, σ, θ^s, θ^f is considered to denote an admissible state as $\mathfrak{N} = \left\{u_i, U_i, q^s_i, q^f_i, e_{ij}, \epsilon, \eta^s, \eta^f, \sigma_{ij}, \sigma, \theta^s, \theta^f\right\}$, having the properties as follows:

(a) $u_i \in C^{1,2}$, $U_i \in C^{1,2}$, $q^s_i \in C^{1,2}$, $q^f_i \in C^{1,2}$, $\eta^s \in C^{0,1}$, $\eta^f \in C^{0,1}$, $\sigma_{ij} \in C^{1,0}$, $\sigma \in C^{1,0}$, $\theta^s \in C^{1,0}$, $\theta^f \in C^{1,0}$.

(b) $\sigma_{ij} = \sigma_{ji}$, $e_{ij} = e_{ji}$ on $\overline{B} \times [0, \infty)$.

Next, the addition and multiplication on the admissible states are defined in the following way:

$$\mathfrak{N} + \mathfrak{N}' = \left\{u_i + u_i', \, U_i + U_i', \, q^s_i + q^{s'}_i, \, q^f_i + q^{f'}_i, ..., \theta^s + \theta^{s'}, \, \theta^f + \theta^{f'}\right\}$$

and

$$\chi\mathfrak{N} = \left\{\chi u_i, \, \chi U_i, \, \chi q^s_i, \, \chi q^f_i, ..., \chi\theta^s, \, \chi\theta^f\right\},$$

where χ is a scalar.

Hence, in view of the above two operations, it is easily observed that the set of all admissible states is a linear space.

A solution of the present initial-boundary value problem is an admissible state $\mathfrak{N}$ if all the field Eqs. (5.1.1)-(5.1.12), the initial conditions (5.1.16) and the boundary conditions (5.1.17) are satisfied by $\mathfrak{N}$.

5.1.3 Alternative Formulation

In this subsection, we present an alternative formulation of the above initial-boundary value problem combining the initial conditions with the field equations. In this regard, the convolution of two functions of space and time is first introduced as follows:

$$[g_1 * g_2](x,t) = \int_0^t g_1(x,t-s)\, g_2(x,s)\, ds, \quad (x,t) \in \overline{B} \times [0,\infty),$$

where g_1 and g_2 are defined on $\overline{B} \times [0,\infty)$ and continuous on $[0,\infty)$ for each $x \in B$. The convolution has the well-known properties, which are as follows:

$$g_1 * g_2 = g_2 * g_1, \tag{5.1.18}$$

$$g_1 * (g_2 + g_3) = (g_1 * g_2) + (g_1 * g_3), \tag{5.1.19}$$

$$g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3, \tag{5.1.20}$$

$$g_1 * g_2 = 0 \implies g_1 = 0 \text{ or } g_2 = 0. \tag{5.1.21}$$

Now, let the functions $m(\ )$ and $r(\ )$ be such that

$$m(t) = t, \;\; r(t) = 1, \quad t \in [0,\infty). \tag{5.1.22}$$

Furthermore, the functions f_i^s, f_i^f, W^s, W^f, L_i^s, and L_i^f defined on $\overline{B} \times [0,\infty)$ are introduced as

$$f_i^s = m * \rho^s F_i^s + \rho_{11}\left(t\dot{u}_{i0} + u_{i0}\right) + \rho_{12}\left(t\dot{U}_{i0} + U_{i0}\right), \tag{5.1.23}$$

$$f_i^f = m * \rho^f F_i^f + \rho_{12}\left(t\dot{u}_{i0} + u_{i0}\right) + \rho_{22}\left(t\dot{U}_{i0} + U_{i0}\right), \tag{5.1.24}$$

$$W^s = r * \frac{\rho H^s}{T_0} + R_{11} u_{i0,i} + R_{21} U_{i0,i} + \frac{F_{11}}{T_0}\theta_0^s, \tag{5.1.25}$$

$$W^f = r * \frac{\rho H^f}{T_0} + R_{12} u_{i0,i} + R_{22} U_{i0,i} + \frac{F_{22}}{T_0}\theta_0^f, \tag{5.1.26}$$

$$L_i^s = \tau_q^s t q_{i0}^s, \tag{5.1.27}$$

$$L_i^f = \tau_q^f t q_{i0}^f. \tag{5.1.28}$$

In order to achieve the results, we will further employ the properties which are as follows:

$$m * \ddot{w}(x,t) = w(x,t) - \left[t\dot{w}(x,0) + w(x,0)\right], \tag{5.1.29}$$

$$r * \dot{w}(x,t) = w(x,t) - w(x,0), \tag{5.1.30}$$

$$m * \dot{w}(x,t) = r * (r * \dot{w}(x,t)) = r * [w(x,t) - w(x,0)] = r * w(x,t) - tw(x,0), \tag{5.1.31}$$

where $w(x,t)$ and $\dot{w}(x,t)$ are defined on $\overline{B} \times [0,\infty)$ and they are continuous and differentiable on $[0,\infty)$.

Consequently, the following theorem is established:

Theorem-4.1.3.1: The functions u_i, U_i, q_i^s, q_i^f, η^s, η^f, σ_{ij}, σ, θ^s, θ^f satisfy the Eqs. (5.1.1)-(5.1.4), (5.1.9), (5.1.10) and the initial conditions (5.1.16) if and only if

$$m * \sigma_{ij,j} + f_i^s = \rho_{11} u_i + \rho_{12} U_i, \tag{5.1.32}$$

$$m * \sigma_{,i} + f_i^f = \rho_{12} u_i + \rho_{22} U_i, \tag{5.1.33}$$

$$\rho\eta^s = -r * \frac{q_{i,i}^s}{T_0} + W^s, \tag{5.1.34}$$

$$\rho\eta^f = -r * \frac{q_{i,i}^f}{T_0} + W^f, \tag{5.1.35}$$

$$m * q_i^s + \tau_q^s r * q_i^s = -K^s m * \theta_{,i}^s + L_i^s, \tag{5.1.36}$$

$$m * q_i^f + \tau_q^f r * q_i^f = -K^f m * \theta_{,i}^f + L_i^f, \tag{5.1.37}$$

where f_i^s, f_i^f, W^s, W^f, L_i^s and L_i^f are given by Eqs. (5.1.23)-(5.1.28).

Proof. First of all, it is supposed that the field Eqs. (5.1.1)-(5.1.4), (5.1.9), (5.1.10) and the initial conditions (5.1.16) hold good. Now, we take convolution of Eqs. (5.1.1) and (5.1.2) by m and make use of Eqs. (5.1.29) and (5.1.16) to get the Eqs. (5.1.32) and (5.1.33). Similarly, taking convolution of Eqs. (5.1.3) and (5.1.4) by r and utilizing the Eqs. (5.1.30), (5.1.7), (5.1.8) and (5.1.16), the Eqs. (5.1.34) and (5.1.35) are obtained. Further, taking convolution of Eqs. (5.1.9) and (5.1.10) through m and making use of Eqs. (5.1.31) and (5.1.16), we arrive at the Eqs. (5.1.36) and (5.1.37).

Now, if the above arguments are reversed, then it is straight-forward to prove the converse part of this theorem.

Consequently, the above theorem provides an alternative characterization of the mixed problem in the following theorem which is a trivial consequence of Theorem-4.1.3.1:

Theorem-4.1.3.2: Let $\mathfrak{N} = \left\{u_i,\, U_i,\, q_i^s,\, q_i^f,\, e_{ij},\, \epsilon,\, \eta^s,\, \eta^f,\, \sigma_{ij},\, \sigma,\, \theta^s,\, \theta^f\right\}$ be an admissible state. Then $\mathfrak{N}$ is a solution of the mixed problem if and only if it satisfies the Eqs. (5.1.32)-(5.1.37), (5.1.5)-(5.1.8), (5.1.11), (5.1.12) and the boundary conditions given in Eq. (5.1.17).

5.1.4 Variational Theorem

This subsection discusses the variational principle on generalized poro-thermoelasticity theory with one thermal relaxation parameter for an isotropic and homogeneous porous medium. In order to obtain the variational theorem, the theorem and the alternative formulation presented in the previous subsections are taken into account.

From the term functional, we mean a real-valued function that holds a subset of a linear space as a domain. It is considered that X represents a linear space whose subspace is Z and we define a functional Ω on Z.

For $a, a' \in Z$, let

$$a + \kappa a' \in Z, \;\; \text{for all}\, \kappa \in \mathbb{R}. \tag{5.1.38}$$

Then the variation of Ω is defined in the following way:

$$\delta_{a'}\Omega\left\{a\right\} = \frac{d}{d\kappa}\Omega\left\{a+\kappa a'\right\}\Big|_{\kappa=0} \tag{5.1.39}$$

The variation of $\Omega\left\{.\right\}$ is zero at a over Z such that

$$\delta\Omega\left\{a\right\} = 0, \quad \text{over } Z \tag{5.1.40}$$

if and only if $\delta_{a'}\Omega\left\{a\right\}$ exists and is equal to zero for all a' consistent with Eq. (5.1.38). Then the following theorem is established:

Theorem-4.1.4.1: Let Σ defines the set of all admissible states. If for any $\Lambda = \left\{u_i,\, U_i,\, q_i^s,\, q_i^f,\, e_{ij},\, \epsilon,\, \eta^s,\, \eta^f,\, \sigma_{ij},\, \sigma,\, \theta^s,\, \theta^f\right\} \in \Sigma$ and for every $t \in [0,\infty)$, we define the functional $\Omega_t\left\{\Lambda\right\}$ on Σ as

$$\begin{aligned}
\Omega_t\left\{\Lambda\right\} =& \int_B m * \left[\frac{\lambda}{2}e_{rr} * e_{ss} + \mu e_{ij} * e_{ij} + \frac{T_0}{2F_{11}}\left(\rho\eta^s - R_{11}e_{kk} - R_{21}\epsilon\right) * \left(\rho\eta^s - R_{11}e_{kk} - R_{21}\epsilon\right)\right] dB \\
&+ \int_B m * \left[\frac{R}{2}\epsilon * \epsilon + \frac{T_0}{2F_{22}}\left(\rho\eta^f - R_{12}e_{kk} - R_{22}\epsilon\right) * \left(\rho\eta^f - R_{12}e_{kk} - R_{22}\epsilon\right)\right] dB \\
&+ \int_B m * \left[Q\epsilon * e_{kk}\right] dB - \int_B m * \left[\sigma_{ij} * e_{ij}\right] dB - \int_B m * \left[\sigma * \epsilon\right] dB \\
&+ \frac{1}{2}\int_B \rho_{11} u_i * u_i dB + \int_B \rho_{12} u_i * U_i dB + \frac{1}{2}\int_B \rho_{22} U_i * U_i dB \\
&- \int_B \left[m * \sigma_{ij,j} + f_i^s\right] * u_i dB - \int_B \left[m * \sigma_{,i} + f_i^f\right] * U_i dB \\
&+ \int_B m * \left[W^s - \rho\eta^s\right] * \theta^s dB + \int_B m * \left[W^f - \rho\eta^f\right] * \theta^f dB \\
&+ \frac{1}{2K^sT_0}\int_B r * \left(m + \tau_q^s r\right) * q_i^s * q_i^s dB + \frac{1}{K^sT_0}\int_B r * \left[K^s m * \theta_{,i}^s - L_i^s\right] * q_i^s dB \\
&+ \frac{1}{2K^fT_0}\int_B r * \left(m + \tau_q^f r\right) * q_i^f * q_i^f dB + \frac{1}{K^fT_0}\int_B r * \left[K^f m * \theta_{,i}^f - L_i^f\right] * q_i^f dB \\
&+ \int_{\partial B_1} m * d_i * \overline{u}_i dA + \int_{\partial B_2} m * \left(d_i - \overline{d}_i\right) * u_i dA
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\partial B_3} m * p_i * \overline{U}_i dA + \int_{\partial B_4} m * (p_i - \overline{p}_i) * U_i dA \\
&+\frac{1}{T_0}\int_{\partial B_5} m * r * q^s * \overline{\theta}^s dA + \frac{1}{T_0}\int_{\partial B_6} m * r * (q^s - \overline{q}^s) * \theta^s dA \\
&+\frac{1}{T_0}\int_{\partial B_6} m * r * q^f * \overline{\theta}^f dA + \frac{1}{T_0}\int_{\partial B_8} m * r * \left(q^f - \overline{q}^f\right) * \theta^f dA, \qquad (5.1.41)
\end{aligned}
$$

then

$$
\delta\Omega_t\left\{\Lambda\right\} = 0, \quad t \in [0, \infty), \qquad (5.1.42)
$$

if and only if Λ satisfies the present mixed initial-boundary value problem represented by Eqs. (5.1.32)-(5.1.37), (5.1.5)-(5.1.8), (5.1.11), (5.1.12) and the boundary conditions (5.1.17).

Proof. We consider a $\Lambda' = \left\{u_i',\, U_i',\, q_i^{s'},\, q_i^{f'},\, e_{ij}',\, \epsilon',\, \eta^{s'},\, \eta^{f'},\, \sigma_{ij}',\, \sigma',\, \theta^{s'},\, \theta^{f'}\right\} \in \Sigma$, from which it is concluded that $\Lambda + \kappa\Lambda' \in \Sigma$ for every real κ. Then, the Eq. (5.1.41) with the use of Eqs. (5.1.18)-(5.1.20), (5.1.39) and the divergence theorem yields

$$
\begin{aligned}
\delta_{\Lambda'}\Omega_t\left\{\Lambda\right\} = &\int_B m * \Big[2\mu e_{ij} + \lambda e_{kk}\delta_{ij} + Q\epsilon\delta_{ij} - \frac{T_0 R_{11}}{F_{11}}\left(\rho\eta^s - R_{11}e_{kk} - R_{21}\epsilon\right)\delta_{ij} \\
&-\frac{T_0 R_{12}}{F_{22}}\left(\rho\eta^f - R_{12}e_{kk} - R_{22}\epsilon\right)\delta_{ij} - \sigma_{ij}\Big] * e_{ij}' dB \\
&+\int_B m * \Big[R\epsilon + Qe_{kk} - \frac{T_0 R_{22}}{F_{22}}\left(\rho\eta^f - R_{12}e_{kk} - R_{22}\epsilon\right) \\
&-\frac{T_0 R_{21}}{F_{11}}\left(\rho\eta^s - R_{11}e_{kk} - R_{21}\epsilon\right) - \sigma\Big] * \epsilon' dB \\
&+\int_B m * \left[\frac{T_0}{F_{11}}\left(\rho\eta^s - R_{11}e_{kk} - R_{21}\epsilon\right) - \theta^s\right] * \rho\eta^{s'} dB \\
&+\int_B m * \left[\frac{T_0}{F_{22}}\left(\rho\eta^f - R_{12}e_{kk} - R_{22}\epsilon\right) - \theta^f\right] * \rho\eta^{f'} dB \\
&-\int_B \left[m * \sigma_{ij,j} + f_i^s - \rho_{11}u_i - \rho_{12}U_i\right] * u_i' dB \\
&-\int_B \left[m * \sigma_{,i} + f_i^f - \rho_{12}u_i - \rho_{22}U_i\right] * U_i' dB
\end{aligned}
$$

$$
\begin{aligned}
&+\int_B m * \left[u_{(i,j)} - e_{ij}\right] * \sigma'_{ij} dB \\
&+\int_B m * \left[U_{i,i} - \epsilon\right] * \sigma' dB \\
&+\int_B m * \left[W^s - r * \frac{q^s_{i,i}}{T_0} - \rho\eta^s\right] * \theta^{s'} dB \\
&+\int_B m * \left[W^f - r * \frac{q^f_{i,i}}{T_0} - \rho\eta^f\right] * \theta^{f'} dB \\
&+\frac{1}{K^s T_0}\int_B r * \left[m * q^s_i + \tau^s_q r * q^s_i + K^s m * \theta^s_{,i} - L^s_i\right] * q^{s'}_i dB \\
&+\frac{1}{K^f T_0}\int_B r * \left[m * q^f_i + \tau^f_q r * q^f_i + K^f m * \theta^f_{,i} - L^f_i\right] * q^{f'}_i dB \\
&-\int_{\partial B_1} m * \left(u_i - \overline{u}_i\right) * d'_i dA + \int_{\partial B_2} m * \left(d_i - \overline{d}_i\right) * u'_i dA \\
&-\int_{\partial B_3} m * \left(U_i - \overline{U}_i\right) * p'_i dA + \int_{\partial B_4} m * \left(p_i - \overline{p}_i\right) * U'_i dA \\
&-\frac{1}{T_0}\int_{\partial B_5} m * r * \left(\theta^s - \overline{\theta}^s\right) * q^{s'} dA + \frac{1}{T_0}\int_{\partial B_6} m * r * \left(q^s - \overline{q}^s\right) * \theta^{s'} dA \\
&-\frac{1}{T_0}\int_{\partial B_7} m * r * \left(\theta^f - \overline{\theta}^f\right) * q^{f'} dA + \frac{1}{T_0}\int_{\partial B_8} m * r * \left(q^f - \overline{q}^f\right) * \theta^{f'} dA. \qquad (5.1.43)
\end{aligned}
$$

To prove the first part of this theorem, let us first consider that Λ satisfies the present mixed initial-boundary value problem. Subsequently, considering Theorem-4.1.3.2, Eq. (5.1.43) with the choice of Λ', we are led to the Eq. (5.1.42). This ends the proof of the first part of Theorem-4.1.4.1.

On the other hand, it is supposed that Eq. (5.1.42) holds. Then, the following is found:

$$
\delta_{\Lambda'}\Omega_t\left\{\Lambda\right\} = 0, \;\; t \in [0, \infty)\,, \;\; \text{for all } \Lambda' \in \Sigma. \qquad (5.1.44)
$$

In order to prove further, the Lemmas (1)-(3) given in Gurtin (1964) are used. We first take $\Lambda' = \{u'_i, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$ and consider that u'_i as well as all of its space derivatives vanish on $\partial B \times [0, \infty)$. Consequently, Eqs. (5.1.43) and (5.1.44) yield

$$
\int_B \left[m * \sigma_{ij,j} + f^s_i - \rho_{11} u_i - \rho_{12} U_i\right] * u'_i \, dB = 0, \;\; t \in [0, \infty)\,.
$$

Thus, the use of Lemma 1 (Gurtin (1964)) and Eq. (5.1.21) leads to Eq. (5.1.32).

We further consider $\Lambda' = \{u_i', 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$ assuming that u_i' vanishes on $\partial B \times [0, \infty)$. Therefore, using Eqs. (5.1.43), (5.1.44), (5.1.32), and Lemma 2 (Gurtin (1964)), the following is deduced:

$$m * \left(d_i - \overline{d}_i\right) = 0 \text{ on } \partial B_2 \times [0, \infty) .$$

Hence, using Eq. (5.1.21), we obtain that one of the boundary conditions holds.

Similarly, by choosing suitable form of Λ' and taking into account the three Lemmas (1)-(3) (Gurtin (1964)), it is concluded that rest of the field Eqs. (5.1.33)-(5.1.37), (5.1.5)-(5.1.8), (5.1.11)-(5.1.12) and other boundary conditions (5.1.17) hold true. Thus, Theorem-4.1.3.2 verifies that Λ satisfies the mixed initial-boundary value problem. Hence, this proves the Theorem-4.1.4.1.

5.1.5 Continuous Dependence Result

In this subsection, the result concerning continuous dependence of solutions upon initial data and external supply terms (heat source and body force) is derived in the present context. To obtain the continuous dependence result, we use the following consequence of the second law of thermodynamics in the context of porothermoelasticity for the solid phase (Shivay and Mukhopadhyay (2021a), Green and Laws (1972)):

$$\frac{d}{dt}\int_B \rho\eta^s dB - \int_B \frac{\rho H^s}{T^s} dB + \int_{\partial B} \frac{q_i^s n_i}{T^s} dA \geq 0. \tag{5.1.45}$$

Here $T^s > 0$ and it will be equal to reference temperature T_0 on the equilibrium condition. Using the Gauss divergence theorem in Eq. (5.1.45), it is found that

$$\frac{q_i^s}{T^s} T^s_{,i} \leq q^s_{i,i} + (\rho T^s \dot{\eta}^s - \rho H^s). \tag{5.1.46}$$

Similarly, the following equation is obtained for the fluid phase:

$$\frac{q_i^f}{T^f}T_{,i}^f \leq q_{i,i}^f + (\rho T^f \dot{\eta}^f - \rho H^f), \tag{5.1.47}$$

where $T^f > 0$, which is equal to reference temperature T_0 on the equilibrium condition. Adding Eqs. (5.1.46) and (5.1.47), we obtain

$$\frac{q_i^s}{T^s}T_{,i}^s + \frac{q_i^f}{T^f}T_{,i}^f \leq q_{i,i}^s + q_{i,i}^f + (\rho T^s \dot{\eta}^s + \rho T^f \dot{\eta}^f - \rho H^s - \rho H^f), \quad T^s = T_0 + \theta^s,\ T^f = T_0 + \theta^f. \tag{5.1.48}$$

Now, considering the linearized form of Eq. (5.1.48) and using Eqs. (5.1.3)-(5.1.4), the following dissipative inequality is acquired:

$$\int_B \left(\frac{q_i^s}{T_0}\theta_{,i}^s + \frac{q_i^f}{T_0}\theta_{,i}^f \right) dB \leq 0. \tag{5.1.49}$$

Eq. (5.1.49) yields

$$\left[\int_{\partial B} \frac{(q^s\theta^s)}{T_0} dA + \int_{\partial B} \frac{(q^f\theta^f)}{T_0} dA - \int_B \frac{q_{i,i}^s\theta^s}{T_0} dB - \int_B \frac{q_{i,i}^f\theta^f}{T_0} dB \right] \leq 0. \tag{5.1.50}$$

In case $q^s = 0$ and $q^f = 0$ on the boundary ∂B, the inequality (5.1.50) takes the form

$$\int_B \left(\frac{q_{i,i}^s\theta^s}{T_0} + \frac{q_{i,i}^f\theta^f}{T_0} \right) dB \geq 0. \tag{5.1.51}$$

Further, taking into consideration of the inequality (5.1.51), Eqs. (5.1.3)-(5.1.4) and (5.1.9)-(5.1.10), we arrive at

$$\int_B \left\{ \theta^s \left[\tau_q^s \frac{\partial}{\partial t}\left(\rho\dot{\eta}^s - \frac{\rho H^s}{T_0} \right) - \frac{K^s\theta_{,ii}^s}{T_0} \right] + \theta^f \left[\tau_q^f \frac{\partial}{\partial t}\left(\rho\dot{\eta}^f - \frac{\rho H^f}{T_0} \right) - \frac{K^f\theta_{,ii}^f}{T_0} \right] \right\} dB \geq 0. \tag{5.1.52}$$

The inequality (5.1.52) is used for obtaining the continuous dependence result.

Now, for convenience, the non-dimensional parameters are emplyoed which are as follows:

$$\begin{aligned}
(x', u', U') &= cn\,(x, u, U)\,,\\
\left(t', \tau_q^{s'}, \tau_q^{f'}\right) &= c^2 n\,\left(t, \tau_q^s, \tau_q^f\right),\\
\left(\theta^{s'}, \theta^{f'}\right) &= \frac{R_{11}}{(\lambda+2\mu)}\left(\theta^s, \theta^f\right),\\
\left(\sigma'_{ij}, \sigma'\right) &= \frac{1}{(\lambda+2\mu)}\left(\sigma_{ij}, \sigma\right),\\
\left(q_i^{s'}, q_i^{f'}\right) &= \frac{R_{11}}{(\lambda+2\mu)cn}\left(\frac{q_i^s}{K^s}, \frac{q_i^f}{K^f}\right),
\end{aligned} \tag{5.1.53}$$

where $n = F_{11}/K^s$ and $c = \sqrt{(\lambda+2\mu)/\rho_{11}}$.

Now, using the above non-dimensional form of parameters, the basic governing Eqs. (5.1.1)-(5.1.10) can be written by the following forms (the primes have been dropped for simplicity):

$$\sigma_{ij,j} + a_1 F_i^s = \ddot{u}_i + a_2 \ddot{U}_i, \tag{5.1.54}$$

$$\sigma_{,i} + a_3 F_i^f = a_2 \ddot{u}_i + a_4 \ddot{U}_i, \tag{5.1.55}$$

$$a_5 H^s - a_6 \dot{\eta}^s = q_{i,i}^s, \tag{5.1.56}$$

$$a_7 H^f - a_8 \dot{\eta}^f = q_{i,i}^f, \tag{5.1.57}$$

$$\sigma_{ij} = 2\mu_1 e_{ij} + \lambda_1 e_{kk}\delta_{ij} + \left(a_9\epsilon - \theta^s - a_{10}\theta^f\right)\delta_{ij}, \tag{5.1.58}$$

$$\sigma = a_{11}\epsilon + a_9 e_{kk} - a_{12}\theta^f - a_{13}\theta^s, \tag{5.1.59}$$

$$a_{14}\eta^s = a_{15}e_{kk} + a_{16}\epsilon + \theta^s, \tag{5.1.60}$$

$$a_{17}\eta^f = a_{18}e_{kk} + a_{19}\epsilon + \theta^f, \tag{5.1.61}$$

$$q_i^s + \tau_q^s \dot{q}_i^s = -\theta^s_{,i}, \tag{5.1.62}$$

$$q_i^f + \tau_q^f \dot{q}_i^f = -\theta^f_{,i}, \tag{5.1.63}$$

where

$$\begin{aligned}
&\lambda_1 = \frac{\lambda}{\lambda+2\mu}, \quad \mu_1 = \frac{\mu}{\lambda+2\mu}, \quad a_1 = \frac{\rho^s}{(\lambda+2\mu)cn}, \quad a_2 = \frac{\rho_{12}}{\rho_{11}}, \quad a_3 = \frac{\rho^f}{(\lambda+2\mu)cn},\\
&a_4 = \frac{\rho_{22}}{\rho_{11}}, \quad a_5 = \frac{R_{11}\rho}{n^2c^2(\lambda+2\mu)K^s}, \quad a_6 = \frac{R_{11}\rho T_0}{n(\lambda+2\mu)K^s}, \quad a_7 = \frac{R_{11}\rho}{n^2c^2(\lambda+2\mu)K^f},\\
&a_8 = \frac{R_{11}\rho T_0}{n(\lambda+2\mu)K^f}, \quad a_9 = \frac{Q}{\lambda+2\mu}, \quad a_{10} = \frac{R_{12}}{R_{11}}, \quad a_{11} = \frac{R}{\lambda+2\mu}, \quad a_{12} = \frac{R_{22}}{R_{11}},\\
&a_{13} = \frac{R_{21}}{R_{11}}, \quad a_{14} = \frac{T_0R_{11}\rho}{F_{11}(\lambda+2\mu)}, \quad a_{15} = \frac{T_0R_{11}^2}{F_{11}(\lambda+2\mu)}, \quad a_{16} = \frac{T_0R_{11}R_{21}}{F_{11}(\lambda+2\mu)},\\
&a_{17} = \frac{T_0R_{11}\rho}{F_{22}(\lambda+2\mu)}, \quad a_{18} = \frac{T_0R_{11}R_{12}}{F_{22}(\lambda+2\mu)}, \quad a_{19} = \frac{T_0R_{11}R_{22}}{F_{22}(\lambda+2\mu)}.
\end{aligned}$$

Now, let us consider two solutions $X^{(\alpha)} = \left\{u_i^{(\alpha)},\, U_i^{(\alpha)},\, \theta^{s(\alpha)},\, \theta^{f(\alpha)}\right\}, \alpha = 1, 2$ corresponding to the following external data having same boundary conditions:

$$\begin{aligned}
\zeta^{(\alpha)} = \Big\{&F_i^{s(\alpha)},\, F_i^{f(\alpha)},\, H^{s(\alpha)},\, H^{f(\alpha)},\, \overline{d}_i,\, \overline{p}_i,\, \overline{u}_i,\, \overline{U}_i,\, \overline{q}^s,\, \overline{q}^f,\, \overline{\theta}^s,\, \overline{\theta}^f,\\
&u_{i0}^{(\alpha)},\, U_{i0}^{(\alpha)},\, \dot{u}_{i0}^{(\alpha)},\, \dot{U}_{i0}^{(\alpha)},\, \theta_0^{s(\alpha)},\, \theta_0^{f(\alpha)},\, q_{i0}^{s(\alpha)},\, q_{i0}^{f(\alpha)}\Big\}.
\end{aligned}$$

Then if we define $u_i = u_i^{(2)} - u_i^{(1)}$, $U_i = U_i^{(2)} - U_i^{(1)}$, $\theta^s = \theta^{s(2)} - \theta^{s(1)}$ and $\theta^f = \theta^{f(2)} - \theta^{f(1)}$, then $X = \left\{u_i, U_i, \theta^s, \theta^f\right\}$ is a solution of the mixed problem corresponding to the external data system

$$\begin{aligned}
\zeta = \Big\{&F_i^s,\, F_i^f,\, H^s,\, H^f,\, \overline{d}_i = 0,\, \overline{p}_i = 0,\, \overline{u}_i = 0,\, \overline{U}_i = 0,\, \overline{q}^s = 0,\, \overline{q}^f = 0,\, \overline{\theta}^s = 0,\, \overline{\theta}^f = 0,\\
&u_{i0},\, U_{i0},\, \dot{u}_{i0},\, \dot{U}_{i0},\, \theta_0^s,\, \theta_0^f,\, q_{i0}^{s(\alpha)},\, q_{i0}^{f(\alpha)}\Big\},
\end{aligned} \tag{5.1.64}$$

where $F_i^s = F_i^{s(2)} - F_i^{s(1)}$, $F_i^f = F_i^{f(2)} - F_i^{f(1)}$, $H^s = H^{s(2)} - H^{s(1)}$, $H^f = H^{f(2)} - H^{f(1)}$, $u_{i0} = u_{i0}^{(2)} - u_{i0}^{(1)}$, $U_{i0} = U_{i0}^{(2)} - U_{i0}^{(1)}$, $\dot{u}_{i0} = \dot{u}_{i0}^{(2)} - \dot{u}_{i0}^{(1)}$, $\dot{U}_{i0} = \dot{U}_{i0}^{(2)} - \dot{U}_{i0}^{(1)}$, $\theta_0^s = \theta_0^{s(2)} - \theta_0^{s(1)}$, $\theta_0^f = \theta_0^{f(2)} - \theta_0^{f(1)}$, $q_{i0}^s = q_{i0}^{s(2)} - q_{i0}^{s(1)}$, $q_{i0}^f = q_{i0}^{f(2)} - q_{i0}^{f(1)}$.

We consider that Γ denotes this problem and we further introduce the function Θ on $[0, t_1]$ such that

$$\Theta = \frac{1}{2}\int_B \left[a_{15}\dot{u}_i\dot{u}_i + 2a_{15}a_2\dot{u}_i\dot{U}_i + a_{15}a_4\dot{U}_i\dot{U}_i + 2Y\right] dB, \tag{5.1.65}$$

where

$$2Y = \left[a_{15}\lambda_1 e_{kk}e_{jj} + 2a_{15}\mu_1 e_{ij}e_{ij} + a_{15}a_{11}\epsilon^2 + (\theta^s)^2 + \frac{a_6}{a_{17}}(\theta^f)^2 + 2a_9a_{15}e_{kk}\epsilon\right]. \tag{5.1.66}$$

Clearly, Y is a positive definite quadratic form in the variables e_{ij}, ϵ, θ^s and θ^f in view of Eq. (5.1.15). Therefore, there exist two positive constants y_1 and y_2 in such a way that

$$y_1\left(e_{ij}e_{ij} + \epsilon^2 + (\theta^s)^2 + (\theta^f)^2\right) \le Y \le y_2\left(e_{ij}e_{ij} + \epsilon^2 + (\theta^s)^2 + (\theta^f)^2\right), \tag{5.1.67}$$

for any $t \in [0, t_1]$ and for all the variables.

Using Eqs. (5.1.58)-(5.1.61) and (5.1.66), we find

$$\dot{Y} = a_{15}\sigma_{ij}\dot{e}_{ij} + a_{15}\sigma\dot{\epsilon} + a_6\theta^s\dot{\eta}^s + a_6\theta^f\dot{\eta}^f. \tag{5.1.68}$$

Then from Eqs. (5.1.65) and (5.1.68), the following is achieved:

$$\begin{aligned}\dot{\Theta} = \int_B [&a_{15}\dot{u}_i\ddot{u}_i + a_{15}a_2\dot{u}_i\ddot{U}_i + a_{15}a_2\dot{U}_i\ddot{u}_i + a_{15}a_4\dot{U}_i\ddot{U}_i + a_{15}\sigma_{ij}\dot{e}_{ij} + a_{15}\sigma\dot{\epsilon} \\ &+ a_6\theta^s\dot{\eta}^s + a_6\theta^f\dot{\eta}^f]dB.\end{aligned} \tag{5.1.69}$$

Now the following lemma is first proved using the above derived results. Then, the continuous dependence result is derived based on this lemma.

Lemma-4.1.5.1. Let $\left\{u_i,\, U_i,\, \theta^s,\, \theta^f\right\}$ be a solution of the problem Γ. Then

$$\dot{\Theta} \leq \int_B \left(a_{15}a_1 F_i^s \dot{u}_i + a_{15}a_3 F_i^f \dot{U}_i + a_5 \theta^s H^s + a_5 \theta^f H^f\right) dB. \tag{5.1.70}$$

Proof. First, we consider the integral in view of Eq. (5.1.64) and Gauss divergence theorem as

$$\begin{aligned}
&\int_B \left[a_{15}\sigma_{ij}\dot{u}_i + a_{15}\sigma\dot{U}_j - \theta^s\left(1+\tau_q^s\frac{\partial}{\partial t}\right)q_j^s - \frac{a_6}{a_8}\theta^f\left(1+\tau_q^f\frac{\partial}{\partial t}\right)q_j^f\right]_{,j} dB \\
&= \int_{\partial B}\left[a_{15}\overline{d}_i\dot{u}_i + a_{15}\overline{p}_j\dot{U}_j - \theta^s\left(1+\tau_q^s\frac{\partial}{\partial t}\right)\overline{q}^s - \frac{a_6}{a_8}\theta^f\left(1+\tau_q^f\frac{\partial}{\partial t}\right)\overline{q}^f\right] dA = 0.
\end{aligned} \tag{5.1.71}$$

Consequently, from Eqs. (5.1.11), (5.1.12), (5.1.56), (5.1.57), (5.1.62) and (5.1.63), it is obtained

$$\begin{aligned}
&\int_B \Big[a_{15}\sigma_{ij}\dot{e}_{ij} + a_{15}\sigma_{ij,j}\dot{u}_i + a_{15}\sigma\dot{\epsilon} + a_{15}\sigma_{,j}\dot{U}_j + \theta^s_{,j}\left(\theta^s_{,j}\right) + \theta^s(a_6\dot{\eta}^s - a_5H^s) \\
&+\theta^s\left(1+\tau_q^s\frac{\partial}{\partial t}\right)(a_6\dot{\eta}^s - a_5H^s) + \frac{a_6}{a_8}\theta^f_{,j}\left(\theta^f_{,j}\right) + \frac{a_6}{a_8}\theta^f\left(a_8\dot{\eta}^f - a_7H^f\right) \\
&\quad + \frac{a_6}{a_8}\theta^f\left(\tau_q^f\frac{\partial}{\partial t}\right)\left(a_8\dot{\eta}^f - a_7H^f\right)\Big] dB = 0.
\end{aligned} \tag{5.1.72}$$

Now we will employ the dimensionless form of inequality (5.1.52) which is as follows:

$$\int_B\left[\theta^s\left(\tau_q^s\frac{\partial}{\partial t}\right)(a_6\dot{\eta}^s - a_5H^s) + \frac{a_6}{a_8}\theta^f\left(\tau_q^f\frac{\partial}{\partial t}\right)\left(a_8\dot{\eta}^f - a_7H^f\right)\right] dB \geq \theta^s\theta^s_{,ii} + \frac{a_6}{a_8}\theta^f\theta^f_{,ii}. \tag{5.1.73}$$

Using Eqs. (5.1.54), (5.1.55) and inequality (5.1.73) in Eq. (5.1.72), we obtain

$$\int_B [a_{15}\sigma_{ij}\dot{e}_{ij} + a_{15}\sigma\dot{\epsilon} + a_{15}\dot{u}_i\ddot{u}_i + a_{15}a_2\dot{u}_i\ddot{U}_i + a_{15}a_2\dot{U}_i\ddot{u}_i + a_{15}a_4\dot{U}_i\ddot{U}_i - a_{15}a_1F_i^s\dot{u}_i$$
$$-a_{15}a_3F_i^f\dot{U}_i + \theta^s(a_6\dot{\eta}^s - a_5H^s) + \frac{a_6}{a_8}\theta^f(a_8\dot{\eta}^f - a_7H^f)]dB$$
$$+\int_B [\theta^s\theta^s_{,ii} + \theta^s_{,i}\left(\theta^s_{,i}\right) + \frac{a_6}{a_8}\theta^f\theta^f_{,ii} + \frac{a_6}{a_8}\theta^f_{,i}(\theta^f_{,i})]dB \leq 0. \tag{5.1.74}$$

Since $\overline{\theta}^s = 0$ and $\overline{\theta}^f = 0$ on ∂B, therefore, applying the divergence theorem in the preceding inequality, the third integral yields

$$\int_B \left[\theta^s\theta^s_{,ii} + \theta^s_{,i}\left(\theta^s_{,i}\right) + \frac{a_6}{a_8}\theta^f\theta^f_{,ii} + \frac{a_6}{a_8}\theta^f_{,i}\left(\theta^f_{,i}\right)\right]dB = \int_B \left[\left(\theta^s\theta^s_{,i}\right)_{,i} + \frac{a_6}{a_8}\left(\theta^f\theta^f_{,i}\right)_{,i}\right]dB$$
$$= \int_{\partial B}\left[\overline{\theta}^s n_i\theta^s_{,i} + \frac{a_6}{a_8}\overline{\theta}^f n_i\theta^f_{,i}\right]dA = 0. \tag{5.1.75}$$

Combining Eqs. (5.1.74) and (5.1.75), the following is acquired:

$$\int_B (a_{15}\sigma_{ij}\dot{e}_{ij} + a_{15}\sigma\dot{\epsilon} + a_{15}\dot{u}_i\ddot{u}_i + a_{15}a_2\dot{u}_i\ddot{U}_i + a_{15}a_2\dot{U}_i\ddot{u}_i$$
$$+a_{15}a_4\dot{U}_i\ddot{U}_i + a_6\theta^s\dot{\eta}^s + a_6\theta^f\dot{\eta}^f)dB$$
$$\leq \int_B [a_{15}a_1F_i^s\dot{u}_i + a_{15}a_3F_i^f\dot{U}_i + a_5\theta^sH^s + a_5\theta^fH^f]dB. \tag{5.1.76}$$

Finally, the inequality (5.1.76) and Eq. (5.1.69) lead to the required inequality (5.1.70).

Now, on the basis of this lemma, the following continuous dependence result is derived in the present context.

Theorem-4.1.5.1: Let D_1 and D_2 are strictly positive constants and consider that $\{u_i, U_i, \theta^s, \theta^f\}$ be a solution of the problem Γ. Then, there exist the positive constants ψ_1 and ψ_2 such that

$$G_1(t) \leq \psi_1G_1(0) + \psi_2\int_0^t G_2(s)ds, \quad t \in [0, t_1]. \tag{5.1.77}$$

Proof. To obtain the main results, we first define the functions G_1 and G_2 on $[0, t_1]$ as follows:

$$G_1 = \left(\int_B \left[\dot{u}_i \dot{u}_i + \dot{u}_i \dot{U}_i + \dot{U}_i \dot{U}_i + (\theta^s)^2 + (\theta^f)^2 + e_{ij} e_{ij} + \epsilon^2 \right] dB \right)^{1/2}, \tag{5.1.78}$$

$$G_2 = \left(\int_B \left[(a_{15} a_1)^2 F_i^s F_i^s + (a_{15} a_3)^2 F_i^f F_i^f + (a_5 H^s)^2 + (a_5 H^f)^2 \right] dB \right)^{1/2}. \tag{5.1.79}$$

Now, making use of the Cauchy-Schwarz inequality in Eq. (5.1.70), it is obtained that

$$\begin{aligned} \dot{\Theta} \leq & \left(\int_B \left[\dot{u}_i \dot{u}_i + \dot{U}_i \dot{U}_i + (\theta^s)^2 + (\theta^f)^2 \right] dB \right)^{1/2} \\ & \times \left(\int_B \left[(a_{15} a_1)^2 F_i^s F_i^s + (a_{15} a_3)^2 F_i^f F_i^f + (a_5 H^s)^2 + (a_5 H^f)^2 \right] dB \right)^{1/2} \\ \leq & G_1 G_2, \end{aligned} \tag{5.1.80}$$

which implies that

$$\Theta(t) \leq \Theta(0) + \int_0^t G_1(s) G_2(s) ds, \quad t \in [0, t_1]. \tag{5.1.81}$$

Eqs. (5.1.65) and (5.1.67) lead to the inequalities as follows:

$$\Theta(t) \geq \chi_1 G_1^2(t), \quad \Theta(0) \leq \chi_2 G_1^2(0), \tag{5.1.82}$$

where

$$\chi_1 = \frac{1}{2} \min(1, D_1, D_2, 2y_1),$$
$$\chi_2 = \frac{1}{2} \max(1, D_1, D_2, 2y_2).$$

Now, in view of Eqs. (5.1.81) and (5.1.82), the following inequality is acquired:

$$G_1^2(t) \leq \psi_1^2 G_1^2(0) + 2\psi_2 \int_0^t G_2(s) G_1(s) ds, \quad t \in [0, t_1], \tag{5.1.83}$$

where

$$\psi_1 = \left(\frac{\chi_2}{\chi_1}\right)^{1/2}, \quad \psi_2 = \left(\frac{1}{2\chi_1}\right).$$

Taking into account of the Gronwall's inequality, we therefore obtain the desired inequality (5.1.77).

Moreover, inequality (5.1.83) yields

$$G_1(t) \leq \psi_1 G_1(0)\exp\left(\psi_2 \int_0^t \frac{G_2(s)}{G_1(s)} ds\right), \tag{5.1.84}$$

$$\dot{G}_1(t) \leq \psi_2 G_2(t). \tag{5.1.85}$$

Hence, in view of the above two inequalities, if we consider null initial conditions and absence of supply terms then it is concluded that $G_1(t) \leq 0$ and $\dot{G}_1(t) \leq 0$. The definition of $G_1(t)$ given by Eq. (5.1.78) therefore implies the uniqueness of solution of the present problem.

5.1.6 Conclusion

The main goal of this subchapter is to discuss some theoretical results (variational principle and continuous dependence result) in the context of generalized poro-thermoelasticity theory with one relaxation parameter. A convolution type variational principle without the use of Laplace transform is obtained. This type of variational principle has an advantage that it recovers all the governing equations along with proper initial conditions and boundary conditions, i.e. a single variational equation stands for a whole set of governing equations, initial and boundary conditions. Also, the importance of variational principles stems from the fact that these principles provide a theoretical basis for numerical techniques such as finite-element and boundary element methods. Further, a continuous dependence result for solutions depending on initial data and external supply terms (heat source and body force) is derived. Consequently, uniqueness of solution of the problem is also established.

5.2 Application of Legendre Wavelet Collocation Method to the Analysis of Poro-Thermoelastic Coupling with Variable Thermal Conductivity

5.2.1 Introduction[2]

In this subchapter, thermo-mechanical interactions within the porous medium are analyzed by taking temperature-dependent thermal conductivity into account. Coupled thermoelastic problems lack an analytical closed-form solution, hence numerical techniques like the finite difference method, finite element method, Chebyshev collocation method, and other approaches are frequently utilized to solve them. Mukhopadhyay and Kumar (2009) used the finite difference scheme to derive the solution for an infinitely long annular cylinder under LS thermoelasticity. Bagri and Eslami (2008), Abbas and Youssef (2012), and Shivay and Mukhopadhyay (2020) employed the finite element method to solve many thermoelastic problems. Further, the Chebyshev collocation method for solving generalized thermoelasticity theories is developed by Alihemmati et al. (2021b; 2021a). In this respect, some other methods (see Kiani and Eslami (2017), Alzahrani et al. (2020), Alihemmati and Beni (2022), Zeverdejani and Kiani (2022)) have been introduced to address the thermoelasticity problems. In the area of engineering and applied research, the wavelet technique has gained popularity among researchers. For the purpose of solving differential and integral equations, approximating functions and various types of wavelets have been employed. In particular, orthogonal wavelets are frequently employed in the relevant literature to handle various kinds of differential equations and some elasticity problems (see Kim et al. (2003), Fan and Qiao (2009), Castro et al. (2010), Heydari et al. (2014), Xu and Xu (2018), He et

[2]The content of this subchapter is published in ***Computers and Mathematics with Applications***, 146, 1-11 (2023)

al. (2019)).

The present subchapter therefore aims to propose a numerical discretization approach on the basis of Legendre wavelet for modeling thermo-mechanical coupled interactions in a poroelastic material with the purpose of providing a simple and efficient alternative method to other numerical methods. Variable material property like temperature-dependent thermal conductivity is considered under the Lord-Shulman porothermoelasticity (LSPTE) theory (Youssef (2007)). Due to the variable thermal conductivity, the governing equations in the present context are non-linear partial differential equations that have been linearized using the Kirchhoff transformation. By using finite difference approach, the time domain is discretized, while Legendre wavelet method is used to approximate the space domain. In order to solve further, the collocation approach is employed to simplify the problem to the solution of a system of algebraic equations. The influences of porosity, variable thermal conductivity, and thermal relaxation parameters on the distributions of different field variables like displacement, temperature, and stress for solid and fluid phases are analyzed and represented graphically. Certain key findings of the current porothermoelastic model in view of the temperature-dependent thermal conductivity are highlighted.

5.2.2 Basic Equations

Following the subchapter 5.1, the basic governing equations of generalized poro-thermoelastic model with one-relaxation parameter for an isotropic and homogeneous medium without any body forces and heat sources can be expressed below.

Equations of motion:

$$\mu u_{i,jj} + (\lambda + 2\mu)u_{j,ij} + QU_{i,jj} - R_{11}\theta^s_{,i} - R_{12}\theta^f_{,i} = \rho_{11}\ddot{u}_i + \rho_{12}\ddot{U}_i, \tag{5.2.1}$$

$$RU_{j,ij} + Qu_{j,ij} - R_{21}\theta^s_{,i} - R_{22}\theta^f_{,i} = \rho_{12}\ddot{u}_i + \rho_{22}\ddot{U}_i. \tag{5.2.2}$$

The equations of heat conduction:

$$K^s\theta^s_{,ii} = \left(\frac{\partial}{\partial t} + \tau^s_q \frac{\partial^2}{\partial t^2}\right)\left(F_{11}\theta^s + T_0 R_{11} e_{kk} + T_0 R_{21}\epsilon\right), \tag{5.2.3}$$

$$K^f\theta^f_{,ii} = \left(\frac{\partial}{\partial t} + \tau^f_q \frac{\partial^2}{\partial t^2}\right)\left(F_{22}\theta^f + T_0 R_{12} e_{kk} + T_0 R_{22}\epsilon\right). \tag{5.2.4}$$

The constitutive relations:

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e_{kk}\delta_{ij} + \left(Q\epsilon - R_{11}\theta^s - R_{12}\theta^f\right)\delta_{ij}, \tag{5.2.5}$$

$$\sigma = R\epsilon + Q e_{kk} - R_{21}\theta^s - R_{22}\theta^f. \tag{5.2.6}$$

The geometrical relations:

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = u_{(i,j)}, \tag{5.2.7}$$

$$\epsilon = U_{i,i}. \tag{5.2.8}$$

We take into account that the thermal conductivity has the following linear function of temperature (Hetnarski (1986)):

$$K^s(\theta^s) = K^s_0(1 + K_1\theta^s), \tag{5.2.9}$$

$$K^f(\theta^f) = K^f_0(1 + K_1\theta^f), \tag{5.2.10}$$

where K_1 is a small parameter to the present study, K^s_0 and K^f_0 denote the temperature independent thermal conductivity of the solid phase and the fluid phase, respectively.

Now, Kirchhoff's transformation (Hetnarski (1986)) is used to solve the coupling equations due to nonlinear terms in Eqs. (5.2.3) and (5.2.4), thus the following mappings are introduced:

$$v^s = \frac{1}{K_0^s}\int_0^{\theta^s} K^s(\theta_1)d\theta_1, \tag{5.2.11}$$

$$v^f = \frac{1}{K_0^f}\int_0^{\theta^f} K^f(\theta_1)d\theta_1. \tag{5.2.12}$$

As a result, we have the following relations:

$$K_0^s v_{,i}^s = K^s(\theta^s)\theta_{,i}^s, \quad K_0^s v_{,ii}^s = (K^s(\theta^s)\theta_{,i}^s)_{,i}, \tag{5.2.13}$$

$$K_0^s \frac{\partial v^s}{\partial t} = K^s(\theta^s)\frac{\partial \theta^s}{\partial t}, \; v^s = \theta^s + \frac{1}{2}K_1(\theta^s)^2, \tag{5.2.14}$$

and

$$K_0^f v_{,i}^f = K^f(\theta^f)\theta_{,i}^f, \quad K_0^f v_{,ii}^f = (K^f(\theta^f)\theta_{,i}^f)_{,i}, \tag{5.2.15}$$

$$K_0^f \frac{\partial v^f}{\partial t} = K^f(\theta^f)\frac{\partial \theta^f}{\partial t}, \; v^f = \theta^f + \frac{1}{2}K_1(\theta^f)^2. \tag{5.2.16}$$

After obtaining v^s and v^f, the temperature increments θ^s and θ^f can be found according to Eqs. (5.2.14) and (5.2.16) as

$$\theta^s = \frac{-1+\sqrt{1+2K_1 v^s}}{K_1}, \quad \theta^f = \frac{-1+\sqrt{1+2K_1 v^f}}{K_1}.$$

Therefore, applying the Kirchhoff's transformation to Eqs. (5.2.1)-(5.2.6), we get

$$\mu u_{i,jj} + (\lambda+2\mu)u_{j,ij} + QU_{i,jj} - \frac{R_{11}}{1+K_1\theta^s}v_{,i}^s - \frac{R_{12}}{1+K_1\theta^f}v_{,i}^f = \rho_{11}\ddot{u}_i + \rho_{12}\ddot{U}_i, \tag{5.2.17}$$

$$RU_{j,ij} + Qu_{j,ij} - \frac{R_{21}}{1+K_1\theta^s}v_{,i}^s - \frac{R_{22}}{1+K_1\theta^f}v_{,i}^f = \rho_{12}\ddot{u}_i + \rho_{22}\ddot{U}_i. \tag{5.2.18}$$

$$K_0^s v^s_{,ii} = \left(\frac{\partial}{\partial t} + \tau_q^s \frac{\partial^2}{\partial t^2}\right)\left(\frac{F_{11}}{1+K_1\theta^s} v^s + T_0 R_{11} e_{kk} + T_0 R_{21}\epsilon\right), \tag{5.2.19}$$

$$K_0^f v^f_{,ii} = \left(\frac{\partial}{\partial t} + \tau_q^f \frac{\partial^2}{\partial t^2}\right)\left(\frac{F_{22}}{1+K_1\theta^f} v^f + T_0 R_{12} e_{kk} + T_0 R_{22}\epsilon\right). \tag{5.2.20}$$

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e_{kk}\delta_{ij} + \left(Q\epsilon - R_{11}\frac{\sqrt{1+2K_1 v^s}-1}{K_1} - R_{12}\frac{\sqrt{1+2K_1 v^f}-1}{K_1}\right)\delta_{ij}, \tag{5.2.21}$$

$$\sigma = R\epsilon + Qe_{kk} - R_{21}\frac{\sqrt{1+2K_1 v^s}-1}{K_1} - R_{22}\frac{\sqrt{1+2K_1 v^f}-1}{K_1}. \tag{5.2.22}$$

Since $|K_1\theta^s| << 1$, $\left|K_1\theta^f\right| << 1$, $|K_1 v^s| << 1$ and $\left|K_1 v^f\right| << 1$, neglecting these terms for linearity, Eqs. (5.2.17)-(5.2.22) take the following forms:

$$\mu u_{i,jj} + (\lambda+2\mu)u_{j,ij} + QU_{i,jj} - R_{11}v^s_{,i} - R_{12}v^f_{,i} = \rho_{11}\ddot{u}_i + \rho_{12}\ddot{U}_i, \tag{5.2.23}$$

$$RU_{j,ij} + Qu_{j,ij} - R_{21}v^s_{,i} - R_{22}v^f_{,i} = \rho_{12}\ddot{u}_i + \rho_{22}\ddot{U}_i. \tag{5.2.24}$$

$$K_0^s v^s_{,ii} = \left(\frac{\partial}{\partial t} + \tau_q^s \frac{\partial^2}{\partial t^2}\right)\left(F_{11} v^s + T_0 R_{11} e_{kk} + T_0 R_{21}\epsilon\right), \tag{5.2.25}$$

$$K_0^f v^f_{,ii} = \left(\frac{\partial}{\partial t} + \tau_q^f \frac{\partial^2}{\partial t^2}\right)\left(F_{22} v^f + T_0 R_{12} e_{kk} + T_0 R_{22}\epsilon\right). \tag{5.2.26}$$

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e_{kk}\delta_{ij} + \left(Q\epsilon - R_{11}v^s - R_{12}v^f\right)\delta_{ij}, \tag{5.2.27}$$

$$\sigma = R\epsilon + Qe_{kk} - R_{21}v^s - R_{22}v^f. \tag{5.2.28}$$

In the case of classical poro-thermoelasticity (CPTE) theory, we recover the corresponding equations from above if we take the values of the thermal relaxation parameter in the following way:

$$\tau_q^s = \tau_q^f = 0.$$

5.2.2.1 Problem formulation: Application to a problem of layer

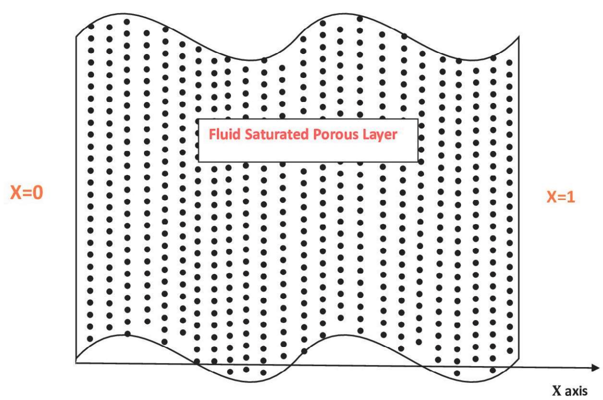

Figure 5.2.1: Model of a fluid-saturated porous layer.

A layer ($0 \leq x \leq 1$) with one-dimensional disturbances is considered that propagate along the x-axis as shown by Fig. 5.2.1. Therefore, all the functions are assumed to be dependent only on the space variable x and time t. Also for simplifications, the following non-dimensional parameters are introduced:

$$\left.\begin{aligned} \left(t', \tau_q^{s'}, \tau_q^{f'}\right) &= c^2\eta\left(t, \tau_q^s, \tau_q^f\right), \\ \left(x', u', U'\right) &= c\eta\left(x, u, U\right), \\ \left(\sigma'_{ij}, \sigma'\right) &= \tfrac{1}{(\lambda+2\mu)}\left(\sigma_{ij}, \sigma\right), \\ \left(v^{s'}, v^{f'}, \theta^{s'}, \theta^{f'}\right) &= \tfrac{R_{11}}{(\lambda+2\mu)}\left(v^s, v^f, \theta^s, \theta^f\right), \end{aligned}\right\} \tag{5.2.29}$$

where $\eta = F_{11}/K_0^s$ and $c = \sqrt{(\lambda+2\mu)/\rho_{11}}$.

By considering these non-dimensional form of parameters and dropping the primes for simplicity, Eqs. (5.2.23)-(5.2.28) yield the following system of equations:

$$\frac{\partial^2 u}{\partial x^2} + b_1 \frac{\partial^2 U}{\partial x^2} - \frac{\partial v^s}{\partial x} - b_2 \frac{\partial v^f}{\partial x} = \frac{\partial^2 u}{\partial t^2} + b_3 \frac{\partial^2 U}{\partial t^2}, \tag{5.2.30}$$

$$b_1 \frac{\partial^2 u}{\partial x^2} + b_4 \frac{\partial^2 U}{\partial x^2} - b_5 \frac{\partial v^s}{\partial x} - b_6 \frac{\partial v^f}{\partial x} = b_7 \frac{\partial^2 u}{\partial t^2} + b_8 \frac{\partial^2 U}{\partial t^2}, \tag{5.2.31}$$

$$\frac{\partial^2 v^s}{\partial x^2} = \left(\frac{\partial}{\partial t} + \tau_q^s \frac{\partial^2}{\partial t^2}\right)\left(v^s + b_9 \frac{\partial u}{\partial x} + b_{10} \frac{\partial U}{\partial x}\right), \tag{5.2.32}$$

$$\frac{\partial^2 v^f}{\partial x^2} = \left(\frac{\partial}{\partial t} + \tau_q^f \frac{\partial^2}{\partial t^2}\right)\left(b_{11} v^f + b_{12} \frac{\partial u}{\partial x} + b_{13} \frac{\partial U}{\partial x}\right), \tag{5.2.33}$$

$$\sigma_{xx} = \frac{\partial u}{\partial x} + b_1 \frac{\partial U}{\partial x} - v^s - b_2 v^f, \tag{5.2.34}$$

$$\sigma = b_1 \frac{\partial u}{\partial x} + b_4 \frac{\partial U}{\partial x} - b_5 v^s - b_6 v^f, \tag{5.2.35}$$

where

$$b_1 = \frac{Q}{\lambda + 2\mu},\ \ b_2 = \frac{R_{22}}{R_{11}},\ \ b_3 = \frac{\rho_{12}}{\rho_{11}},\ \ b_4 = \frac{R}{\lambda + 2\mu},\ \ b_5 = \frac{R_{21}}{R_{11}},$$
$$b_6 = \frac{R_{22}}{R_{11}},\ \ b_7 = \frac{\rho_{12}}{\rho_{11}},\ \ b_8 = \frac{\rho_{22}}{\rho_{11}},\ \ b_9 = \frac{T_0 R_{11}^2}{F_{11}(\lambda + 2\mu)},$$
$$b_{10} = \frac{T_0 R_{11} R_{21}}{F_{11}(\lambda + 2\mu)},\ \ b_{11} = \frac{F_{22}}{\eta K_0^f},\ \ b_{12} = \frac{T_0 R_{11} R_{12}}{\eta K_0^f(\lambda + 2\mu)},\ \ b_{13} = \frac{T_0 R_{11} R_{22}}{\eta K_0^f(\lambda + 2\mu)}.$$

5.2.2.2 Initial and boundary conditions

The homogeneous initial conditions are considered for the present context as

$$u(x,0) = 0,\ \ U(x,0),\ \ \dot{u}(x,0) = 0,\ \ \dot{U}(x,0) = 0,$$

$$\theta^s(x,0) = 0,\ \ \theta^f(x,0) = 0,\ \ \dot{\theta}^s(x,0) = 0,\ \ \dot{\theta}^f(x,0) = 0.$$

In view of mappings defined by Eqs. (5.2.11) and (5.2.12), the initial conditions are reduced to

$$v^s(x,0) = 0, \;\; v^f(x,0) = 0, \;\; \dot{v}^s(x,0) = 0, \;\; \dot{v}^f(x,0) = 0. \tag{5.2.36}$$

For boundary conditions, the surface of the layer at $x = 0$ is assumed to be stress free and is suddenly exposed to a thermal shock of the following forms (Hobiny (2020), Hetnarski et al. (2009)):

$$\theta^s(0,t) = (1-\beta)\theta_0\mathscr{H}(t), \;\; \theta^f(0,t) = \beta\theta_0\mathscr{H}(t), \;\; \sigma_{xx}(0,t) = 0, \;\; \sigma(0,t) = 0,$$

where θ_0 is constant and $\mathscr{H}(t)$ represents the Heaviside unit step function.

Additionally, the remaining boundary conditions are taken as

$$u(1,t) = 0, \;\; U(1,t) = 0, \;\; \frac{d\theta^s(1,t)}{dx} = 0, \;\; \frac{d\theta^f(1,t)}{dx} = 0.$$

Therefore, the layer is fixed and thermally insulated at another end of the boundary of layer (at $x = 1$).

Using Eqs. (5.2.11) and (5.2.12), we thus arrive at the following boundary conditions:

$$\begin{aligned} &v^s(0,t) = (1-\beta)v_0\mathscr{H}(t), \;\; v^f(0,t) = \beta v_0\mathscr{H}(t), \;\; \sigma(0,t) = 0, \;\; \sigma_{xx}(0,t) = 0, \\ &\frac{dv^s(1,t)}{dx} = 0, \;\; \frac{dv^f(1,t)}{dx} = 0, \;\; u(1,t) = 0, \;\; U(1,t) = 0, \end{aligned} \tag{5.2.37}$$

where $v_0 = \theta_0 + \frac{K_1}{2}\theta_0^2$.

5.2.3 Solution Method

In order to achieve the solution, we employ a finite-difference approach to discretize the time domain and the Legendre wavelet method to approximate the space domain. The details of Legendre wavelet method is already descibed in subchapter 3.2.

5.2.3.1 Time discretization

For time discretization as in Oruç, et al. (2015), the interval $[0, T]$ is divided into N sub intervals of equal length $\Delta t = \frac{T}{N}$ and define

$$t_j = j\Delta t, \quad 0 \leq j \leq N, \ j \in \mathbb{Z},$$

where T is the final time for t.

Further, in view of Eqs. (5.2.7) and (5.2.8), the following new variables are introduced for the sake of simplification:

$$\frac{\partial v^s}{\partial t} = l^s, \ \ \frac{\partial v^f}{\partial t} = l^f, \ \ \frac{\partial u}{\partial t} = w^s, \ \ \frac{\partial U}{\partial t} = w^f, \ \ \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) = z^s, \ \ \frac{\partial}{\partial t}\left(\frac{\partial U}{\partial x}\right) = z^f. \tag{5.2.38}$$

Making use of the above equation, Eqs. (5.2.30)-(5.2.33) yield the following system of equations:

$$\frac{\partial^2 u}{\partial x^2} + b_1\frac{\partial^2 U}{\partial x^2} - \frac{\partial v^s}{\partial x} - b_2\frac{\partial v^f}{\partial x} = \frac{\partial w^s}{\partial t} + b_3\frac{\partial w^f}{\partial t}, \tag{5.2.39}$$

$$b_1\frac{\partial^2 u}{\partial x^2} + b_4\frac{\partial^2 U}{\partial x^2} - b_5\frac{\partial v^s}{\partial x} - b_6\frac{\partial v^f}{\partial x} = b_7\frac{\partial w^s}{\partial t} + b_8\frac{\partial w^f}{\partial t}, \tag{5.2.40}$$

$$\frac{\partial^2 v^s}{\partial x^2} = \left(1 + \tau_q^s\frac{\partial}{\partial t}\right)\left(l^s + b_9 z^s + b_{10} z^f\right), \tag{5.2.41}$$

$$\frac{\partial^2 v^f}{\partial x^2} = \left(1 + \tau_q^f\frac{\partial}{\partial t}\right)\left(b_{11} l^f + b_{12} z^s + b_{13} z^f\right). \tag{5.2.42}$$

Using the finite difference method to approximate the time derivatives and averaging the other terms in the above Eqs. (5.2.38)-(5.2.42), the following equations are acquired:

$$\frac{v^s_{j+1} - v^s_j}{\Delta t} = \frac{l^s_{j+1} + l^s_j}{2}, \quad \frac{v^f_{j+1} - v^f_j}{\Delta t} = \frac{l^f_{j+1} + l^f_j}{2},$$
$$\frac{u_{j+1} - u_j}{\Delta t} = \frac{w^s_{j+1} + w^s_j}{2}, \quad \frac{U_{j+1} - U_j}{\Delta t} = \frac{w^f_{j+1} + w^f_j}{2},$$

$$\frac{(u_x)_{j+1}-(u_x)_j}{\Delta t}=\frac{z^s_{j+1}+z^s_j}{2},\quad \frac{(U_x)_{j+1}-(U_x)_j}{\Delta t}=\frac{z^f_{j+1}+z^f_j}{2},$$

$$\frac{(u_{xx})_{j+1}+(u_{xx})_j}{2}+b_1\frac{(U_{xx})_{j+1}+(U_{xx})_j}{2}-\frac{(v^s_x)_{j+1}+(v^s_x)_j}{2}-b_2\frac{(v^f_x)_{j+1}+(v^f_x)_j}{2}$$
$$=\frac{w^s_{j+1}-w^s_j}{\Delta t}+b_3\frac{w^f_{j+1}-w^f_j}{\Delta t},$$

$$b_1\frac{(u_{xx})_{j+1}+(u_{xx})_j}{2}+b_4\frac{(U_{xx})_{j+1}+(U_{xx})_j}{2}-b_5\frac{(v^s_x)_{j+1}+(v^s_x)_j}{2}-b_6\frac{(v^f_x)_{j+1}+(v^f_x)_j}{2}$$
$$=b_7\frac{w^s_{j+1}-w^s_j}{\Delta t}+b_8\frac{w^f_{j+1}-w^f_j}{\Delta t},$$

$$\frac{(v^s_{xx})_{j+1}+(v^s_{xx})_j}{2}=\frac{l^s_{j+1}+l^s_j}{2}+\tau^s_q\frac{l^s_{j+1}-l^s_j}{\Delta t}+b_9\frac{z^s_{j+1}+z^s_j}{2}+b_9\tau^s_q\frac{z^s_{j+1}-z^s_j}{\Delta t}$$
$$+b_{10}\frac{z^f_{j+1}+z^f_j}{2}+b_{10}\tau^s_q\frac{z^f_{j+1}-z^f_j}{\Delta t},$$

$$\frac{(v^f_{xx})_{j+1}+(v^f_{xx})_j}{2}=b_{11}\frac{l^f_{j+1}+l^f_j}{2}+b_{11}\tau^f_q\frac{l^f_{j+1}-l^f_j}{\Delta t}+b_{12}\frac{z^s_{j+1}+z^s_j}{2}+b_{12}\tau^f_q\frac{z^s_{j+1}-z^s_j}{\Delta t}$$
$$+b_{13}\frac{z^f_{j+1}+z^f_j}{2}+b_{13}\tau^f_q\frac{z^f_{j+1}-z^f_j}{\Delta t}.$$

After simplifying the above equations, we arrive at

$$2v^s_{j+1}-\Delta t l^s_{j+1}=\Delta t l^s_j+2v^s_j, \tag{5.2.43}$$

$$2v^f_{j+1}-\Delta t l^f_{j+1}=\Delta t l^f_j+2v^f_j, \tag{5.2.44}$$

$$2u_{j+1}-\Delta t w^s_{j+1}=\Delta t w^s_j+2u_j, \tag{5.2.45}$$

$$2U_{j+1}-\Delta t w^f_{j+1}=\Delta t w^f_j+2U_j, \tag{5.2.46}$$

$$2(u_x)_{j+1}-\Delta t z^s_{j+1}=\Delta t z^s_j+2(u_x)_j, \tag{5.2.47}$$

$$2(U_x)_{j+1}-\Delta t z^f_{j+1}=\Delta t z^f_j+2(U_x)_j, \tag{5.2.48}$$

$$\Delta t\left\{(u_{xx})_{j+1}+b_1(U_{xx})_{j+1}-(v^s_x)_{j+1}-b_2(v^f_x)_{j+1}\right\}-2\left\{w^s_{j+1}+b_3w^f_{j+1}\right\}$$
$$=-\Delta t\left\{(u_{xx})_j+b_1(U_{xx})_j-(v^s_x)_j-b_2(v^f_x)_j\right\}-2\left\{w^s_j+b_3w^f_j\right\}, \tag{5.2.49}$$

$$\Delta t\left\{b_1(u_{xx})_{j+1}+b_4(U_{xx})_{j+1}-b_5(v^s_x)_{j+1}-b_6(v^f_x)_{j+1}\right\}-2\left\{b_7w^s_{j+1}+b_8w^f_{j+1}\right\}$$
$$=-\Delta t\left\{b_1(u_{xx})_j+b_4(U_{xx})_j-b_5(v^s_x)_j-b_6(v^f_x)_j\right\}-2\left\{b_7w^s_j+b_8w^f_j\right\}, \tag{5.2.50}$$

$$\Delta t(v^s_{xx})_{j+1} - B_1 l^s_{j+1} - B_2 z^s_{j+1} - B_3 z^f_{j+1} = -\Delta t(v^s_{xx})_j + B_4 l^s_j + B_5 z^s_j + B_6 z^f_j, \quad (5.2.51)$$

$$\Delta t(v^f_{xx})_{j+1} - B_7 l^f_{j+1} - B_8 z^s_{j+1} - B_9 z^f_{j+1} = -\Delta t(v^f_{xx})_j + B_{10} + B_{11} z^s_j + B_{12} z^f_j, \quad (5.2.52)$$

where $v^s_{j+1} = v^s(x, t_{j+1})$ and so forth.

Moreover

$$B_1 = (\Delta t + 2\tau^s_q), \quad B_2 = b_9(\Delta t + 2\tau^s_q), \quad B_3 = b_{10}(\Delta t + 2\tau^s_q), \quad B_4 = (\Delta t - 2\tau^s_q),$$

$$B_5 = b_9(\Delta t - 2\tau^s_q), \;\; B_6 = b_{10}(\Delta t - 2\tau^s_q), \;\; B_7 = b_{11}(\Delta t + 2\tau^f_q), \;\; B_8 = b_{12}(\Delta t + 2\tau^f_q),$$

$$B_9 = b_{13}(\Delta t + 2\tau^f_q), \;\; B_{10} = b_{11}(\Delta t - 2\tau^f_q) l^f_j, \;\; B_{11} = b_{12}(\Delta t - 2\tau^f_q), \;\; B_{12} = b_{13}(\Delta t - 2\tau^f_q).$$

5.2.3.2 Space discretization by Legendre wavelets

In order to apply the Legendre wavelet method, we expand the highest derivatives of unknown functions appeared in the Eqs. (5.2.43)-(5.2.52) into the Legendre wavelets (see subsection 3.2.5). Therefore, the following is assumed for the solutions of above equations:

$$(v^s_{xx})_{j+1}(x) = \sum_{i=1}^{\hat{n}} c^s_i \Psi_i(x) = (C^s)^T \Psi(x), \quad (5.2.53)$$

where $\Psi(x)$ is $\hat{n} = 2^{k-1}N$ column vectors of Legendre wavelets which is defined in subsection 3.2.5, $(C^s)^T$ is a row vector and subscript xx indicates the second-order derivative with respect to x. Now, integrating Eq. (5.2.53) with respect to x from 0 to x, it is obtained that

$$(v^s_x)_{j+1}(x) = \sum_{i=1}^{\hat{n}} c^s_i p_{i,1}(x) + (v^s_x)_{j+1}(0). \quad (5.2.54)$$

where $p_{i,1}(x) = \int_0^x \Psi_i(x')dx'$.

Since $(v^s_x)_{j+1}(0)$ is unknown so by putting $x = 1$ in Eq. (5.2.54) and considering Eq.

(5.2.37), Eq. (5.2.54) can be rewritten as

$$\left(v_x^s\right)_{j+1}(x) = \sum_{i=1}^{\hat{n}} c_i^s p_{i,1}(x) - \sum_{i=1}^{\hat{n}} c_i^s p_{i,1}(1). \tag{5.2.55}$$

Again integrating Eq. (5.2.55) from 0 to x and using boundary condition given by Eq. (5.2.37), we achieve

$$\left(v^s\right)_{j+1}(x) = \sum_{i=1}^{\hat{n}} c_i^s p_{i,2}(x) - x\sum_{i=1}^{\hat{n}} c_i^s p_{i,1}(1) + (1-\beta)v_0\mathscr{H}(t_{j+1}), \tag{5.2.56}$$

where $p_{i,2}(x) = \int_0^x p_{i,1}(x')dx'$.

Now, it is considered that

$$(u_{xx})_{j+1}(x) = \sum_{i=1}^{\hat{n}} d_i^s \Psi_i(x) = (D^s)^T \Psi(x). \tag{5.2.57}$$

Integrating Eq. (5.2.57) with respect to x from 0 to x, we get

$$(u_x)_{j+1}(x) = \sum_{i=1}^{\hat{n}} d_i^s p_{i,1}(x) + f(t_{j+1}), \tag{5.2.58}$$

where $f(t_{j+1}) = u_x(0, t_{j+1})$ which is obtained by Eqs. (5.2.34), (5.2.35) and boundary condition (5.2.37).

Again integrating Eq. (5.2.58) from 0 to x and using boundary condition given by Eq. (5.2.37), the following is obtained:

$$(u)_{j+1}(x) = \sum_{i=1}^{\hat{n}} d_i^s p_{i,2}(x) + xf(t_{j+1}) + (u)_{j+1}(0). \tag{5.2.59}$$

By setting $x = 1$ in Eq. (5.2.59) and considering Eq. (5.2.37), Eq. (5.2.59) can be rewritten as follows:

$$(u)_{j+1}(x) = \sum_{i=1}^{\hat{n}} d_i^s p_{i,2}(x) + x f_1(t_{j+1}) - \sum_{i=1}^{\hat{n}} d_i^s p_{i,2}(1) - f(t_{j+1}). \tag{5.2.60}$$

Now, we approximate other functions $l^s(x)$, $w^s(x)$ and $z^s(x)$ by the Legendre wavelet in the following forms:

$$(l^s)_{j+1}(x) = \sum_{i=1}^{\hat{n}} L_i^s \Psi_i(x), \tag{5.2.61}$$

$$(w^s)_{j+1}(x) = \sum_{i=1}^{\hat{n}} W_i^s \Psi_i(x), \tag{5.2.62}$$

$$(z^s)_{j+1}(x) = \sum_{i=1}^{\hat{n}} Z_i^s \Psi_i(x). \tag{5.2.63}$$

Similarly, the functions in view of the fluid phase can be expanded into the series of Legendre wavelet as

$$\left(v_{xx}^f\right)_{j+1}(x) = \sum_{i=1}^{\hat{n}} c_i^f \Psi_i(x) = (C^f)^T \Psi(x), \tag{5.2.64}$$

$$\left(v_x^f\right)_{j+1}(x) = \sum_{i=1}^{\hat{n}} c_i^f p_{i,1}(x) - \sum_{i=1}^{\hat{n}} c_i^f p_{i,1}(1), \tag{5.2.65}$$

$$\left(v^f\right)_{j+1}(x) = \sum_{i=1}^{\hat{n}} c_i^f p_{i,2}(x) - x \sum_{i=1}^{\hat{n}} c_i^f p_{i,1}(1) + \beta v_0 \mathscr{H}(t_{j+1}), \tag{5.2.66}$$

$$(U_{xx})_{j+1}(x) = \sum_{i=1}^{\hat{n}} d_i^f \Psi_i(x) = (D^f)^T \Psi(x), \tag{5.2.67}$$

$$(U_x)_{j+1}(x) = \sum_{i=1}^{\hat{n}} d_i^f p_{i,1}(x) + g(t_{j+1}), \tag{5.2.68}$$

where $g(t_{j+1}) = U_x(0, t_{j+1})$.

$$(U)_{j+1}(x) = \sum_{i=1}^{\hat{n}} d_i^f p_{i,2}(x) + x f_2(t_{j+1}) - \sum_{i=1}^{\hat{n}} d_i^f p_{i,2}(1) - g(t_{j+1}), \tag{5.2.69}$$

and

$$\left(l^f\right)_{j+1}(x) = \sum_{i=1}^{\hat{n}} L_i^f \Psi_i(x), \tag{5.2.70}$$

$$\left(w^f\right)_{j+1}(x) = \sum_{i=1}^{\hat{n}} W_i^f \Psi_i(x), \tag{5.2.71}$$

$$\left(z^f\right)_{j+1}(x) = \sum_{i=1}^{\hat{n}} Z_i^f \Psi_i(x). \tag{5.2.72}$$

Now, substituting Eqs. (5.2.53), (5.2.55)-(5.2.58) and (5.2.60)-(5.2.72) into Eqs. (5.2.43)-(5.2.52) and discretizing the results at the collocation points $\frac{2i-1}{2\hat{n}}$, $i = 1, 2, ..., \hat{n}$, a system of algebraic equations is achieved that yields wavelet coefficients as its solution. Then, by plugging these wavelet coefficients in corresponding Eqs. (5.2.53), (5.2.55)-(5.2.58) and (5.2.60)-(5.2.72), the numerical solutions at each time level can be constructed consecutively. This computation starts with the use of initial conditions. In this manner, this iteration process is carried out repeatedly until the desired time level is attained.

5.2.4 Numerical Results

In this subsection, we implement the above-described proposed scheme based on the Legendre wavelet-collocation method and illustrate the problem by finding the numerical solution of the problem. The effects of the temperature-dependent thermal conductivity, porosity, and thermal relaxation parameters on the physical quantities are discussed. For the purpose of computational work, the material of sandstone saturated with kerosene is considered. Therefore, the basic parameters for this two-phase system are taken in the following way (Sherief and Hussein (2012)):

$$Q = 75.79 \times 10^6\text{Pa},\;\; R = 33.25 \times 10^6\text{Pa},\;\; \lambda = 453.69 \times 10^6\text{Pa},\;\; \mu = 280.94 \times 10^6\text{Pa},$$

$$C_E^s = 1.1715K\text{J/kg}^o\text{C},\;\; C_E^f = 2.092\text{J/kg}^o\text{C},\;\; K_0^s = 1.83\text{W/m}^o\text{C},\;\; K_0^f = 0.148\text{W/m}^o\text{C},$$

$$\alpha_t^s = 2.257 \times 10^{-6}/^o\text{C},\;\; \alpha_t^f = 0.932 \times 10^{-6}/^o\text{C},\;\; \beta = 0.26,\;\; \rho_{12} = -0.001\rho,$$

$$\alpha_t^{sf} = -0.33 \times 10^{-6}/^o\text{C},\;\; \alpha_t^{fs} = -0.88 \times 10^{-6}/^o\text{C},\;\; \rho^s = 23603\ \text{kg/m}^3,\;\; \rho^f = 723\ \text{kg/m}^3,$$

$$\tau_q^s = 0.03\text{s},\;\; \tau_q^f = 0.001\text{s}.$$

In view of the data presented above, MATLAB software is used to perform numerical computation for the solution of various physical fields in the present context. The calculations are carried out for $t = 0.1$ and $\theta_0 = 1$. In order to obtain the numerical solution and represent it graphically, the value of $k = 1$, $N = 5$ and $\Delta t = 0.01$ are chosen.

The distributions of displacement, temperature, and stress of solid and fluid phases along the x-axis are displayed by various figures. Figs. 5.2.2-5.2.10 show that the solid and fluid phase displacements start from the negative value at the first boundary of the layer and after attaining their maximum values, they tend to zero according to the boundary conditions. As seen in the figures, solid and fluid temperatures begin with their maximum values at $x = 0$ and steadily decrease to zero as we get closer to the layer boundary at $x = 1$. Therefore, this temperature field is a decreasing function of x. Further, we observe that stresses in both the solid and fluid phases starting with zero, decrease until they reach minimum absolute values, and then again start increasing to approach zero value. Hence, these figures demonstrate the successful implementation of the proposed numerical scheme and indicate that the boundary conditions are satisfied by all the field variables. We also have the following observations about the effects of temperature-dependent thermal conductivity, porosity, and thermal relaxation parameters on displacements, temperatures, and stresses:

5.2.4.1 Verification of numerical results

In order to validate the current formulation and numerical results, the findings of the current work is compared with well-known data from the literature. We compare the outcomes of the current study with the results of Sherief and Hussein (2012) by taking into account the same boundary conditions as provided in Sherief and Hussein (2012). Additionally, we establish a comparison between the results of the present work and the results predicted by Hobiny (2020) by considering the same material and same data as given in Hobiny (2020). A good agreement between the results of the present case and the corresponding results of Sherief and Hussein (2012), Hobiny (2020) is observed, which indicates that the present numerical method is effective and accurate.

5.2.4.2 Effect of temperature-dependent thermal conductivity

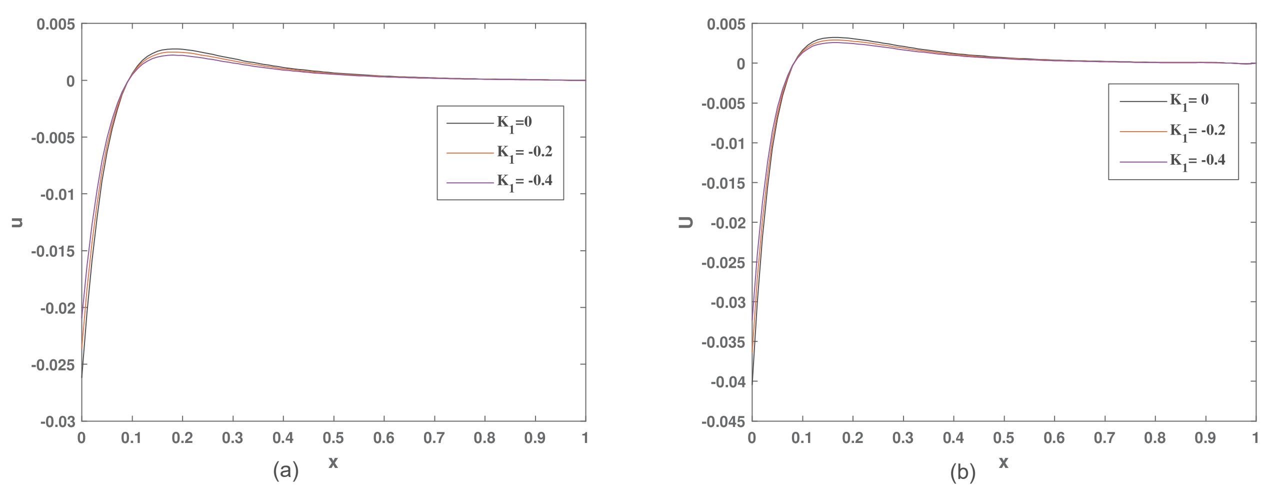

Figure 5.2.2: (a) Displacement distribution due to solid phase at $t = 0.1$ and $\beta = 0.26$. (b) Displacement distribution due to fluid phase at $t = 0.1$ and $\beta = 0.26$.

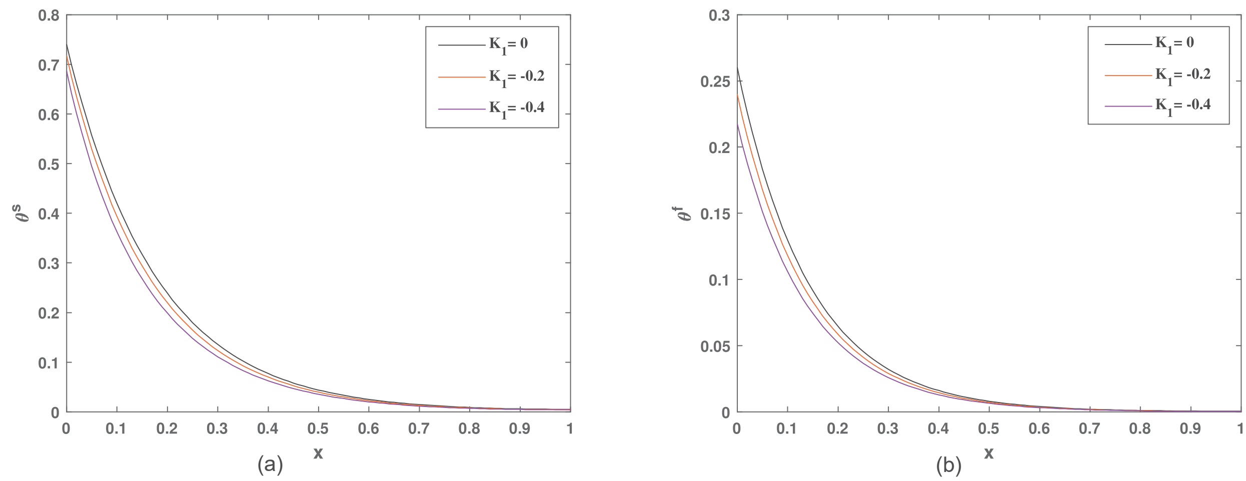

Figure 5.2.3: (a) Temperature distribution due to solid phase at $t = 0.1$ and $\beta = 0.26$. (b) Temperature distribution due to fluid phase at $t = 0.1$ and $\beta = 0.26$.

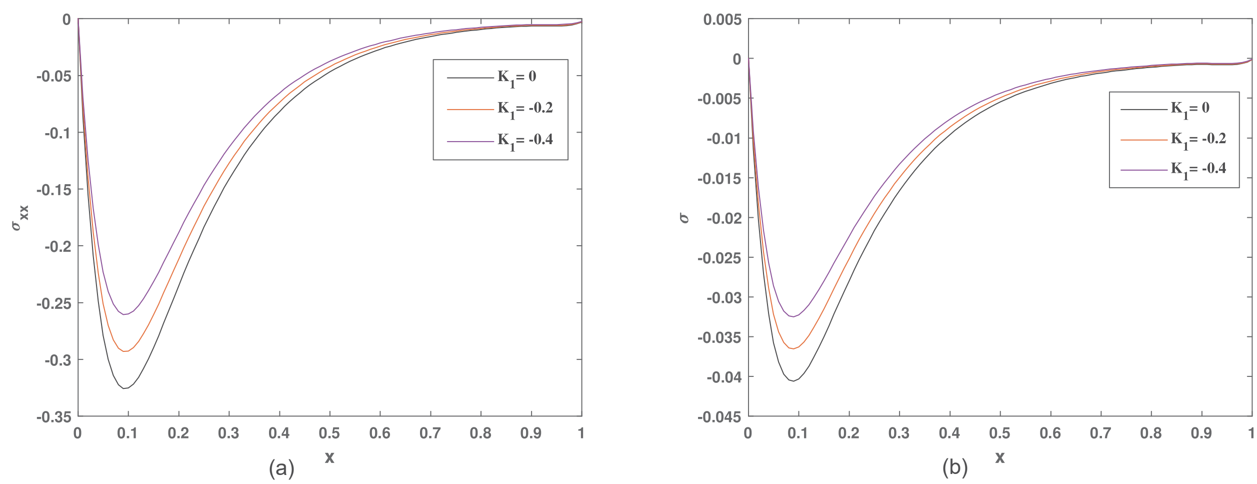

Figure 5.2.4: (a) Stress distribution due to solid phase at $t = 0.1$ and $\beta = 0.26$. (b) Stress distribution due to solid phase at $t = 0.1$ and $\beta = 0.26$.

Figs. 5.2.2-5.2.4 depict the influence of temperature-dependent thermal conductivity on the displacement, temperature, and stress of solid and fluid phases at $t = 0.1$ and $\beta = 0.26$. From Figs. 5.2.2 and 5.2.3, it is observed that the absolute value of displacement and temperature of solid and fluid phases decrease as the value of temperature dependency parameter K_1 approaches higher negative values. However, the stress in both the solid and fluid phases increases as the value of K_1 decreases. This is clearly verified in Fig. 5.2.4. Hence, we get to the conclusion that higher temperature and lower stress will occur if the temperature-dependent thermal conductivity is ignored. Additionally, it is worth noting that the parameter K_1 has a greater impact on temperature and stress as compared to displacement. This indicates that temperature-dependent thermal conductivities (K^s, K^f) have a significant effect on displacements, temperatures, and stresses.

5.2.4.3 Effect of porosity

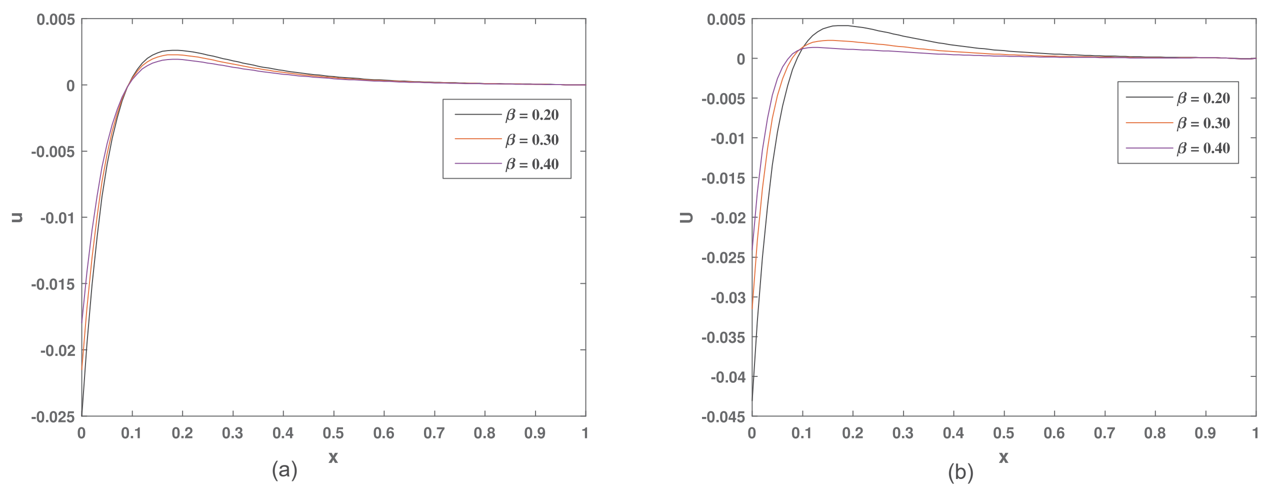

Figure 5.2.5: (a) Displacement distribution due to solid phase at $t = 0.1$ and $K_1 = -0.25$. (b) Displacement distribution due to fluid phase at $t = 0.1$ and $K_1 = -0.25$.

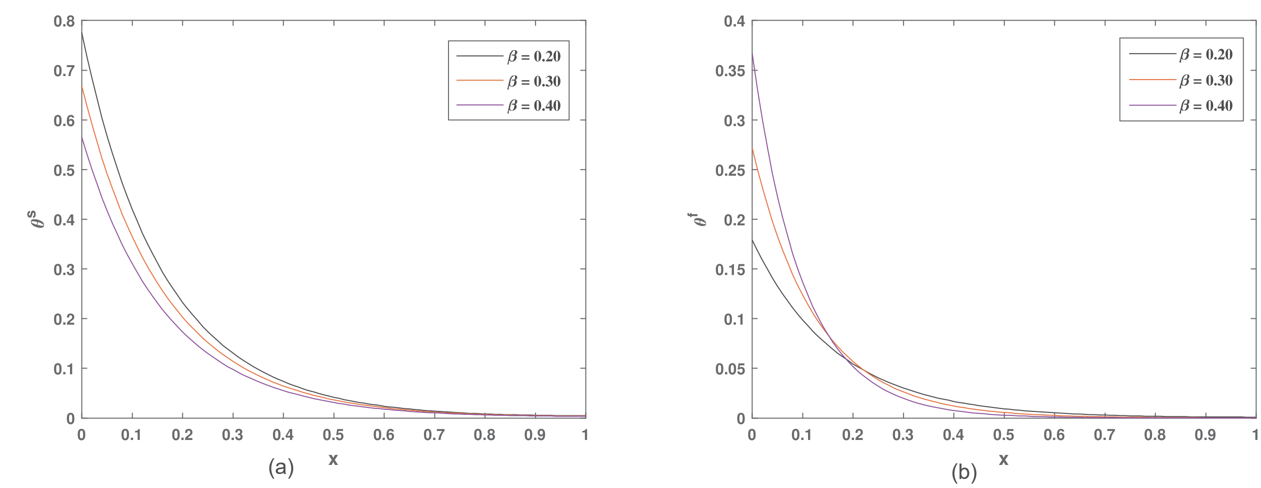

Figure 5.2.6: (a) Temperature distribution due to solid phase at $t = 0.1$ and $K_1 = -0.25$. (b) Temperature distribution due to fluid phase at $t = 0.1$ and $K_1 = -0.25$.

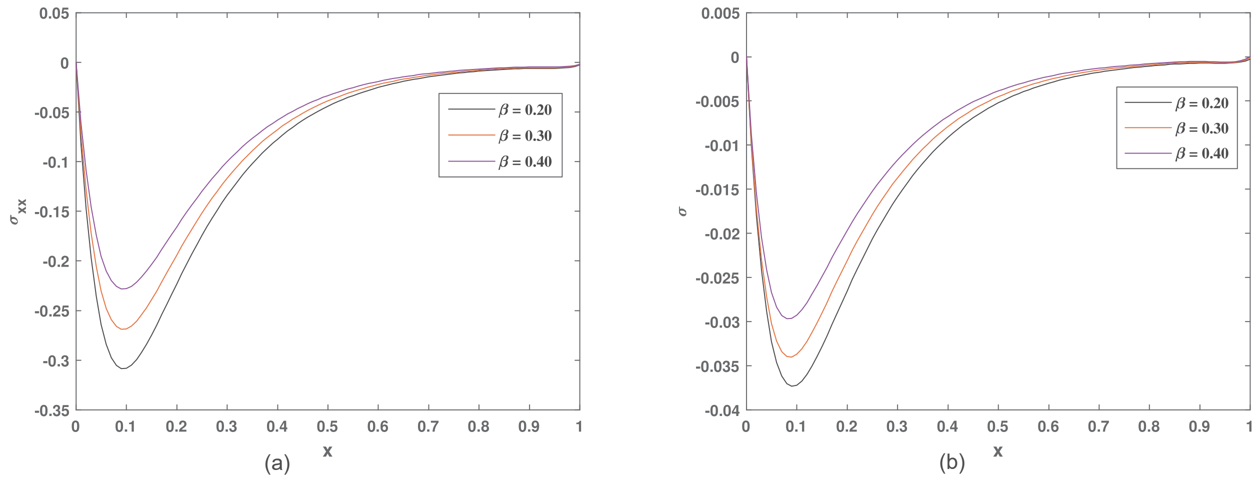

Figure 5.2.7: (a) Stress distribution due to solid phase at $t = 0.1$ and $K_1 = -0.25$. (b) Stress distribution due to solid phase at $t = 0.1$ and $K_1 = -0.25$.

The effect of porosity on the variation of displacement, temperature, and stress in both the solid and fluid phases is demonstrated by Figs. 5.2.5-5.2.7. Like the case of temperature-dependent thermal conductivity parameter, porosity is seen to have a very strong influence on the variations of all the studied fields. Fig. 5.2.5 reveals that as the porosity increases, the displacement of the solid and fluid phases attain higher values at the layer's boundary $x = 0$, however, the peak values are lower in the case of higher porosity values. Similar to the nature of displacement, the temperature of the fluid phase is initially high and then decreases abruptly for higher values of porosity, whereas the solid phase temperature decreases with the increasing value of β as seen in Fig. 5.2.6. Further, we observe that stresses of the solid and fluid phase increase when the value of β increases. One can observe this clearly in Fig. 5.2.7.

5.2.4.4 Effect of thermal relaxation parameters

Figs. 5.2.8-5.2.10 describe the influences of thermal relaxation parameters (τ_q^s, τ_q^f) on the displacement, temperature, and stress of solid and fluid phases at different instants

$t = 0.15,\ 0.25$ when $\beta = 0.26$ and $K_1 = -0.25$. We compare the Lord-Shulman porothermoelasticity (LSPTE) and classical porothermoelasticity (CPTE) theories by taking different values of thermal relaxation parameters and time. The values $\tau_q^s = \tau_q^f = 0$ correspond to the classical case of poro-thermoelastic model and by taking $\tau_q^s > 0$, $\tau_q^f > 0$, we obtain the case of LSPTE model. Unlike the fluid phase temperature which exhibits almost a similar trend in both models, a significant difference is observed in the values of other field variables. Therefore, it is clear that the thermal relaxation parameters have a prominent effect on solid temperature and stresses, while have a slight effect on displacements. According to these figures, we observe that the effect of thermal shock can be seen in all field variables and it can be concluded that the effective region of influence increases with time. Hence, in contrast to LSPTE predicting the finite speed of heat signals, CPTE theory predicts infinite speed behavior and has a larger effective domain of influence.

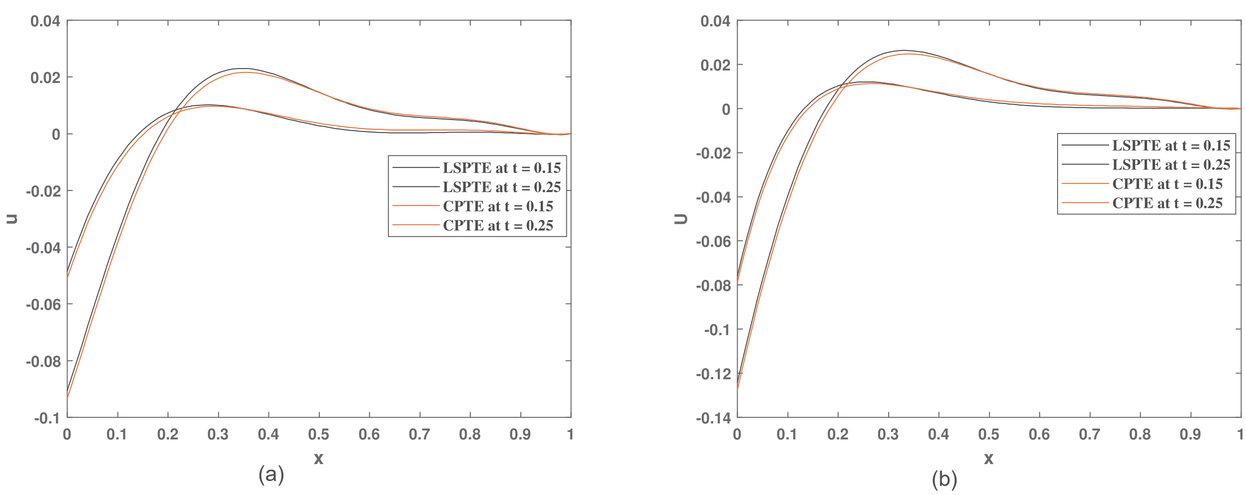

Figure 5.2.8: (a) Displacement distribution due to solid phase for different models. (b) Displacement distribution due to fluid phase for different models.

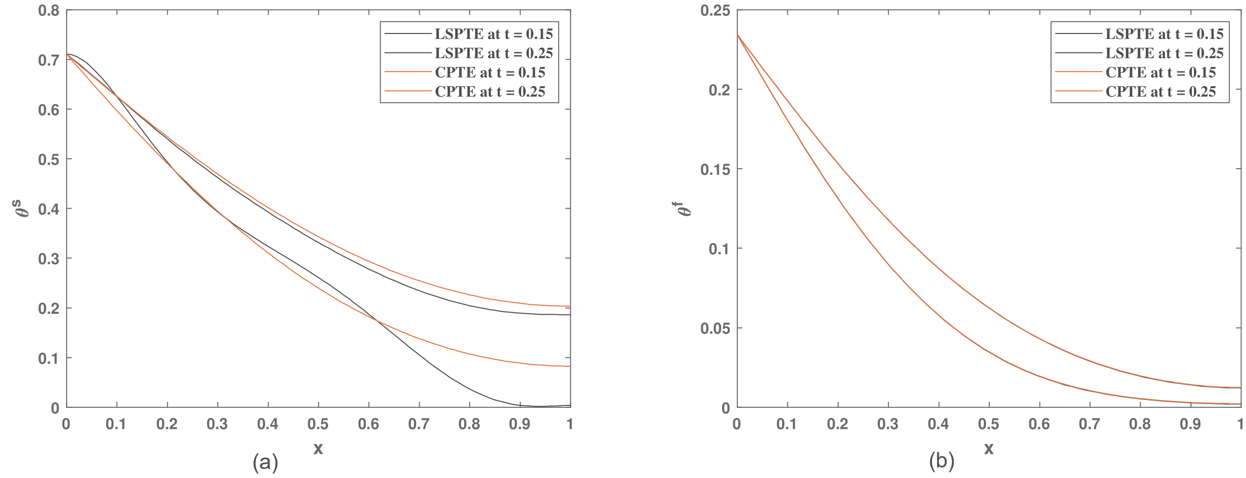

Figure 5.2.9: (a) Temperature distribution due to solid phase for different models. (b) Temperature distribution due to fluid phase for different models.

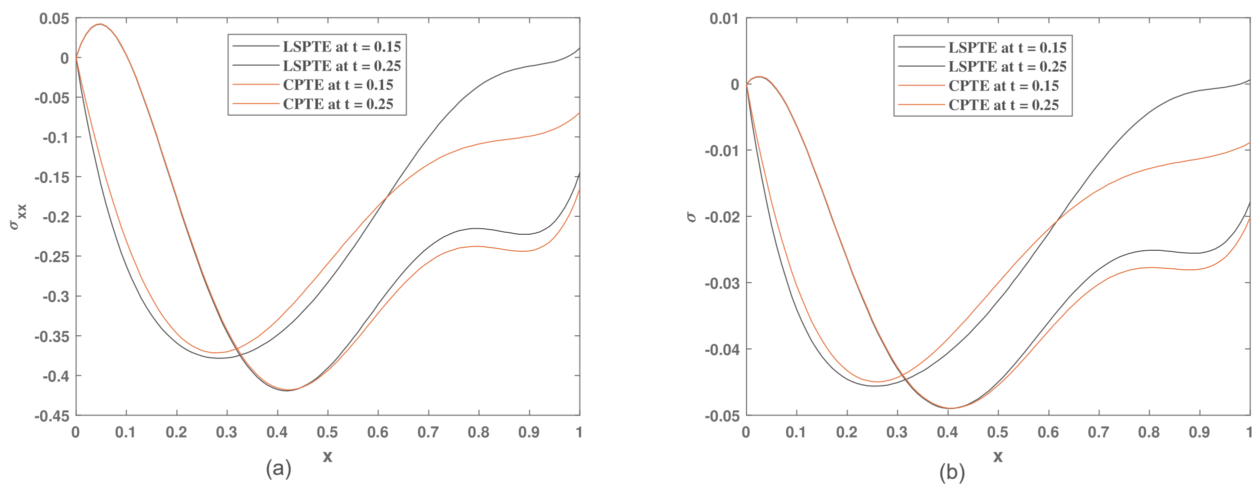

Figure 5.2.10: (a) Stress distribution due to solid phase for different models. (b) Stress distribution due to solid phase for different models.

5.2.5 Conclusion

In the present subchapter, the poro-thermoelastic interactions inside a one-dimensional layer defined in the region $0 \leq x \leq 1$ are investigated in the context of Lord-Shulman poro-thermoelasticity (LSPTE) theory. To analyze the coupling of thermoelasticity and porosity, the Legendre wavelet collocation method is implemented and the numerical solution of the problem is obtained. For all field variables, i.e. displacement, temperature, and stress of solid and fluid phases, the significant effects of temperature-dependent thermal conductivity, thermal relaxation parameters, and porosity are examined. The behavior of these field variables is graphically presented along with a number of key facts. Main findings of the present investigation can be summarized as follows:

- Major achievement of this subchapter is to propose an alternative numerical scheme based on Legendre wavelet collocation that can solve any poro-thermoleastic problem satisfactorily like the integral transform method, finite element method, finite difference method, etc.

- Since boundary conditions are handled here automatically throughout the solution procedure, there is no need to exert additional effort to impose them. Also, the current method gives satisfactory outcomes even with fewer collocation points and a simpler framework.

- It can be seen that the magnitude of physical field variables is greater for the solid phase quantities than for quantities of the fluid phase.

- In the case when thermal conductivity is considered to be dependent on the temperature, temperature and stress fields of solid and fluid phases decrease, whereas no significant difference is observed in the profile of solid and fluid phase displacements. Hence, the temperature-dependent parameter (K_1) has a greater impact on temperatures and stresses as compared to the displacements.

- Thermal relaxation parameters almost have no effect on the displacements in both the phases and on the fluid phase temperature while they have significant effects on these fields of the solid phase. Also, it is observed that the trend of variation of all the studied field variables changes with time indicating that the domain of influence increases with time.

- The porosity has a great effect on the distributions of displacements, temperatures, and stresses. As the value of porosity increases, the absolute value of all the field variables decreases. Thus, temperature-dependent thermal conductivity, porosity and thermal relaxation parameters play a significant role in poro-thermoelastic interactions.

- It is worth to be mentioned here that as compared to the present poro-thermoelasticity theory (LSPTE) which admits a finite speed of heat signals, the effective region of influence for the theory of classical poro-thermoelasticity is found to be wider due to the infinite speed behavior of classical theory.