References

[1] A. B. Abubakar, P. Kumam, M. Malik, and A. H. Ibrahim. A hybrid conjugate gradient based approach for solving unconstrained optimization and motion control problems. *Mathematics and Computers in Simulation*, 201:640–657, 2022.

[2] I. Ahmad, D. Singh, and B. A. Dar. Optimality and duality in non-differentiable interval valued multiobjective programming. *International Journal of Mathematics in Operational Research*, 11(3):332–356, 2017.

[3] M. Al-Baali. Descent property and global convergence of the Fletcher—Reeves method with inexact line search. *IMA Journal of Numerical Analysis*, 5(1):121–124, 1985.

[4] M. Alonso and L. Rodríguez-Marín. Set-relations and optimality conditions in set-valued maps. *Nonlinear Analysis: Theory, Methods & Applications*, 63(8):1167–1179, 2005.

[5] D. T. V. An and J.-C. Yao. Differential stability of convex optimization problems with possibly empty solution sets. *Journal of Optimization Theory and Applications*, 181(1):126–143, 2019.

[6] N. Andrei. *Nonlinear Conjugate Gradient Methods for Unconstrained Optimization*. Springer, Berlin, 2020.

[7] L. Q. Anh, D. P. Thanh, and T. D. T. Thi. A new scalarization approach and applications in set optimization. *Optimization*, 72(4):1093–1117, 2023.

[8] Q. H. Ansari, E. Köbis, and P. K. Sharma. Characterizations of set relations with respect to variable domination structures via oriented distance function. *Optimization*, 67(9):1389–1407, 2018.

[9] Q. H. Ansari and P. K. Sharma. Set order relations, set optimization, and Ekeland's variational principle. In *Optimization, Variational Analysis and Applications: IFSOVAA-2020*, volume 355, pages 103–165, Singapore, 2021. Springer.

[10] Anshika and D. Ghosh. Interval-valued value function and its application in interval optimization problems. *Computational and Applied Mathematics*, 41(4):137, 2022.

[11] Anshika, D. Ghosh, R. Mesiar, H.-R. Yao, and R. S. Chauhan. Generalized-Hukuhara subdifferential analysis and its application in nonconvex composite interval optimization problems. *Information Sciences*, 622:771–793, 2023.

[12] Anshika, K. Kumar, and D. Ghosh. Generalized Hukuhara Dini Hadamard ϵ-subdifferential and H_ϵ-subgradient and their applications in interval optimization. *Journal of Applied and Numerical Optimization*, 6(2):177–202, 2024.

[13] T. Antczak. Optimality conditions and duality results for nonsmooth vector optimization problems with the multiple interval-valued objective function. *Acta Mathematica Scientia*, 37(4):1133–1150, 2017.

[14] Y. Araya. New types of nonlinear scalarizations in set optimization. In *Proceedings of the 3th Asian Conference on Nonlinear Analysis and Optimization*, pages 1–15, Yokohama, 2012.

[15] J. P. Aubin and H. Frankowska. *Set-Valued Analysis*. Springer, Birkhäuser Boston, 1990.

[16] C. Audet and W. Hare. Algorithmic construction of the subdifferential from directional derivatives. *Set-Valued and Variational Analysis*, 26(3):431–447, 2018.

[17] V. Azhmyakov. On the set-valued approach to optimal control of sliding mode processes. *Journal of the Franklin Institute*, 349(4):1323–1336, 2012.

[18] M. Bartholomew-Biggs. *Nonlinear Optimization with Financial Applications*. Springer Science & Business Media, Heidelberg, 2006.

[19] A. Beck. *First-Order Methods in Optimization*. SIAM, Philadelphia, 2017.

[20] B. Bede and S. G. Gal. Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations. *Fuzzy Sets and Systems*, 151(3):581–599, 2005.

[21] D. P. Bertsekas. On the Goldstein-Levitin-Polyak gradient projection method. *IEEE Transactions on Automatic Control*, 21(2):174–184, 1976.

[22] D. P. Bertsekas. *Convex Optimization Algorithms.* Athena Scientific, Belmont, 2015.

[23] S. K. Bharati and S. R. Singh. Solution of multiobjective linear programming problems in interval-valued intuitionistic fuzzy environment. *Soft Computing*, 23(1):77–84, 2019.

[24] A. K. Bhurjee and S. K. Padhan. Optimality conditions and duality results for non-differentiable interval optimization problems. *Journal of Applied Mathematics and Computing*, 50:59–71, 2016.

[25] A. K. Bhurjee and G. Panda. Efficient solution of interval optimization problem. *Mathematical Methods of Operations Research*, 76(3):273–288, 2012.

[26] A. K. Bhurjee and G. Panda. Parametric multi-objective fractional programming problem with interval uncertainty. *International Journal of Operational Research*, 35(1):132–145, 2019.

[27] J. R. Birge and L. Qi. Subdifferential convergence in stochastic programs. *SIAM Journal on Optimization*, 5(2):436–453, 1995.

[28] R. E. Boche. Complex interval arithmetic with some applications. Master's thesis, San Jose State College, 1966.

[29] J. Borwein. Proper efficient points for maximizations with respect to cones. *SIAM Journal on Control and Optimization*, 15(1):57–63, 1977.

[30] J. Borwein. A note on ϵ-subgradients and maximal monotonicity. *Pacific Journal of Mathematics*, 103(2):307–314, 1982.

[31] G. Bouza, E. Quintana, and C. Tammer. A unified characterization of nonlinear scalarizing functionals in optimization. *Vietnam Journal of Mathematics*, 47(3):683–713, 2019.

[32] G. Bouza, E. Quintana, and C. Tammer. A steepest descent method for set optimization problems with set-valued mappings of finite cardinality. *Journal of Optimization Theory and Applications*, 190(3):711–743, 2021.

[33] A. Brøndsted and R. T. Rockafellar. On the subdifferentiability of convex functions. *Proceedings of the American Mathematical Society*, 16(4):605–611, 1965.

[34] J. V. Burke and S. Deng. Weak sharp minima revisited part I: basic theory. *Control and Cybernetics*, 31:439–469, 2002.

[35] J. V. Burke and S. Deng. Weak sharp minima revisited, part II: application to linear regularity and error bounds. *Mathematical Programming*, 104(2):235–261, 2005.

[36] J. V. Burke and M. C. Ferris. Weak sharp minima in mathematical programming. *SIAM Journal on Control and Optimization*, 31(5):1340–1359, 1993.

[37] P. H. Calamai and J. J. Moré. Projected gradient methods for linearly constrained problems. *Mathematical Programming*, 39(1):93–116, 1987.

[38] G. A. Carrizo, N. S. Fazzio, and M. L. Schuverdt. A nonmonotone projected gradient method for multiobjective problems on convex sets. *Journal of the Operations Research Society of China*, 12(2):410–427, 2024.

[39] Y. Chalco-Cano, W. A. Lodwick, and A. Rufián-Lizana. Optimality conditions of type KKT for optimization problem with interval-valued objective function via generalized derivative. *Fuzzy Optimization and Decision Making*, 12(3):305–322, 2013.

[40] Y. Chalco-Cano, A. Rufián-Lizana, H. Román-Flores, and M.-D. Jiménez-Gamero. Calculus for interval-valued functions using generalized Hukuhara derivative and applications. *Fuzzy Sets and Systems*, 219:49–67, 2013.

[41] S. Chanas and D. Kuchta. A concept of the optimal solution of the transportation problem with fuzzy cost coefficients. *Fuzzy Sets and Systems*, 82(3):299–305, 1996.

[42] S. Chanas and D. Kuchta. Multiobjective programming in optimization of interval objective functions–a generalized approach. *European Journal of Operational Research*, 94(3):594–598, 1996.

[43] R. S. Chauhan, D. Ghosh, and Q. H. Ansari. Generalized Hukuhara Hadamard derivative of interval-valued functions and its applications to interval optimization. *Soft Computing*, 28(5):4107–4123, 2024.

[44] R. S. Chauhan, D. Ghosh, J. Ramík, and A. K. Debnath. Generalized Hukuhara-Clarke derivative of interval-valued functions and its properties. *Soft Computing*, 25:14629–14643, 2021.

[45] G.-Y. Chen, X. Huang, and X. Yang. *Vector Optimization: Set-Valued and Variational Analysis*, volume 541. Springer, Berlin, 2005.

[46] J. Chen, Q. H. Ansari, and J.-C. Yao. Characterizations of set order relations and constrained set optimization problems via oriented distance function. *Optimization*, 66(11):1741–1754, 2017.

[47] S. H. Chen, J. Wu, and Y. D. Chen. Interval optimization for uncertain structures. *Finite Elements in Analysis and Design*, 40(11):1379–1398, 2004.

[48] A. R. Conn, K. Scheinberg, and L. N. Vicente. *Introduction to Derivative-Free Optimization*. SIAM, Philadelphia, 2009.

[49] H. W. Corley. Existence and Lagrangian duality for maximizations of set-valued functions. *Journal of Optimization Theory and Applications*, 54:489–501, 1987.

[50] H. W. Corley. Optimality conditions for maximizations of set-valued functions. *Journal of Optimization Theory and Applications*, 58:1–10, 1988.

[51] R. Correa, A. Hantoute, and M. A. López-Cerdá. Moreau–Rockafellar-type formulas for the subdifferential of the supremum function. *SIAM Journal on Optimization*, 29(2):1106–1130, 2019.

[52] L. Cromme. Strong uniqueness: A far-reaching criterion for the convergence analysis of iterative procedures. *Numerische Mathematik*, 29(2):179–193, 1978.

[53] Y.-H. Dai and Y. Yuan. A nonlinear conjugate gradient method with a strong global convergence property. *SIAM Journal on Optimization*, 10(1):177–182, 1999.

[54] A. Dhara and J. Dutta. *Optimality Conditions in Convex Optimization: A Finite-Dimensional View*. CRC Press, Boca Raton, 2011.

[55] L. M. G. Drummond and A. N. Iusem. A projected gradient method for vector optimization problems. *Computational Optimization and Applications*, 28(1):5–29, 2004.

[56] L. M. G. Drummond and B. F. Svaiter. A steepest descent method for vector optimization. *Journal of Computational and Applied Mathematics*, 175(2):395–414, 2005.

[57] M. Ehrgott, J. Ide, and A. Schöbel. Minmax robustness for multi-objective optimization problems. *European Journal of Operational Research*, 239(1):17–31, 2014.

[58] G. Eichfelder and J. Jahn. Vector and set optimization. In *Multiple Criteria Decision Analysis: State of the Art Surveys*, pages 695–737, Heidelberg, 2016. Springer.

[59] G. Eichfelder, J. Niebling, and S. Rocktäschel. An algorithmic approach to multiobjective optimization with decision uncertainty. *Journal of Global Optimization*, 77(1):3–25, 2020.

[60] G. Eichfelder and S. Rocktäschel. Solving set-valued optimization problems using a multiobjective approach. *Optimization*, 72(3):789–820, 2023.

[61] N. S. Fazzio and M. L. Schuverdt. Convergence analysis of a nonmonotone projected gradient method for multiobjective optimization problems. *Optimization Letters*, 13:1365–1379, 2019.

[62] M. C. Ferris. *Weak sharp minima and penalty functions in mathematical programming*. PhD thesis, University of Cambridge, Cambridge, 1988.

[63] M. C. Ferris. Iterative linear programming solution of convex programs. *Journal of Optimization Theory and Applications*, 65(1):53–65, 1990.

[64] R. Fletcher. Unconstrained optimization. In *Practical Methods of Optimization*, volume 1. John Wiley and Sons, New York, 1980.

[65] R. Fletcher and C. M. Reeves. Function minimization by conjugate gradients. *The Computer Journal*, 7(2):149–154, 1964.

[66] J. Fliege and B. F. Svaiter. Steepest descent methods for multicriteria optimization. *Mathematical Methods of Operations Research*, 51(3):479–494, 2000.

[67] E. H. Fukuda and L. M. G. Drummond. On the convergence of the projected gradient method for vector optimization. *Optimization*, 60(8-9):1009–1021, 2011.

[68] E. H. Fukuda and L. M. G. Drummond. A survey on multiobjective descent methods. *Pesquisa Operacional*, 34:585–620, 2014.

[69] E. M. Gafni and D. P. Bertsekas. Convergence of a gradient projection method. Technical report, Laboratory for Information and Decision Systems, MIT, Cambridge, 1982.

[70] A. M. Geoffrion. Proper efficiency and the theory of vector maximization. *Journal of Mathematical Analysis and Applications*, 22(3):618–630, 1968.

[71] C. Gerstewitz. Nichtkonvexe dualität in der vektoroptimierung. *Wissenschaftliche Zeitschrift der TH Leuna-Merseburg*, 25(3):357–364, 1983.

[72] D. Ghosh. A Newton method for capturing efficient solutions of interval optimization problems. *Opsearch*, 53(3):648–665, 2016.

[73] D. Ghosh. Newton method to obtain efficient solutions of the optimization problems with interval-valued objective functions. *Journal of Applied Mathematics and Computing*, 53(1):709–731, 2017.

[74] D. Ghosh, R. S. Chauhan, R. Mesiar, and A. K. Debnath. Generalized Hukuhara Gâteaux and Fréchet derivatives of interval-valued functions and their application in optimization with interval-valued functions. *Information Sciences*, 510:317–340, 2020.

[75] D. Ghosh, A. K. Debnath, R. S. Chauhan, and R. Mesiar. Generalized-Hukuhara subgradient and its application in optimization problem with interval-valued functions. *Sādhanā*, 47(2):1–16, 2022.

[76] D. Ghosh, A. K. Debnath, and W. Pedrycz. A variable and a fixed ordering of intervals and their application in optimization with interval-valued functions. *International Journal of Approximate Reasoning*, 121:187–205, 2020.

[77] D. Ghosh, A. Singh, K. K. Shukla, and K. Manchanda. Extended Karush-Kuhn-Tucker condition for constrained interval optimization problems and its application in support vector machines. *Information Sciences*, 504:276–292, 2019.

[78] S. Ghosh, D. Ghosh, and Anshika. Normal and tangent cones for set of intervals and their application in optimization with functions of interval variables. *Soft Computing*, 27(15):10737–10758, 2023.

[79] J. C. Gilbert and J. Nocedal. Global convergence properties of conjugate gradient methods for optimization. *SIAM Journal on Optimization*, 2(1):21–42, 1992.

[80] B. M. Glover and B. D. Craven. A Fritz John optimality condition using the approximate subdifferential. *Journal of Optimization Theory and Applications*, 82(2):253–265, 1994.

[81] A. A. Goldstein. On gradient projection. In *Proc. 12th Ann. Allerton Conference and Circuits and Systems*, pages 38–40, Allerton Park, USA, 1974.

[82] A. Göpfert, H. Riahi, C. Tammer, and C. Zalinescu. *Variational Methods in Partially Ordered Spaces*. Springer, New York, 2006.

[83] P. Grohs and S. Hosseini. ε-subgradient algorithms for locally Lipschitz functions on Riemannian manifolds. *Advances in Computational Mathematics*, 42(2):333–360, 2016.

[84] C. Günther, E. Köbis, and N. Popovici. Computing minimal elements of finite families of sets w.r.t. preorder relations in set optimization. *Journal of Applied and Numerical Optimization*, 1(2):131–144, 2019.

[85] C. Günther, E. Köbis, and N. Popovici. On strictly minimal elements w.r.t. preorder relations in set-valued optimization. *Applied Set-Valued Analysis and Optimization*, 1(3):205–219, 2019.

[86] C. Günther and N. Popovici. New algorithms for discrete vector optimization based on the Graef-Younes method and cone-monotone sorting functions. *Optimization*, 67(7):975–1003, 2018.

[87] Y. Guo, G. Ye, W. Liu, D. Zhao, and S. Treanţă. On symmetric gH-derivative: Applications to dual interval-valued optimization problems. *Chaos, Solitons & Fractals*, 158:112068, 2022.

[88] Y. Guo, G. Ye, D. Zhao, and W. Liu. gH-symmetrically derivative of interval-valued functions and applications in interval-valued optimization. *Symmetry*, 11(10):1203, 2019.

[89] C. Gutiérrez, L. Huerga, B. Jiménez, and V. Novo. Proper approximate solutions and ε-subdifferentials in vector optimization: basic properties and limit behaviour. *Nonlinear Analysis: Theory, Methods & Applications*, 79:52–67, 2013.

[90] C. Gutiérrez, B. Jiménez, and V. Novo. A chain rule for ϵ-subdifferentials with applications to approximate solutions in convex Pareto problems. *Journal of Mathematical Analysis and Applications*, 310(1):309–327, 2005.

[91] W. W. Hager and H. Zhang. A new conjugate gradient method with guaranteed descent and an efficient line search. *SIAM Journal on Optimization*, 16(1):170–192, 2005.

[92] A. H. Hamel, F. Heyde, A. Löhne, B. Rudloff, and C. Schrage. Set optimization—a rather short introduction. In *Set Optimization and Applications-The State of the Art: From Set Relations to Set-Valued Risk Measures*, pages 65–141, Heidelberg, 2015. Springer.

[93] A. H. Hamel, F. Heyde, and B. Rudloff. Set-valued risk measures for conical market models. *Mathematics and Financial Economics*, 5(1):1–28, 2011.

[94] A. H. Hamel and A. Löhne. A set optimization approach to zero-sum matrix games with multi-dimensional payoffs. *Mathematical Methods of Operations Research*, 88:369–397, 2018.

[95] A. H. Hamel and D. Visetti. The value functions approach and Hopf-Lax formula for multiobjective costs via set optimization. *Journal of Mathematical Analysis and Applications*, 483(1):123605, 2020.

[96] E. R. Hansen. Interval arithmetic in matrix computations, part I. *Journal of the Society for Industrial and Applied Mathematics, Series B: Numerical Analysis*, 2(2):308–320, 1965.

[97] E. R. Hansen. A generalized interval arithmetic. In *Interval Mathematics: Proceedings of the International Symposium Karlsruhe, West Germany*, pages 7–18. Springer, 1975.

[98] E. R. Hansen. Global optimization using interval analysis: the one-dimensional case. *Journal of Optimization Theory and Applications*, 29(3):331–344, 1979.

[99] E. R. Hansen. Global optimization using interval analysis—the multi-dimensional case. *Numerische Mathematik*, 34(3):247–270, 1980.

[100] E. R. Hansen and S. Sengupta. Bounding solutions of systems of equations using interval analysis. *BIT Numerical Mathematics*, 21(2):203–211, 1981.

[101] A. Hantoute, M. A. López, and C. Zălinescu. Subdifferential calculus rules in convex analysis: a unifying approach via pointwise supremum functions. *SIAM Journal on Optimization*, 19(2):863–882, 2008.

[102] E. Hernández and L. Rodríguez-Marín. Nonconvex scalarization in set optimization with set-valued maps. *Journal of Mathematical Analysis and Applications*, 325(1):1–18, 2007.

[103] J.-B. Hiriart-Urruty and C. Lemaréchal. *Fundamentals of Convex Analysis*. Springer, Berlin, 2004.

[104] J.-B. Hiriart-Urruty and C. Lemaréchal. *Convex Analysis and Minimization Algorithms I*. Springer Berlin, Heidelberg, 2013.

[105] J.-B. Hiriart-Urruty and C. Lemaréchal. *Convex Analysis and Minimization Algorithms II: Advanced Theory and Bundle Methods*. Springer-Verlag, Berlin, Heidelberg, 1993.

[106] J.-B. Hiriart-Urruty and R. Phelps. ε-subdifferential calculus. *In Convex Analysis and Optimization, Pitman, New York*, pages 43–92, 1982.

[107] M. Hukuhara. Intégration des applications mesurables dont la valeur est un compact convexe. *Funkcialaj Ekvacioj*, 10(3):205–223, 1967.

[108] M. Ida. Portfolio selection problem with interval coefficients. *Applied Mathematics Letters*, 16(5):709–713, 2003.

[109] J. Ide and E. Köbis. Concepts of efficiency for uncertain multi-objective optimization problems based on set order relations. *Mathematical Methods of Operations Research*, 80:99–127, 2014.

[110] J. Ide, E. Köbis, D. Kuroiwa, A. Schöbel, and C. Tammer. The relationship between multi-objective robustness concepts and set-valued optimization. *Fixed Point Theory and Applications*, 83(2014), 2014.

[111] M. Inuiguchi and Y. Kume. Goal programming problems with interval coefficients and target intervals. *European Journal of Operational Research*, 52(3):345–360, 1991.

[112] M. Inuiguchi and M. Sakawa. Minimax regret solution to linear programming problems with an interval objective function. *European Journal of Operational Research*, 86(3):526–536, 1995.

[113] A. D. Ioffe. Approximate subdifferentials and applications. I: The finite-dimensional theory. *Transactions of the American Mathematical Society*, 281(1):389–416, 1984.

[114] H. Ishibuchi and H. Tanaka. Multiobjective programming in optimization of the interval objective function. *European Journal of Operational Research*, 48(2):219–225, 1990.

[115] J. Jahn. *Vector Optimization*. Springer Berlin, Heidelberg, 2009.

[116] J. Jahn. A derivative-free descent method in set optimization. *Computational Optimization and Applications*, 60:393–411, 2015.

[117] J. Jahn. Vectorization in set optimization. *Journal of Optimization Theory and Applications*, 167:783–795, 2015.

[118] J. Jahn. A derivative-free rooted tree method in nonconvex set optimization. *Pure and Applied Functional Analysis*, 3(4):603–623, 2018.

[119] J. Jahn and T. X. D. Ha. New order relations in set optimization. *Journal of Optimization Theory and Applications*, 148(2):209–236, 2011.

[120] M. Jana and G. Panda. Solution of nonlinear interval vector optimization problem. *Operational Research*, 14(1):71–85, 2014.

[121] A. Jayswal, I. Ahmad, and J. Banerjee. Nonsmooth interval-valued optimization and saddle-point optimality criteria. *Bulletin of the Malaysian Mathematical Sciences Society*, 39(4):1391–1411, 2016.

[122] V. Jeyakumar and B. M. Glover. Characterizing global optimality for DC optimization problems under convex inequality constraints. *Journal of Global Optimization*, 8:171–187, 1996.

[123] C. Jiang, X. Han, G. R. Liu, and G. P. Liu. A nonlinear interval number programming method for uncertain optimization problems. *European Journal of Operational Research*, 188(1):1–13, 2008.

[124] L. Jiang, J. Cao, and L. Xiong. Generalized multiobjective robustness and relations to set-valued optimization. *Applied Mathematics and Computation*, 361:599–608, 2019.

[125] B. Jiménez, V. Novo, and A. Vílchez. Six set scalarizations based on the oriented distance: properties and application to set optimization. *Optimization*, 69(3):437–470, 2020.

[126] A. Jourani. Hoffman's error bound, local controllability, and sensitivity analysis. *SIAM Journal on Control and Optimization*, 38(3):947–970, 2000.

[127] W. M. K. Kahan. A more complete interval arithmetic. In *Lecture notes for a summer course*. University of Michigan, Ann Arbor, 1968.

[128] E. Karaman, M. Soyertem, İ. Atasever Güvenç, D. Tozkan, M. Küçük, and Y. Küçük. Partial order relations on family of sets and scalarizations for set optimization. *Positivity*, 22:783–802, 2018.

[129] M. M. Karkhaneei and N. Mahdavi-Amiri. Nonconvex weak sharp minima on Riemannian manifolds. *Journal of Optimization Theory and Applications*, 183:85–104, 2019.

[130] E. Kaucher. Interval analysis in the extended interval space IR. In *Fundamentals of Numerical Computation (Computer-Oriented Numerical Analysis)*, pages 33–49. Springer, Vienna, 1980.

[131] A. A. Khan, C. Tammer, and C. Zalinescu. *Set-valued Optimization*. Springer Berlin, Heidelberg, 2016.

[132] E. Köbis and M. A. Köbis. Treatment of set order relations by means of a nonlinear scalarization functional: a full characterization. *Optimization*, 65(10):1805–1827, 2016.

[133] E. Köbis and T. T. Le. Numerical procedures for obtaining strong, strict and ideal minimal solutions of set optimization problems. *Applied Analysis and Optimization*, 2(3):423–440, 2018.

[134] E. Köbis, C. Tammer, and D. Kuroiwa. Generalized set order relations and their numerical treatment. *Applied Analysis and Optimization*, 1(1):45–65, 2017.

[135] G. Kumar and D. Ghosh. Ekeland's variational principle for interval-valued functions. *Computational and Applied Mathematics*, 42(1):28, 2023.

[136] G. Kumar and J.-C. Yao. Frećhet subdifferential calculus for interval-valued functions and its applications in nonsmooth interval optimization. *Journal of Nonlinear and Variational Analysis*, 7(5):811–837, 2024.

[137] K. Kumar, Anshika, and D. Ghosh. Generalized Hukuhara subdifferentiability for convex interval-valued functions and its applications in nonsmooth interval optimization. In *Fuzzy, Rough and Intuitionistic Fuzzy Set Approaches for Data Handling: Theory and Applications*, pages 237–256. Springer, Singapore, 2023.

[138] K. Kumar, D. Ghosh, and G. Kumar. Weak sharp minima for interval-valued functions and its primal-dual characterizations using generalized Hukuhara subdifferentiability. *Soft Computing*, 26(19):10253–10273, 2022.

[139] D. Kuroiwa. Some criteria in set-valued optimization. Investigation on nonlinear analysis and convex analysis. *Surikaisekikenkyusho Kokyuroku*, 985:171–176, 1997.

[140] D. Kuroiwa. On set-valued optimization. *Nonlinear Analysis: Theory, Methods & Applications*, 47(2):1395–1400, 2001.

[141] D. Kuroiwa, T. Tanaka, and T. X. D. Ha. On cone convexity of set-valued maps. *Nonlinear Analysis: Theory, Methods & Applications*, 30(3):1487–1496, 1997.

[142] V. Levin. Some applications of set-valued mappings in mathematical economics. *Journal of Mathematical Economics*, 20(1):69–87, 1991.

[143] A. S. Lewis and J.-S. Pang. Error bounds for convex inequality systems. In *Generalized Convexity, Generalized Monotonicity: Recent Results*, pages 75–110, Boston, MA, 1998. Springer.

[144] C. Li, B. S. Mordukhovich, J. Wang, and J.-C. Yao. Weak sharp minima on Riemannian manifolds. *SIAM Journal on Optimization*, 21(4):1523–1560, 2011.

[145] G. Li. Global error bounds for piecewise convex polynomials. *Mathematical Programming*, 137(1):37–64, 2013.

[146] L. Li. Optimality conditions for nonlinear optimization problems with interval-valued objective function in admissible orders. *Fuzzy Optimization and Decision Making*, 22(2):247–265, 2023.

[147] X. W. Liu and Q. L. Da. A satisfactory solution for interval number linear programming. *Journal of Systems Engineering*, 14(2):123–128, 1999.

[148] Z.-Q. Luo and P. Tseng. On the linear convergence of descent methods for convex essentially smooth minimization. *SIAM Journal on Control and Optimization*, 30(2):408–425, 1992.

[149] V. Lupulescu. Hukuhara differentiability of interval-valued functions and interval differential equations on time scales. *Information Sciences*, 248:50–67, 2013.

[150] L. H. Ma. *Research on method and application of robust optimization for uncertain system.* PhD thesis, Zhejiang University, Hangzhou, 2002.

[151] M. M. Mäkelä and P. Neittaanmäki. *Nonsmooth Optimization: Analysis and Algorithms with Applications to Optimal Control.* World Scientific, Singapore, 1992.

[152] S. Markov. Calculus for interval functions of a real variable. *Computing*, 22:325–337, 1979.

[153] E. Mastrogiacomo and M. Rocca. Set optimization of set-valued risk measures. *Annals of Operations Research*, 296(1):291–314, 2021.

[154] S.-Y. Matsushita and L. Xu. Finite termination of the proximal point algorithm in banach spaces. *Journal of Mathematical Analysis and Applications*, 387(2):765–769, 2012.

[155] K. Mittal, P. Gautam, and V. Vetrivel. Finite convergence and sharp minima for quasi-equilibrium problems. *Journal of Optimization Theory and Applications*, pages 1–24, 2024.

[156] R. E. Moore. *Interval Analysis.* Prentice-Hall, Englewood Cliffs, NJ, 1966.

[157] F. Mráz. Calculating the exact bounds of optimal values in LP with interval coefficients. *Annals of Operations Research*, 81:51–62, 1998.

[158] P. Neame, N. Boland, and D. Ralph. An outer approximate subdifferential method for piecewise affine optimization. *Mathematical Programming*, 87(1):57–86, 2000.

[159] N. Neukel. Order relations of sets and its application in socio-economics. *Applied Mathematical Sciences*, 7(115), 2013.

[160] A. Neumaier. *Interval Methods for Systems of Equations.* Cambridge University Press, Cambridge, UK, 1990.

[161] J. Nocedal and S. J. Wright. *Numerical Optimization.* Springer, New York, 1999.

[162] L. R. L. Pérez and L. F. Prudente. Nonlinear conjugate gradient methods for vector optimization. *SIAM Journal on Optimization*, 28(3):2690–2720, 2018.

[163] L. R. L. Pérez and L. F. Prudente. A Wolfe line search algorithm for vector optimization. *ACM Transactions on Mathematical Software*, 45(4):1–23, 2019.

[164] E. Polak and G. Ribiere. Note sur la convergence de méthodes de directions conjuguées. *Revue Française D'informatique et de Recherche Opérationnelle. Série rouge*, 3(16):35–43, 1969.

[165] B. T. Polyak. The conjugate gradient method in extremal problems. *USSR Computational Mathematics and Mathematical Physics*, 9(4):94–112, 1969.

[166] B. T. Polyak. Sharp minima. institute of control sciences lecture notes. In *IIASA workshop on Generalized Lagrangians and Their Applications, IIASA, Laxenburg, Austria*, 1979.

[167] V. Postolică. Vectorial optimization programs with multifunctions and duality. *Annales Mathématiques du Québec*, 10(1):85–102, 1986.

[168] Z. Quan, F. Zhiping, and P. Dehui. A ranking approach for interval numbers in uncertain multiple attribute decision making problems. *Systems Engineering-Theory & Practice*, 5:129–133, 1999.

[169] F. N. Ris. Tools for the analysis of interval arithmetic. In *Interval Mathematics: Proceedings of the International Symposium Karlsruhe, West Germany*, pages 75–98. Springer, 1975.

[170] R. T. Rockafellar and R. J.-B. Wets. *Variational Analysis.* Springer Science & Business Media, Berlin, 2009.

[171] J. B. Rosen. Nonlinear programming. the gradient projection method. *Bulletin of the American Mathematical Society*, 6(3):25–26, 1957.

[172] J. B. Rosen. The gradient projection method for nonlinear programming. Part I. Linear constraints. *Journal of the Society for Industrial and Applied Mathematics*, 8(1):181–217, 1960.

[173] J. B. Rosen. The gradient projection method for nonlinear programming. Part II. Nonlinear constraints. *Journal of the Society for Industrial and Applied Mathematics*, 9(4):514–532, 1961.

[174] M. Schmidt, A. Schöbel, and L. Thom. Min-ordering and max-ordering scalarization methods for multi-objective robust optimization. *European Journal of Operational Research*, 275(2):446–459, 2019.

[175] A. Schwartz and E. Polak. Family of projected descent methods for optimization problems with simple bounds. *Journal of Optimization Theory and Applications*, 92:1–31, 1997.

[176] A. Sengupta, T. K. Pal, and D. Chakraborty. Interpretation of inequality constraints involving interval coefficients and a solution to interval linear programming. *Fuzzy Sets and Systems*, 119(1):129–138, 2001.

[177] T. Shaocheng. Interval number and fuzzy number linear programmings. *Fuzzy Sets and Systems*, 66(3):301–306, 1994.

[178] F. Shi, G. Ye, W. Liu, and D. Zhao. Optimality conditions for nonsmooth fuzzy optimization models under the gH-weak subdifferentiability. *Computational and Applied Mathematics*, 43(1):10, 2024.

[179] L. Stefanini and B. Bede. Generalized Hukuhara differentiability of interval-valued functions and interval differential equations. *Nonlinear Analysis: Theory, Methods & Applications*, 71(3-4):1311–1328, 2009.

[180] L. Stefanini and B. Bede. Generalized fuzzy differentiability with LU-parametric representation. *Fuzzy Sets and Systems*, 257:184–203, 2014.

[181] R. E. Steuer. Algorithms for linear programming problems with interval objective function coefficients. *Mathematics of Operations Research*, 6(3):333–348, 1981.

[182] M. Studniarski. Weak sharp minima in multiobjective optimization. *Control and Cybernetics*, 36(4):925–937, 2007.

[183] T. Sunaga. Theory of an interval algebra and its application to numerical analysis. *RAAG memoirs*, 2:29–46, 1958.

[184] A. Svensson. Lower semicontinuity of intersections of set-valued maps and applications on bilevel games. *Set-Valued and Variational Analysis*, 32(3):24, 2024.

[185] H. Tanaka and K. Asai. Fuzzy linear programming problems with fuzzy numbers. *Fuzzy Sets and Systems*, 13(1):1–10, 1984.

[186] B. B. Upadhyay, R. K. Pandey, and S. Liao. Newton's method for interval-valued multiobjective optimization problem. *Journal of Industrial and Management Optimization*, 20(4):1633–1661, 2024.

[187] B. B. Upadhyay, R. K. Pandey, J. Pan, and S. Zeng. Quasi-Newton algorithms for solving interval-valued multiobjective optimization problems by using their certain equivalence. *Journal of Computational and Applied Mathematics*, 438:115550, 2024.

[188] D. Van Luu and T. T. Mai. Optimality and duality in constrained interval-valued optimization. *4OR*, 16(3):311–337, 2018.

[189] M. Volle. Calculus rules for global approximate minima and applications to approximate subdifferential calculus. *Journal of Global Optimization*, 5(2):131–157, 1994.

[190] C. Wang, Q. Liu, and X. Yang. Convergence properties of nonmonotone spectral projected gradient methods. *Journal of Computational and Applied Mathematics*, 182(1):51–66, 2005.

[191] C. Wang and N. Xiu. Convergence of the gradient projection method for generalized convex minimization. *Computational Optimization and Applications*, 16(2):111–120, 2000.

[192] J. Wang, C. Li, and J.-C. Yao. Finite termination of inexact proximal point algorithms in Hilbert spaces. *Journal of Optimization Theory and Applications*, 166(1):188–212, 2015.

[193] M. Warmus. Calculus of approximations. *Bulletin de l'Academie Polonaise de Sciences*, 4(5):253–257, 1956.

[194] M. Warmus. Approximations and inequalities in the calculus of approximations. Classification of approximate numbers. *Bulletin de l'Academie Polonaise de Sciences*, 9(4):241–245, 1961.

[195] H. C. Wu. The Karush–Kuhn–Tucker optimality conditions in an optimization problem with interval-valued objective function. *European Journal of Operational Research*, 176(1):46–59, 2007.

[196] H. C. Wu. On interval-valued nonlinear programming problems. *Journal of Mathematical Analysis and Applications*, 338(1):299–316, 2008.

[197] H. C. Wu. Wolfe duality for interval-valued optimization. *Journal of Optimization Theory and Applications*, 138(3):497–509, 2008.

[198] H. C. Wu. Duality theory for optimization problems with interval-valued objective functions. *Journal of Optimization Theory and Applications*, 144(3):615–628, 2010.

[199] X. Y. Wu, G. H. Huang, L. Liu, and J. Li. An interval nonlinear program for the planning of waste management systems with economies-of-scale effects—a case study for the region of Hamilton, Ontario, Canada. *European Journal of Operational Research*, 171(2):349–372, 2006.

[200] Y. D. Xu and S. J. Li. A new nonlinear scalarization function and applications. *Optimization*, 65(1):207–231, 2016.

[201] J. Ye. New uniform parametric error bounds. *Journal of optimization theory and applications*, 98:197–219, 1998.

[202] G. Yuan, Z. Wei, and X. Lu. Global convergence of BFGS and PRP methods under a modified weak Wolfe–Powell line search. *Applied Mathematical Modelling*, 47:811–825, 2017.

[203] J. Zeng and S. J. Li. An Ekeland's variational principle for set-valued mappings with applications. *Journal of Computational and Applied Mathematics*, 230(2):477–484, 2009.

[204] J. Zhou and C. Wang. New characterizations of weak sharp minima. *Optimization Letters*, 6(8):1773–1785, 2012.

[205] G. Zoutendijk. Nonlinear programming, computational methods. *Integer and Nonlinear Programming*, pages 37–86, 1970.

Appendix A

A.1 Proof of Lemma 1.2

(i). Let $\mathbf{A} = [\underline{a}, \overline{a}], \mathbf{B} = [\underline{b}, \overline{b}],$ and $\mathbf{C} = [\underline{c}, \overline{c}]$. We have

$$r \leq \underline{a} \text{ and } r \leq \overline{a}. \tag{A.1}$$

Similarly, by $\mathbf{A} \preceq \mathbf{B} \ominus_{gH} \mathbf{C}$, we have

$$\underline{a} \leq \min\left\{\underline{b} - \underline{c}, \overline{b} - \overline{c}\right\} \text{ and } \overline{a} \leq \max\left\{\underline{b} - \underline{c}, \overline{b} - \overline{c}\right\}. \tag{A.2}$$

- Case 1. Let $\min\left\{\underline{b} - \underline{c}, \overline{b} - \overline{c}\right\} = \overline{b} - \overline{c}$ and $\max\left\{\underline{b} - \underline{c}, \overline{b} - \overline{c}\right\} = \underline{b} - \underline{c}$. Then, from (A.1) and (A.2), we get

$$\overline{c} + r \leq \overline{b} \text{ and } \underline{c} + r \leq \underline{b}.$$

  Hence, $\mathbf{C} \oplus [r, r] \preceq \mathbf{B}$.

- Case 2. When $\min\left\{\underline{b} - \underline{c}, \overline{b} - \overline{c}\right\} = \underline{b} - \underline{c}$ and $\max\left\{\underline{b} - \underline{c}, \overline{b} - \overline{c}\right\} = \overline{b} - \overline{c}$. Then, by equations (A.1) and (A.2), we have

$$\underline{c} + r \leq \underline{b} \text{ and } \overline{c} + r \leq \overline{b}$$

Therefore, again we get the inequality

$$[\underline{c}, \overline{c}] \oplus [r, r] \leq [\underline{b}, \overline{b}],$$

which implies that

$$\mathbf{C} \oplus [r, r] \preceq \mathbf{B}.$$

(ii). Let $\mathbf{A} = [\underline{a}, \overline{a}]$ and $\mathbf{B} = [\underline{b}, \overline{b}]$. Then,

$$\begin{aligned}
& \Big\{(1-\lambda) \odot \mathbf{A} \oplus \lambda \odot \mathbf{B}\Big\} \ominus_{gH} \mathbf{A} \\
= & \Big\{(1-\lambda) \odot [\underline{a}, \overline{a}] \oplus \lambda \odot [\underline{b}, \overline{b}]\Big\} \ominus_{gH} [\underline{a}, \overline{a}] \\
= & \Big[(1-\lambda)\underline{a} + \lambda \underline{b}, (1-\lambda)\overline{a} + \lambda \overline{b}\Big] \ominus_{gH} \Big[\underline{a}, \overline{a}\Big] \text{ as } \lambda \in [0,1] \\
= & \Big[\min\Big\{\lambda \underline{b} - \lambda \underline{a}, \lambda \overline{b} - \lambda \overline{a}\Big\}, \max\Big\{\lambda \underline{b} - \lambda \underline{a}, \lambda \overline{b} - \lambda \overline{a}\Big\}\Big] \\
= & \lambda \odot \Big\{\mathbf{A} \ominus_{gH} \mathbf{B}\Big\}.
\end{aligned}$$

A.2 Proof of Lemma 1.6

Let $\mathbf{T} = [\underline{t}, \overline{t}]$, $\mathbf{U} = [\underline{u}, \overline{u}]$ and $\mathbf{V} = [\underline{v}, \overline{v}]$. Then,

$$\begin{aligned}
\left(\tfrac{1}{2} \odot \mathbf{T} \oplus \tfrac{1}{2} \odot \mathbf{U}\right) \ominus_{gH} \mathbf{V} &= \left(\left[\tfrac{\underline{t}}{2}, \tfrac{\overline{t}}{2}\right] \oplus \left[\tfrac{\underline{u}}{2}, \tfrac{\overline{u}}{2}\right]\right) \ominus_{gH} [\underline{v}, \overline{v}] \\
&= \left[\tfrac{\underline{t}}{2} + \tfrac{\underline{u}}{2}, \tfrac{\overline{t}}{2} + \tfrac{\overline{u}}{2}\right] \ominus_{gH} [\underline{v}, \overline{v}] \\
&= \left[\min\left\{\tfrac{\underline{t}}{2} + \tfrac{\underline{u}}{2} - \underline{v}, \tfrac{\overline{t}}{2} + \tfrac{\overline{u}}{2} - \overline{v}\right\}, \max\left\{\tfrac{\underline{t}}{2} + \tfrac{\underline{u}}{2} - \underline{v}, \tfrac{\overline{t}}{2} + \tfrac{\overline{u}}{2} - \overline{v}\right\}\right].
\end{aligned} \tag{A.3}$$

We have

$$\begin{aligned}
& \tfrac{1}{2} \odot \left(\mathbf{T} \ominus_{gH} \mathbf{V}\right) \oplus \tfrac{1}{2} \odot \left(\mathbf{U} \ominus_{gH} \mathbf{V}\right) \\
= & \tfrac{1}{2} \odot \left[\min\{\underline{t} - \underline{v}, \overline{t} - \overline{v}\}, \max\{\underline{t} - \underline{v}, \overline{t} - \overline{v}\}\right]
\end{aligned} \tag{A.4}$$

$$\oplus \tfrac{1}{2} \odot \left[\min\{\underline{u} - \underline{v}, \overline{u} - \overline{v}\}, \max\{\underline{u} - \underline{v}, \overline{u} - \overline{v}\}\right]. \tag{A.5}$$

We now consider the following four possible cases.

- Case 1. Let $\underline{t} - \underline{v} \leq \overline{t} - \overline{v}$ and $\underline{u} - \underline{v} \leq \overline{u} - \overline{v}$. Then, $\underline{t} + \underline{u} - 2\underline{v} \leq \overline{t} + \overline{u} - 2\overline{v}$, which implies from (A.3) that

$$\left(\tfrac{1}{2} \odot \mathbf{T} \oplus \tfrac{1}{2} \odot \mathbf{U}\right) \ominus_{gH} \mathbf{V} = \left[\tfrac{\underline{t}}{2} + \tfrac{\underline{u}}{2} - \underline{v}, \tfrac{\overline{t}}{2} + \tfrac{\overline{u}}{2} - \overline{v}\right].$$

From (A.4), we have

$$\begin{aligned} \tfrac{1}{2} \odot \left(\mathbf{T} \ominus_{gH} \mathbf{V}\right) \oplus \tfrac{1}{2} \odot \left(\mathbf{U} \ominus_{gH} \mathbf{V}\right) &= \tfrac{1}{2} \odot [\underline{t} - \underline{v}, \overline{t} - \overline{v}] \oplus \tfrac{1}{2} \odot [\underline{u} - \underline{v}, \overline{u} - \overline{v}] \\ &= \left[\tfrac{\underline{t}}{2} + \tfrac{\underline{u}}{2} - \underline{v}, \tfrac{\overline{t}}{2} + \tfrac{\overline{u}}{2} - \overline{v}\right] \\ &= \left(\tfrac{1}{2} \odot \mathbf{T} \oplus \tfrac{1}{2} \odot \mathbf{U}\right) \ominus_{gH} \mathbf{V}. \end{aligned}$$

- Case 2. Let $\overline{t} - \overline{v} \leq \underline{t} - \underline{v}$ and $\overline{u} - \overline{v} \leq \underline{u} - \underline{v}$. Then, $\overline{t} + \overline{u} - 2\overline{v} \leq \underline{t} + \underline{u} - 2\underline{v}$, which implies from (A.3) that

$$\left(\tfrac{1}{2} \odot \mathbf{T} \oplus \tfrac{1}{2} \odot \mathbf{U}\right) \ominus_{gH} \mathbf{V} = \left[\tfrac{\overline{t}}{2} + \tfrac{\overline{u}}{2} - \overline{v}, \tfrac{\underline{t}}{2} + \tfrac{\underline{u}}{2} - \underline{v}\right].$$

From (A.4), we have

$$\begin{aligned} \tfrac{1}{2} \odot \left(\mathbf{T} \ominus_{gH} \mathbf{V}\right) \oplus \tfrac{1}{2} \odot \left(\mathbf{U} \ominus_{gH} \mathbf{V}\right) &= \tfrac{1}{2} \odot [\overline{t} - \overline{v}, \underline{t} - \underline{v}] \oplus \tfrac{1}{2} \odot [\overline{u} - \overline{v}, \underline{u} - \underline{v}] \\ &= \left[\tfrac{\overline{t}}{2} + \tfrac{\overline{u}}{2} - \overline{v}, \tfrac{\underline{t}}{2} + \tfrac{\underline{u}}{2} - \underline{v}\right] \\ &= \left(\tfrac{1}{2} \odot \mathbf{T} \oplus \tfrac{1}{2} \odot \mathbf{U}\right) \ominus_{gH} \mathbf{V}. \end{aligned}$$

- Case 3. Let $\overline{t} - \overline{v} < \underline{t} - \underline{v}$ and $\underline{u} - \underline{v} < \overline{u} - \overline{v}$. Then, from (A.4), we have

$$\begin{aligned}\tfrac{1}{2} \odot \left(\mathbf{T} \ominus_{gH} \mathbf{V}\right) \oplus \tfrac{1}{2} \odot \left(\mathbf{U} \ominus_{gH} \mathbf{V}\right) =& \tfrac{1}{2} \odot [\overline{t} - \overline{v}, \underline{t} - \underline{v}] \oplus \tfrac{1}{2} \odot [\underline{u} - \underline{v}, \overline{u} - \overline{v}] \\ =& \left[\tfrac{\overline{t}}{2} + \tfrac{\underline{u}}{2} - \tfrac{\overline{v}}{2} - \tfrac{\underline{v}}{2}, \tfrac{\underline{t}}{2} + \tfrac{\overline{u}}{2} - \tfrac{\underline{v}}{2} - \tfrac{\overline{v}}{2}\right].\end{aligned}$$

Let if possible

$$\tfrac{1}{2} \odot \left(\mathbf{T} \ominus_{gH} \mathbf{V}\right) \oplus \tfrac{1}{2} \odot \left(\mathbf{U} \ominus_{gH} \mathbf{V}\right) \prec \left(\tfrac{1}{2} \odot \mathbf{T} \oplus \tfrac{1}{2} \odot \mathbf{U}\right) \ominus_{gH} \mathbf{V}.$$

Then, $\left[\tfrac{\overline{t}}{2} + \tfrac{\underline{u}}{2} - \tfrac{\overline{v}}{2} - \tfrac{\underline{v}}{2}, \tfrac{\underline{t}}{2} + \tfrac{\overline{u}}{2} - \tfrac{\underline{v}}{2} - \tfrac{\overline{v}}{2}\right]$

$$\prec \left[\min\left\{\tfrac{\underline{t}}{2} + \tfrac{\underline{u}}{2} - \underline{v}, \tfrac{\overline{t}}{2} + \tfrac{\overline{u}}{2} - \overline{v}\right\}, \max\left\{\tfrac{\underline{t}}{2} + \tfrac{\underline{u}}{2} - \underline{v}, \tfrac{\overline{t}}{2} + \tfrac{\overline{u}}{2} - \overline{v}\right\}\right]. \tag{A.6}$$

We have two further possible cases.

- Case (i). If $\tfrac{\underline{t}}{2} + \tfrac{\underline{u}}{2} - \underline{v} \leq \tfrac{\overline{t}}{2} + \tfrac{\overline{u}}{2} - \overline{v}$, then from (A.6), we have

$$\begin{aligned}& \tfrac{\underline{t}}{2} + \tfrac{\overline{u}}{2} - \tfrac{\underline{v}}{2} - \tfrac{\overline{v}}{2} \leq \tfrac{\overline{t}}{2} + \tfrac{\overline{u}}{2} - \overline{v} \\ \Longrightarrow\ & \underline{t} - \underline{v} \leq \overline{t} - \overline{v},\end{aligned}$$

  which is impossible as $\overline{t} - \overline{v} < \underline{t} - \underline{v}$.

- Case (ii). If $\tfrac{\overline{t}}{2} + \tfrac{\overline{u}}{2} - \overline{v} \leq \tfrac{\underline{t}}{2} + \tfrac{\underline{u}}{2} - \underline{v}$, then from (A.6), we have

$$\begin{aligned}& \tfrac{\underline{t}}{2} + \tfrac{\overline{u}}{2} - \tfrac{\underline{v}}{2} - \tfrac{\overline{v}}{2} \leq \tfrac{\underline{t}}{2} + \tfrac{\underline{u}}{2} - \underline{v} \\ \Longrightarrow\ & \overline{u} - \overline{v} \leq \underline{u} - \underline{v},\end{aligned}$$

  which is impossible as $\underline{u} - \underline{v} < \overline{u} - \overline{v}$.

- Case 4. Let $\underline{t}-\underline{v}<\overline{t}-\overline{v}$ and $\overline{u}-\overline{v}<\underline{u}-\underline{v}$. Then, from (A.4), we have

$$\begin{aligned}\tfrac{1}{2}\odot\left(\mathbf{T}\ominus_{gH}\mathbf{V}\right)\oplus\tfrac{1}{2}\odot\left(\mathbf{U}\ominus_{gH}\mathbf{V}\right)&=\tfrac{1}{2}\odot[\underline{t}-\underline{v},\overline{t}-\overline{v}]\oplus\tfrac{1}{2}\odot[\overline{u}-\overline{v},\underline{u}-\underline{v}]\\&=\left[\tfrac{\underline{t}}{2}+\tfrac{\overline{u}}{2}-\tfrac{\underline{v}}{2}-\tfrac{\overline{v}}{2},\tfrac{\overline{t}}{2}+\tfrac{\underline{u}}{2}-\tfrac{\overline{v}}{2}-\tfrac{\underline{v}}{2}\right].\end{aligned}$$

Let if possible

$$\tfrac{1}{2}\odot\left(\mathbf{T}\ominus_{gH}\mathbf{V}\right)\oplus\tfrac{1}{2}\odot\left(\mathbf{U}\ominus_{gH}\mathbf{V}\right)\prec\left(\tfrac{1}{2}\odot\mathbf{T}\oplus\tfrac{1}{2}\odot\mathbf{U}\right)\ominus_{gH}\mathbf{V}.$$

Then, $\left[\tfrac{\underline{t}}{2}+\tfrac{\overline{u}}{2}-\tfrac{\underline{v}}{2}-\tfrac{\overline{v}}{2},\tfrac{\overline{t}}{2}+\tfrac{\underline{u}}{2}-\tfrac{\overline{v}}{2}-\tfrac{\underline{v}}{2}\right]$

$$\prec\left[\min\left\{\tfrac{\underline{t}}{2}+\tfrac{\underline{u}}{2}-\underline{v},\tfrac{\overline{t}}{2}+\tfrac{\overline{u}}{2}-\overline{v}\right\},\max\left\{\tfrac{\underline{t}}{2}+\tfrac{\underline{u}}{2}-\underline{v},\tfrac{\overline{t}}{2}+\tfrac{\overline{u}}{2}-\overline{v}\right\}\right].\qquad\text{(A.7)}$$

We have two further possible cases.

- Case (i). If $\tfrac{\underline{t}}{2}+\tfrac{\underline{u}}{2}-\underline{v}\leq\tfrac{\overline{t}}{2}+\tfrac{\overline{u}}{2}-\overline{v}$, then from (A.7), we have

$$\begin{aligned}&\tfrac{\overline{t}}{2}+\tfrac{\underline{u}}{2}-\tfrac{\overline{v}}{2}-\tfrac{\underline{v}}{2}\leq\tfrac{\overline{t}}{2}+\tfrac{\overline{u}}{2}-\overline{v}\\\Longrightarrow\ &\underline{u}-\underline{v}\leq\overline{u}-\overline{v},\end{aligned}$$

which is impossible as $\overline{u}-\overline{v}<\underline{u}-\underline{v}$.

- Case (ii). If $\tfrac{\overline{t}}{2}+\tfrac{\overline{u}}{2}-\overline{v}\leq\tfrac{\underline{t}}{2}+\tfrac{\underline{u}}{2}-\underline{v}$, then from (A.7), we have

$$\begin{aligned}&\tfrac{\overline{t}}{2}+\tfrac{\underline{u}}{2}-\tfrac{\overline{v}}{2}-\tfrac{\underline{v}}{2}\leq\tfrac{\underline{t}}{2}+\tfrac{\underline{u}}{2}-\underline{v}\\\Longrightarrow\ &\overline{t}-\overline{v}\leq\underline{t}-\underline{v},\end{aligned}$$

which is impossible as $\underline{t}-\underline{v}<\overline{t}-\overline{v}$.

Hence, accumulating all cases, for any $\mathbf{T}, \mathbf{U}$ and $\mathbf{V}$ in $I(\mathbb{R})$, we have

$$\left(\tfrac{1}{2} \odot \mathbf{T} \oplus \tfrac{1}{2} \odot \mathbf{U}\right) \ominus_{gH} \mathbf{V} \not\preceq \tfrac{1}{2} \odot \left(\mathbf{T} \ominus_{gH} \mathbf{V}\right) \oplus \tfrac{1}{2} \odot \left(\mathbf{U} \ominus_{gH} \mathbf{V}\right).$$

A.3 Proof of Lemma 1.7

Let $\mathbf{T} = (\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_n)$ with $\mathbf{T}_j = [\underline{t}_j, \overline{t}_j]$ for all $j = 1, 2, \ldots, n$, $\mathbf{U} = [\underline{u}, \overline{u}]$ and $\mathbf{V} = [\underline{v}, \overline{v}]$. Then,

$$\begin{aligned} & x^\top \odot \mathbf{T} \preceq \mathbf{U} \ominus_{gH} \mathbf{V} \oplus \epsilon \\ \implies & \left[\min\left\{\sum_{j=1}^{n} x_j \underline{t}_j, \sum_{j=1}^{n} x_j \overline{t}_j\right\}, \max\left\{\sum_{j=1}^{n} x_j \underline{t}_j, \sum_{j=1}^{n} x_j \overline{t}_j\right\}\right] && \text{(A.8)} \\ & \preceq \left[\min\left\{\underline{u} - \underline{v}, \overline{u} - \overline{v}\right\}, \max\left\{\underline{u} - \underline{v}, \overline{u} - \overline{v}\right\}\right] \oplus \epsilon. && \text{(A.9)} \end{aligned}$$

Here, two cases arise.

- Case 1. Let $\min\left\{\sum\limits_{j=1}^{n} x_j \underline{t}_j, \sum\limits_{j=1}^{n} x_j \overline{t}_j\right\} = \sum\limits_{j=1}^{n} x_j \underline{t}_j$ and $\min\left\{\underline{u} - \underline{v}, \overline{u} - \overline{v}\right\} = \underline{u} - \underline{v}$. Thus, by (A.8), we have

$$\begin{aligned} & \sum_{j=1}^{n} x_j \underline{t}_j \leq \underline{u} - \underline{v} + \epsilon \text{ and } \sum_{j=1}^{n} x_j \overline{t}_j \leq \overline{u} - \overline{v} + \epsilon \\ \implies & \underline{v} - \underline{u} \leq \sum_{j=1}^{n} (-x_j) \underline{t}_j + \epsilon \text{ and } \overline{u} - \overline{v} \leq \sum_{j=1}^{n} (-x_j) \overline{t}_j + \epsilon \\ \implies & \left[\min\left\{\underline{u} - \underline{v}, \overline{u} - \overline{v}\right\}, \max\left\{\underline{u} - \underline{v}, \overline{u} - \overline{v}\right\}\right] \preceq \\ & \left[\min\left\{\sum_{j=1}^{n} (-x_j) \underline{t}_j, \sum_{j=1}^{n} (-x_j) \overline{t}_j\right\}, \max\left\{\sum_{j=1}^{n} (-x_j) \underline{t}_j, \sum_{j=1}^{n} (-x_j) \overline{t}_j\right\}\right] \oplus \epsilon \\ \implies & \mathbf{V} \ominus_{gH} \mathbf{U} \preceq (-x)^\top \odot \mathbf{T} \oplus \epsilon. \end{aligned}$$

- Case 2. Let $\min\left\{\sum\limits_{j=1}^{n} x_j\underline{t}_j, \sum\limits_{j=1}^{n} x_j\bar{t}_j\right\} = \sum\limits_{j=1}^{n} x_j\bar{t}_j$ and $\min\{\underline{u}-\underline{v}, \overline{u}-\overline{v}\} = \underline{u}-\underline{v}$. Thus, by (A.8), we have

$$\begin{aligned}
&\sum_{j=1}^{n} x_j\bar{t}_j \leq \underline{u}-\underline{v}+\epsilon \text{ and } \sum_{j=1}^{n} x_j\underline{t}_j \leq \overline{u}-\overline{v}+\epsilon \\
&\Longrightarrow \underline{v}-\underline{u} \leq \sum_{j=1}^{n}(-x_j)\bar{t}_j+\epsilon \text{ and } \overline{u}-\overline{v} \leq \sum_{j=1}^{n}(-x_j)\underline{t}_j+\epsilon \\
&\Longrightarrow \left[\min\{\underline{u}-\underline{v}, \overline{u}-\overline{v}\}, \max\{\underline{u}-\underline{v}, \overline{u}-\overline{v}\}\right] \preceq \\
&\quad \left[\min\left\{\sum_{j=1}^{n}(-x_j)\underline{t}_j, \sum_{j=1}^{n}(-x_j)\bar{t}_j\right\}, \max\left\{\sum_{j=1}^{n}(-x_j)\underline{t}_j, \sum_{j=1}^{n}(-x_j)\bar{t}_j\right\}\right] \oplus \epsilon \\
&\Longrightarrow \mathbf{V} \ominus_{gH} \mathbf{U} \preceq (-x)^\top \odot \mathbf{T} \oplus \epsilon.
\end{aligned}$$

Hence, by Case 1 and Case 2, for any $x \in \mathbb{R}^n$, $\widehat{\mathbf{T}} \in I(\mathbb{R})^n$, and $\mathbf{U}, \mathbf{V} \in I(\mathbb{R})$,

$$x^\top \odot \mathbf{T} \preceq \mathbf{U} \ominus_{gH} \mathbf{V} \oplus \epsilon \implies \mathbf{V} \ominus_{gH} \mathbf{U} \preceq (-x)^\top \odot \mathbf{T} \oplus \epsilon.$$

A.4 Proof of Lemma 1.8

Note that $x^\top \odot \widehat{\mathbf{T}} \oplus \epsilon$

$$\begin{aligned}
&= \left[\min\left\{\sum_{j=1}^{n} x_j\underline{t}_j, \sum_{j=1}^{n} x_j\bar{t}_j\right\}, \max\left\{\sum_{j=1}^{n} x_j\underline{t}_j, \sum_{j=1}^{n} x_j\bar{t}_j\right\}\right] \oplus \epsilon \\
&\preceq \left[\min\left\{\sum_{j=1}^{n} |x_j| \|\mathbf{T}_j\|_{I(\mathbb{R})}, \sum_{j=1}^{n} |x_j| \|\mathbf{T}_j\|_{I(\mathbb{R})}\right\},\right. \\
&\qquad\qquad \left.\max\left\{\sum_{j=1}^{n} |x_j| \|\mathbf{T}_j\|_{I(\mathbb{R})}, \sum_{j=1}^{n} |x_j| \|\mathbf{T}_j\|_{I(\mathbb{R})}\right\}\right] \oplus \epsilon \\
&= \left[\min\left\{\|x\| \|\widehat{\mathbf{T}}\|_{I(\mathbb{R})^n}, \|x\| \|\widehat{\mathbf{T}}\|_{I(\mathbb{R})^n}\right\}, \max\left\{\|x\| \|\widehat{\mathbf{T}}\|_{I(\mathbb{R})^n}, \|x\| \|\widehat{\mathbf{T}}\|_{I(\mathbb{R})^n}\right\}\right] \oplus \epsilon \\
&\preceq \|x\| \odot \left[\|\widehat{\mathbf{T}}\|_{I(\mathbb{R})^n}+\epsilon, \|\widehat{\mathbf{T}}\|_{I(\mathbb{R})^n}+\epsilon\right].
\end{aligned}$$

List of published/accepted articles

1. **K. Kumar**, D. Ghosh, and G. Kumar. Weak sharp minima for interval-valued functions and its primal-dual characterizations using generalized Hukuhara subdifferentiability. *Soft Computing*, 26, 10253–10273, 2022.

2. **K. Kumar**, D. Ghosh, J. Chen, and J.-C. Yao. Epsilon-subdifferentiability for interval-valued functions and its application in interval optimization problems. *Computational and Applied Mathematics.* (Accepted)

3. **K. Kumar**, Anshika, and D. Ghosh. Generalized Hukuhara subdifferentiability for convex interval-valued functions and its applications in nonsmooth interval optimization. *In Fuzzy, Rough and Intuitionistic Fuzzy Set Approaches for Data Handling: Theory and Applications*, Springer, Singapore, 2023.

4. **K. Kumar**, D. Ghosh, J.-C. Yao, and X. Zhao. Nonlinear conjugate gradient methods for unconstrained set optimization problems whose objective functions have finite cardinality. *Optimization*, 1–40, 2024.
   https://doi.org/10.1080/02331934.2024.2390116

5. D. Ghosh, **K. Kumar**, J.-C. Yao, and X. Zhao. A projected gradient method for constrained set optimization problems with set-valued mappings of finite cardinality. *Engineering Optimization*, 1–30, 2025.
   https://doi.org/10.1080/0305215X.2024.2433077

6. **K. Kumar**, A. Singh, A. Upadhyay, D. Ghosh. An improved nonmonotone quasi-Newton method for multiobjective optimization problems. *Journal of Nonlinear and Convex Analysis.* (Accepted)

7. **K. Kumar**, D. Ghosh, A. Upadhyay, J.-C. Yao, and X. Zhao. Quasi-Newton methods for multiobjective optimization problems: A systematic review. *Applied Set-Valued Analysis and Optimization*, 5(2), 291–321, 2023.

8. Anshika, **K. Kumar**, and D. Ghosh. Generalized Hukuhara Dini Hadamard ϵ-subdifferential and H_ϵ-subgradient and their applications in interval optimization. *Journal of Applied and Numerical Optimization*, 6(2), 177-202, 2024.

9. A. Singh, **K. Kumar**, and D. Ghosh. Improved nonmonotone adaptive trust-region method to solve generalized Nash equilibrium problems. *Journal of Nonlinear and Convex Analysis*, 25(1), 11–29, 2024.

10. A. Upadhyay, D. Ghosh, and **K. Kumar**. Nonmonotone Wolfe-type quasi-Newton methods for multiobjective optimization problems. *Optimization*. (Accepted)

11. Anshika, **K. Kumar**, and D. Ghosh. Generalized Hukuhara global subdifferentiability in interval optimization problems. *In Fuzzy Optimization, Decision-making and Operations Research*, Springer Cham, 2023.