ELEMENTS OF MECHANICAL VIBRATION ANALYSIS & SYNTHESIS

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2019

ISBN: 978-93-5382-954-4

DEDICATED
TO THE MEMORY OF MY BELOVED SON
ER. SANJEEV KUMAR GUPTA, I.R.S.E.E
ASTT. ELECTIRCAL ENGINEER
INDIAN RAILWAYS

Acknowledgements

Writing a book at the age of 83 of life and searching for a publisher to publish is harder than I thought and more rewarding than I could have ever imagined. Having an idea and turning it into a book is as hard as it sounds. The experience is both internally challenging and rewarding. I especially want to thank the individuals who helped make this happen. None of this would have been possible without the support of Prof. Santosh Kumar and Dr. Navin Upadhyay. A very special thanks to library staff, Mr. Kanu Chakraborty and Mr. Abhishek Shukla, for their support in the compilation of the book.

A very special thanks to Dr. Navin Upadhyay for supporting, encouraging and arranging the ISBN to publish this title as e-books and keep it in open domain for wide accessibility. In spite of his busy schedule and workload, he spared time to give this book a shape.

I want to thank God most of all, because without his blessings I would not be able to do any of this.

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CHAPTER I
INTRODUCTION

1.11(a) BACKGROUND

Mechanical engineering design has undergone fundamental changes arising out of space age requirements. Design requirements of the present space technology have led to the emergence of many new technological disciplines. These requirements have also provided sufficient background material for the transformation of already existing areas of technology. It is due to this fact that the subjects of mechanical design and analysis, amongst many others have undergone fundamental changes. Mechanical design requirements of the present day space technology are quite complex and they place demands of high orders of accuracy on various designed functional components of systems. Precise dynamic performance evaluation of designed mechanical systems has become a prerequisite before any action is initiated for their manufacture. The subject of machine vibrations which is concerned with the evaluation of dynamics of mechanical systems, needs methodologies which can successfully deal with emerging situations of present era. One of the aims of this book is to provide elements of study material to meet these requirements. Special emphasis has been given to modern approach for handling vibration problems of complex systems.

In order to fully appreciate the contents of this book it is desirable for the reader to undertake lessons in Engineering mechanics and Mechanics of deformable solids.

1.1.1 IMPORTANCE

Importance of the study of vibrations is primarily due to their occurrence in different forms in a wide variety of situations. Physical phenomena of nature show cyclic or oscillatory character. Machines and structures develop undesirable vibrations under certain conditions. Vibrations are also generated intentionally for carrying out certain engineering and other useful tasks. Some typical examples of vibrations from different fields are given below:

Vibrations of living systems (viz. heart beats are periodic, lungs expand and contract periodically, biorhythms of the body functions are periodic); oscillatory nature of physical phenomena of nature (viz. light ft sound; molecules and atoms exhibit vibrations at their respective levels); vibrations of celestial bodies (viz. earth vibrates); vibrations of the man-made systems (viz. machines vibrate, rockets vibrate, artificial satellites vibrate, structures vibrate, etc.); and so on.

Theory of vibrations has also been successfully applied for accomplishing certain useful engineering tasks. Some of the important examples of these applications are:

Shakers in foundries, pilasters in testing machines, vibratory conveyance of materials (or components), vibratory sieving, vibratory tamping, vibratory pile driving, vibratory forging, vibratory casting, vibratory machining, vibratory welding, vibratory heat transfer, vibration assisted shovels of bulldozers, vibration assisted fruit harvesting, vibratory stress relief, etc. Vibration concepts are also used in the design and development of various instruments meant for the measurement of dynamic phenomena.

1.11 MECHANICAL VIBRATIONS

Subject of mechanical vibrations is concerned with the study of oscillatory phenomenon of mechanical systems. Main objective of the study of this subject is the analysis of dynamic systems with a view to provide measures for the control of undesirable vibrations. Devising means for generating vibrations to accomplish prescribed useful engineering tasks is another objective of its study.

Control of undesirable vibrations of mechanical systems is of paramount importance since their harmful effects can be destructive. Presence of these vibrations can impair functioning of the machines which create them; machines in the vicinity can be damaged; and many kinds of physiological and psychological damage can be caused to human beings present at the site of vibrations.

Production or maintenance of vibrations requires energy input. For this purpose an external supply of energy needed (e.g. examples of useful engineering processes which are based on vibration principle). In case vibrations produced as a result of some internal dynamic process of the system itself then this is possible only when a part of the useful energy of the system is consumed. Vibrations generated are undesirable and must be eliminated.

1.12 HISTORICAL DEVELOPMENT OF VIBRATIONS

The beginning of man's scientific interest in vibrations can be ascribed to the discovery of musical instruments. Early men (musicians and philosophers) applied the laws of sound production to the design and manufacture of instrument. They understood the physics of the vibrating strings enough to design a variety of bowed and plucked instrument. They used the knowledge of the principle of sounding as an amplifier of a tone from a string attached to it.

Pythagoras (582 - 507 B.C.) showed that if two like strings are subjected to equal tensions and one is half the length of the other, then the tones they produce will be an octave apart. Though it was known that sound was related to mechanical vibrations yet it was not until Galileo Galilee (1564-1662) made it clear that the pitch is determined by the vibration frequency. He was familiar with isochronism of a simple pendulum and applied this property for the time measurement.

Sauveur (1653-1716) made use of the phenomenon of beats to demonstrate the relationship between pitch and frequency. Brook Tay1or (1685-1731) was the first to have computed frequencies of vibrating strings which agreed with the experimental findings of Galileo and those of Mercene (1588-1703). John Wal1is (1616-1703) and Sauveur are credited with having first observed the phenomenon of modes. It was Daniel Bernoulli who first proposed the principle of linear superposition of harmonics. Fourier (1882) established the method of harmonic series.

THE STUDY OF STRING VIBRATIONS REPRESENTS THE TRANSITION BETWEEN MUSIC AND MECHANICAL VIBRATIONS.

Euler (1744), D. Bernoulli (1751), Chaladni (1802), Navier (1824) Navier (1827), Cauchey (1828) and Poisson (1829, 1833) have made important contributions in one way or the other towards the advancement of the subject of mechanical vibrations. Lord Rayleigh (1842-1919) made very significant contributions to the theory of vibrations of thin shells and plates. Wohler (1819-1914) provided insight of the fatigue failure due to vibrations. Stodola (1859-1943) made notable contributions to the study of beams, plates and membranes. His principal subject of study was the turbine and he was probably the first who found that vibration analysis was very necessary for workable turbine designs.

H. Poincare (1892) is considered to be the father of the theory of nonlinear vibrations. Andronov and Leontovich (1927) discussed directly the differential equations with periodic coefficients which give rise to parametric vibrations. Exhaustive coverage on the theory of vibrations has been provided by Malkin (1966) as also by Minorsky (1962). The work of the Russian school of Krylot and Bogoliubovl (1937) and Mitropolskyl (1965) paved the way for unprecedented growth of mechanics of vibratory motion. Tondl (l963) using complex variables studied parametric main and combination resonances.

1.2 ELEMENTARY CONCEPTS OF VIBRATION AND SHOCK

Consider that a body (machine/structure are part of a massive system) is held in position by restorative (e.g. elastic) constraints. If this body is momentarily disturbed from its position of equilibrium by direct or indirect force(s) then it will commence a to-and-fro movement. This to-and-fro motion of the body about its equilibrium position is called vibration.

1.21 Periodic Vibrations

Periodic vibration (or oscillation) of a particle or a body is a to-and-fro motion which takes place about a reference position such that the motion repeats itself exactly after fixed period of time. This fixed period of time is called periodic time. The periodic time (T) is measured in seconds. Reciprocal of the periodic time is frequency (f). The units of frequency are in Hertz (or cycles per second). Frequency and the periodic time are related by the expression

$$f = 1/T$$

Simplest kind of periodic notion is a simple harmonic motion (fig.1.1). For a simple harmonic motion (SHM), relation between the displacement (x) of a body from its reference position and the time(t) is given by:

$$x = x_0 sin\omega t \tag{1.1}$$

Where x = displacement from equilibrium position

x_o = amplitude of vibrations

ω = circular frequency (radians per second)

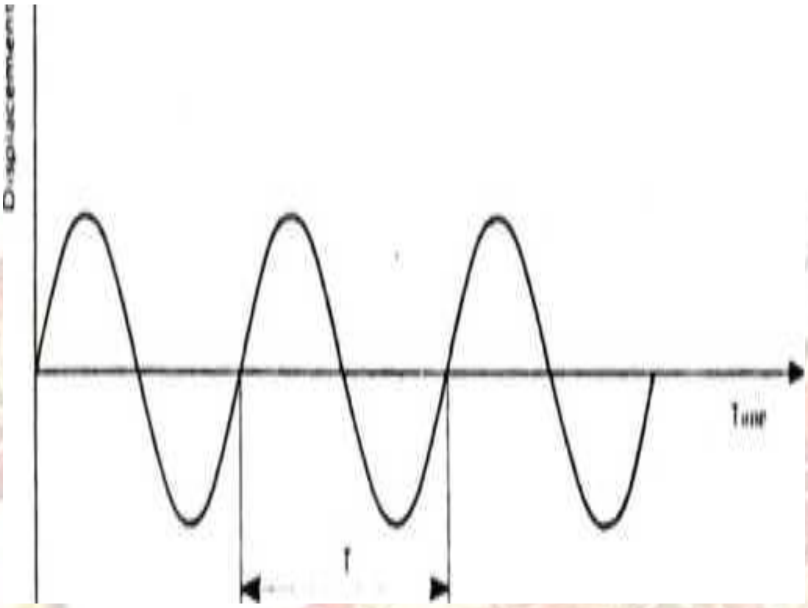


Fig.1.1: A simple harmonic motion(sinusoidal)

Circular frequency (ω) and the periodic time (T) are related by the following expression:

$$\boldsymbol{T = 2\pi/\omega} \text{ seconds} \tag{1.2}$$

This motion (i.e. SHM) can also be described in terms of the velocity (v). The velocity is obtained by differentiating, the equation (1.1) with respect to time. Thus

$$v = \frac{dx}{dt} = \omega x_0 \cos(\omega t) = v_0 \sin(\omega t + \pi/2) \tag{1.1a}$$

Where $v_0 = \omega x_0$.

Similarly, acceleration can also be used for this purpose. Thus

$$a = \frac{dv}{dt} = -\omega^2 x_0 \sin(\omega t) = a_0 \sin(\omega t + \pi) \tag{1.1b}$$

Where $a_0 = \omega^2 x_0$ is amplitude of acceleration

Equations (1.1),(1.la) and (1.1b) describe simple harmonic; motion. It is to be observed from these equations that the form and period of vibrations remain the same whether it is displacement, velocity or acceleration. The velocity leads displacement by n/2 and acceleration leads displacement by π.

1.211 Phase Difference

Consider following two vibrations

$$x_1 = a \sin \omega t \text{ and } x_2 = b \sin(\omega t + \emptyset)$$

These vibrations do not attain maxima of their displacements simultaneously, one lags behind the other by an amount equal to $\emptyset/\omega$. The quantity ($\emptyset$) is called the phase angle or phase difference between these two vibrations. A phase angle has a meaning in that case in which the concerned motions have the same circular frequency.

Most vibrations encountered in practice are not simple harmonic in nature even though they may be periodic. A typical case of a non-harmonic but periodic motion is shown in fig.1.2. It depicts the variation of piston acceleration of a single cylinder reciprocating engine plotted against the time.

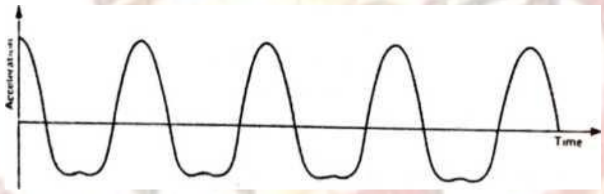


Fig.1.2: Example of a periodic but non-harmonic motion

1.22 Random Vibrations

Random vibrations are those in which the vibratory particle undergo irregular motion cycles that never repeat exactly. This type of vibrations occur quite frequently in nature. Fig.1.3 shows the trace of one such typical case of random vibrations.

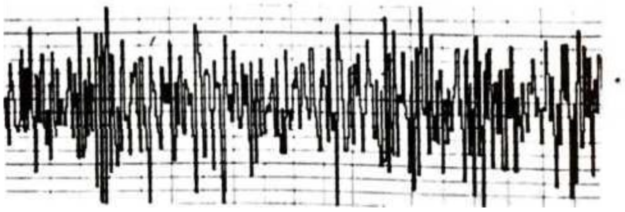


Fig.1.3: A typical random vibration signal

1.23 Shocks

Mechanical shocks are relatively abundant in daily life. Origin of these shocks can be traced to a wide variety of sources, e.g. explosions, rough handling of equipment, hammering action, supersonic motion etc.

Shocks are characterized by the sudden energy release over a short duration of time. A simple shock is a transference of kinetic energy from one system to another system such that this energy transfer takes place in a relatively short time when compared with the natural period of vibration of the system. A transient phenomenon is also of this type. A basic difference between the two is that the latter may last for several periods of vibration of the system. Transient phenomena are also described as complex shocks. Fig. 1.4 shows some of the shock time functions

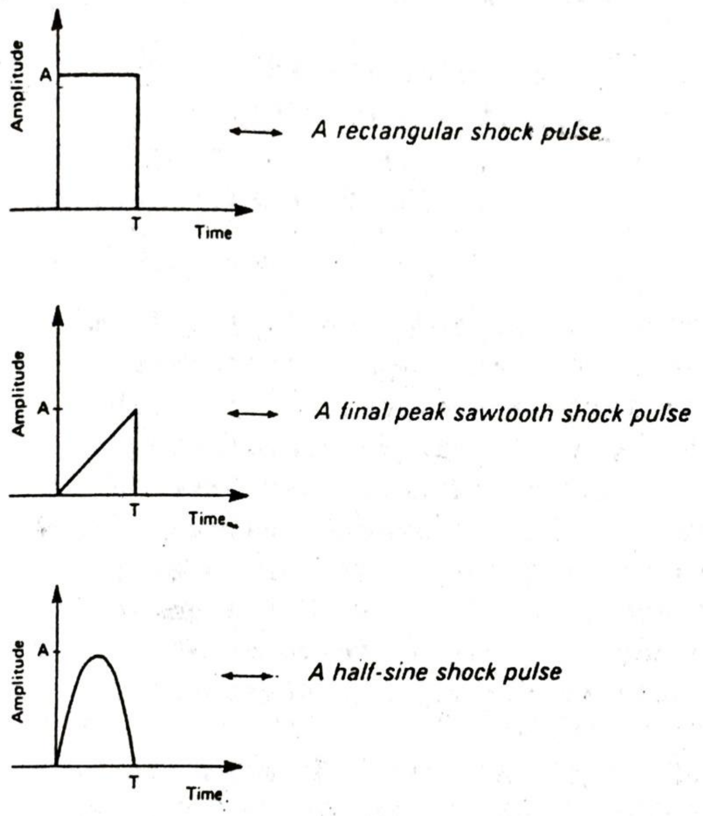


Fig.1.4: **Shock-time functions,(a)rectangular(b)saw-tooth and (c)half-sine pulse**

2 CLASSIFICATION OF VIBRATIONS

There is a wide variety of physical phenomena which give rise to vibrations. These vibrations may or may not be of the same type. As a matter of fact vibrations are of many types. In order to have a logical and comprehensive treatment of the subject it is necessary to put different vibrations in certain well defined categories. Hence, the classification of vibrations becomes necessary. Some of the important methods of classification are given below;

1. Degrees of freedom
2. Nature of the differential equation(s)
3. Mechanism(s) of initiation of vibration
4. Displacement-time diagram of the vibratory system
5. Orientation of the axis of predominant vibrations

1. Classification based *on* degrees of freedom

The number of generalized coordinates needed to define the geometric configuration of a system represents its number of degrees of freedom. Accordingly, a vibratory system can be a single degree of freedom system or a multi-degree of freedom system.

A vibratory system basically consists of mass particles which are interconnected by restorative (e.g. elastic) mechanisms. Geometric configuration of such a system is defined with the help of spatial coordinates attached to its masses. These spatial coordinates may have certain set *of* interrelationships. The difference between the number *of* spatial coordinates and the number of inter-relationships is defined as the number of degrees of freedom of the vibratory system under study. If this difference is equal to one *then* the system is said to possess one degree of freedom.

For the case of a system having n mass particles and no interrelationships between them, it is necessary to use all of its n spatial coordinates for defining its geometric configuration. Such a system is said to possess n degrees of freedom.

The coordinates which define the number of degrees of freedom of a system are called the generalized coordinates.

2. Classification based *on* the **nature of differential equation(s)**

Vibratory systems are mathematically described in terms of their differential equation(s) of motion. These differential equation(s) may be linear, nonlinear or stochastic in nature and a vibratory system described accordingly, i.e. a linear differential equation of motion will represent the case of linear vibrations: a non-linear differential equation of motion will represent the case of nonlinear vibrations and a stochastic differential equation of motion will describe random vibrations.

3. Classification based *on* the mechanism of origin *of* vibrations

Based on the mechanism of the origin of vibrations, the classification is given below:

VIBRATION

Natural	Self-excited	Parametrically Excited or Rheonomic	Forced	Coupled
1. Linear 2. Nonlinear	1. Oscillator type 2. Storage type		1. Linear 2. Nonlinear	

FREE VIBRATIONS (OR NATURAL VIBRATIONS)

A system is said to execute free vibrations if the vibratory motion continues without the aid of an agency. In the case of a conservative system executing free vibrations there is a continuous exchange of potential energy and kinetic energy throughout the cycle of vibrations such that the sum of these two energies remains constant at any given instant of time.

Consider the case of a vibratory system represented following differential equation of motion:

$$m\frac{d^2x}{dt^2} + f(x) = 0 \tag{1.3}$$

where m = mass of the oscillator

x = displacement of the mass from the equilibrium position

$f(x)$ = restoring force function

Nature of the differential equation of motion (1.3) depends on the nature of the restoring force function *f(x)*. A linear restoring force function makes the equation linear and the vibrations characterized by it are linear. In the event of this restoring force function being nonlinear the governing differential equation will be nonlinear and the represented vibrations nonlinear. A linear restoring function is given below:

$$f(x) = k\,x \tag{1.4}$$

where k is the stiffness constant expressed in terms of force per unit length.

A nonlinear restoring function is of the type given below:

$$f(x) = k\,x^n \tag{1.5}$$

A linear differential equation of motion represents linear vibrations and it is nonlinear *if* the defining *differential* equation happens to be nonlinear.

SELF-EXCITED VIBRATIONS

The important characteristic feature of this type of vibrations is the availability of a source of energy which the system draws energy in synchronism with its natural vibrations. The unavoidable energy losses due damping sources are automatically balanced out.

The self-excited vibrations can be oscillator there is a source of energy to the vibratory system governed by a control mechanism and actuated by the system or the vibrator itself) or the storage type (where there is a storage device through which the flow of energy takes place).

Classical examples of self-excited vibrations are: the steam in a steam-engine which acquires alternating property due to the motion of the piston; balance wheel of a clock where the vibratory system is the spring-wheel itself and the source of energy is wound spring and wing flutter of airplanes.

Self-excited vibrations are represented by nonlinear equations of the form given below:

$$m\frac{d^2x}{dt^2} + f(x, \dot{x}) = 0 \qquad (1.6)$$

It is important to note that the designer must be able to diagnose whether the vibrations to be cured are self-excited or not.

PARAMETRICALLY EXCITED VIBRATIONS

These vibrations, characterized by the time variable parameters of the system, may be defined by an equation of the form

$$m\,\dot{x} + c(t)\,\dot{x} + f(t)\,x = 0 \qquad (1.7)$$

An important feature of this class of vibrations defined by equation (1.7) is that the principle of superposition holds good in this case. It is the distinguishing feature of this class of vibrations that the excitation remains inactive so long as the oscillator is in the position of equilibrium. Some slight disturbance from the equilibrium position leads to building up of the vibrations whereas in the case of forced vibrations the oscillations can build up from rest.

FORCED VIBRATIONS

In this case it is necessary that the system is excited by a time variable external force. Vibrations of a single degree of freedom system with external excitation f(t) is given by the following equation of motion:

$$m\,\dot{x} + c\,\dot{x} + k\,x = f(t) \tag{1.8}$$

Forced vibrations can be linear, nonlinear or stochastic depending upon the nature of the defining differential equation of the system. The excitation f(t) is a function of tine which may be periodic, non-periodic or stochastic.

COUPLED VIBRATIONS

In general, vibrating systems have many degrees of freedom. A system having many degrees of freedom will have many model of vibrations. When these modes of oscillations mutually interact then these vibrations are said to be coupled.

CHAPTER II

KINEMATICS AND GEOMETRY OF VIBRATIONS

The study of vibrations begins with the understanding of basic concepts of kinematics and geometry of vibrations. These are discussed in this chapter through the study of topics like the periodic function, effect of the plane transformation, synchronous vibrations; Lissajous figures; sum of non-synchronous vibrations; beats; Fourier analysis and transient time functions.

2.1 CHARACTERISTICS OF VIBRATIONS

2.11 Periodic Function

Periodic function of time is fundamental to the development of the subject of vibrations. In order to understand the meaning of this function, let us consider a physical quantity which varies with time t. We assign a variable x to this quantity. Expressed as a function, this quantity x is

$$x = x(t) \tag{2.1}$$

The function x(t) will be called a periodic function of time if it has the same value for all those values of time which differ by a constant time (T). In other words, if x(t) is a periodic function of time then it will satisfy the following equality

$$x(t) = x(t + nT) \qquad n = 1, 2, 3, \ldots \tag{2.2}$$

An example of a periodic function is x(t) = a sinωt. In case this periodic function x(t) represents a vibration then the smallest value of T for which equation (2.2) is called the period vibration. The function x(t)=a sinωt has a period of 2π. Relationship between the periodic time (T) and the frequency of vibrations (f) is

$$f = 1/T \tag{2.3}$$

In other words, the frequency of vibrations (f) is reciprocal of the periodic time (T).

2.12 Cyclic Motion & Sinusoidal Motion

The terms cyclic motion and sinusoidal motion are routinely used in vibration analysis. As a matter of fact there is relationship between the two. This must be understood. To develop an idea of this relationship, let us consider the case of a Skotch yoke mechanism shown below in fig. 2.1:

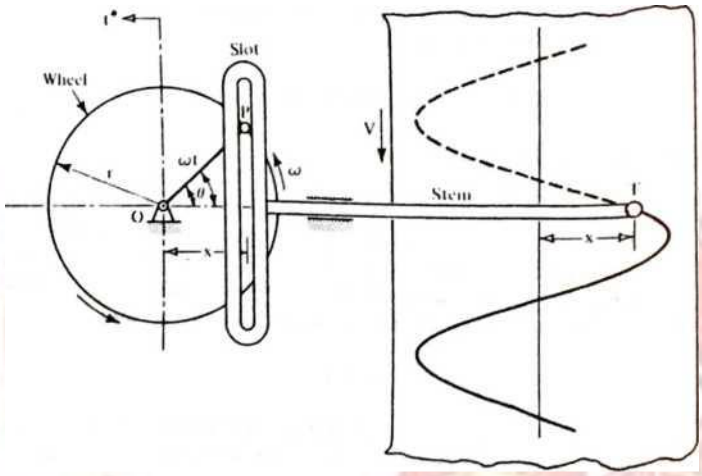


Fig.2.1: Skotch Yoke Mechanism & trace of the tip of stem

The crank OP of the mechanism rotates with an angular velocity ω about the center O. The pin P is made to slide in the vertical slot shown. This slot is attached to a stem reciprocating in the horizontal guide. The displacement x, of the tip of the stem with respect to a middle position, after time t, is given by the following:

$$x = r\cos\theta = r\cos\omega t \tag{2.4}$$

If the recording paper moves with a velocity V (=ωr), then the function x = r cos ωt will be recorded on the paper as a sine curve. Equation (2.4) represents a sinusoidal motion. Thus the cyclic motion and the sinusoidal motions are routine related. The quantity ω which is the angular velocity of the cyclic motion is also known as the circular frequency of the sinusoidal motion. The sinusoidal motion described by the equation (2.4) is a periodic motion having an amplitude r, frequency f(=ω/2π) and periodic time T((=2π/ω).

2.13 Effect of Plane Transformation

In certain cases of vibration representation it is quite possible to have a vibration which is not sinusoidal on a given (x,t) plane but can be made so through a change of the plane (i.e. transformation of coordinate axes).

This is illustrated with the help of a case shown in fig.2.2 wherein the depicted periodic motion is not sinusoidal on the given (x,t) plane. This periodic vibration becomes sinusoidal when it is transformed to another plane (x^*, t^*).

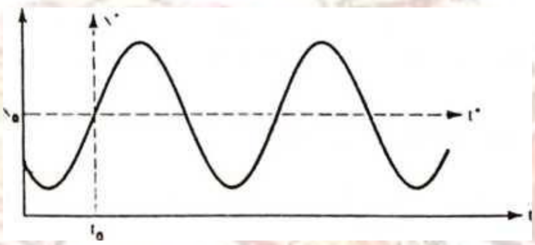


Fig.2.2: Plane transformation

Consider that a motion described on a plane (x,t) is

$$x = x_0 + \ a \sin \omega t + \ \cos \omega t \qquad (2.5)$$

Equation (2.5) does not represent a sinusoidal motion. Let us rewrite this equation in the form given below:

$$x - x_0 = \ \alpha \ \sin \omega\, (t - t_0) = \alpha \ \cos\{\omega(t - t_0) - \pi/2\} \qquad (2.6)$$

$$\text{where} \qquad a \geq b, \alpha \ = \ \sqrt{(a^2 + b^2)}, t_0 = \ (1/\omega)\ tan^{-1}(b/a) \qquad (2.7)$$

Next, we define a plane (x*, t*) such that x* = x-x0 and t*= t-t0.

Above mentioned transformation changes equation (2.6) to the form given below.

$$x^* = \ \alpha \sin \omega t^* \qquad (2.5)$$

Equation (2.8) shows a sinusoidal motion on the new plane.

2.14 Synchronous Vibrations

Two harmonic vibrations having the same frequency are called synchronous vibrations, e.g. two Vibrations represented by (10 sin 2t) and (-2 sin 2t) are synchronous. Two synchronous vibrations need not have their maxima at the same time, e.g. a sin(ωt) and b sin(ωt-α). It is obvious that the latter of these two vibrations is delayed by an angle α. This delay is called phase-lag and angle α is called phase angle. Concept of the phase angle applies to synchronous vibrations.

2.2 COMPOSITION OF TWO HARMONIC MOTIONS

Consider the case of following two sinusoidal coplanar vibrations of equal frequency.

$$x_1 = a\sin(\omega t + \emptyset_1) \tag{a}$$

$$x_2 = b\sin(\omega t + \emptyset_2) \tag{b}$$

where x1 and x2 represent displacements along two mutually orthogonal axes.

These two motions can be combined in the manner given below:

From equation (b): $\sin(\omega t + \emptyset_2) = x_2/b$ and therefore

$\cos(\omega t + \emptyset_2) = (1 - x_2^2/b^2)^{1/2}$

Equation (a) is rewritten in the form given below:

$$x_1/a = a\sin(\omega t + \emptyset_1) = \sin\{(\omega t + \emptyset_2) + (\emptyset_1 - \emptyset_2)\}$$

$$= \sin(\omega t + \emptyset_2)\cos(\emptyset_1 - \emptyset_2) + \cos(\omega t + \emptyset_2)\sin(\emptyset_1 - \emptyset_2) \tag{d}$$

With the help of equations (a) to (c), we convert equation (d) into the following form:

$$\frac{x_1}{a} = \frac{x_2}{b}\cos(\emptyset_1 - \emptyset_2) + (1 - x_2^2/b^2)^{1/2}\sin(\emptyset_1 - \emptyset_2) \tag{e}$$

Upon rearrangement, equation (e) becomes

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 2x_1x_2ab\cos(\emptyset_1 - \emptyset_2) = \sin^2(\emptyset_1 - \emptyset_2) \tag{2.9}$$

Inspection of equation (2.9) shows that it is an equation of an ellipse. The position and dimensions of this ellipse are dependent upon the amplitudes a t b and also on the initial phases of motion $\emptyset_1$ & $\emptyset_2$.

(1) Following particular cases of interest are obtained from the equation (2.9) :

(2) In case $\emptyset_1 = \emptyset_2 = 0$ or π the ellipse degenerates into two coincident straight lines given by the equation $\left(\frac{x_1}{a} - \frac{x_2}{b}\right)^2 = 0$.

(3) If $\emptyset_1 - \emptyset_2 = \pi/2$ then the equation (2.9) is reduced to the equation of an ellipse (the exes of which coincide with the axes of coordinates) given by $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$.

(4) For the case a = b, this ellipse degenerates into a circle $x_1^2 + x_2^2 = a^2$ with its center located at the origin.

In case the frequencies of two vibrations are not equal but form a rational ratio then also it is possible to have a similar treatment. This treatment will give an equation representing some periodic circulating motion. A trace of the path of a point performing two such simultaneous motions is called a Lissajous Figure. Fig.2.3a shows Lissajous figures drawn for the cases of addition of two synchronous Vibrations at right angles to each other for different values of initial phase differences $\emptyset_1 - \emptyset_2$.

In case two vibrations having unequal frequencies are added then the initial phase difference will be dependent on the selected time origin. Consider two vibrations given below

$$\mathrm{x_1 = \ a\sin(n\omega t + \emptyset_1)\ and\ x_2 = \ b\sin(n\omega t + \emptyset_2)}$$

In this case, we select a reduced phase difference (ψ) to be

$$\psi = \emptyset_m = \left(\frac{m}{n}\right)\emptyset_1$$

This phase is independent of the origin of time.

Lissajous figures for m:n=1:2, 1:3 and 2:3 are shown in fig.2.3b, fig.2.3c and fig.2.3d respectively

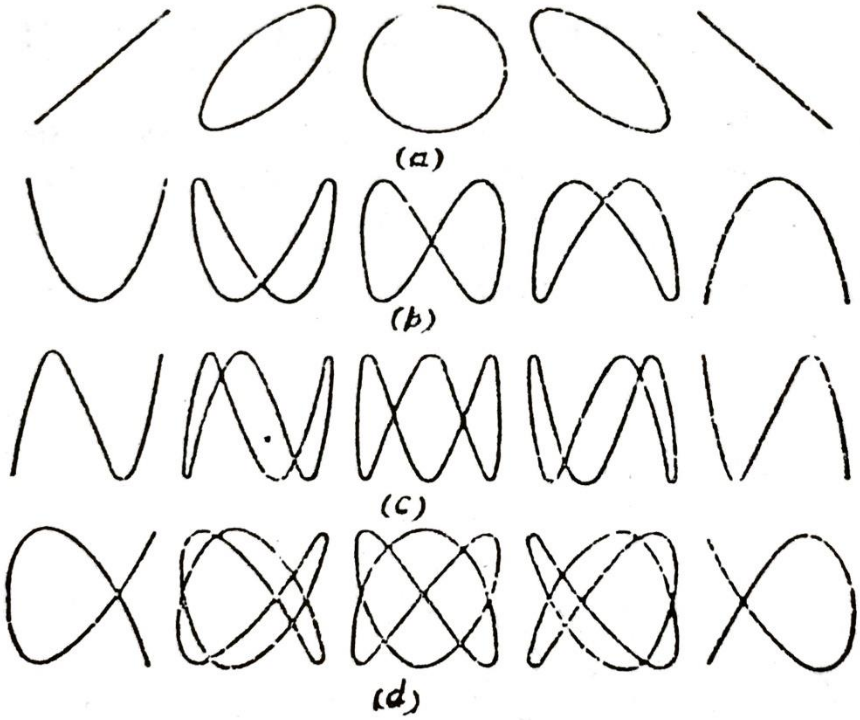


Fig.2.3: Lissajous figures

If the ratio m:n is irrational, the motion of the point is almost periodic and its trajectory fills up completely a rectangle.

An interesting and important case corresponds to that in which two circular circulating motions in phase are added when the points move in opposite directions at equal radii r.

Let the two motions be represented by the following;

$$x_1 = r\, e^{\mathrm{i}(\omega t-\phi)} \text{ and } x_2 = r\, e^{-\mathrm{i}(\omega t-\phi)}$$

Adding these motions, we get the following:

$$x = x_1 + x_2$$

$$= r\, e^{\mathrm{i}(\omega t-\phi)} + r\, e^{-\mathrm{i}(\omega t-\phi)}$$

$$= 2\mathrm{r} \cos(\omega t - \phi)$$

This equation represents the case of a one-dimensional vibration of the same frequency ω but amplitude 2r.

2.3 VECTOR & COMPLEX REPRESENTATION OF VIBRATIONS

Any harmonic motion can be represented by means of a rotating vector. Consider fig.2.4 in which x represents the real axis and y the imaginary axis. OP can be considered to be a rotating vector represented by a complex number

$$z = x + iy \tag{2.10}$$

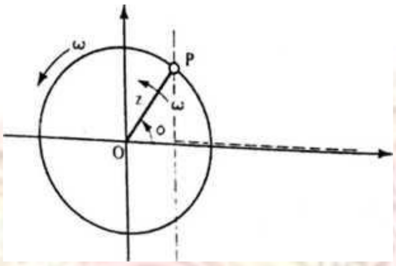


Fig.2.4: Complex representation of vibration

From the theory of complex numbers, we know that

$$z = r\,e^{i\phi} = r\,e^{i\omega t} \tag{2.11}$$

Where r = OP.

The real part of z is written as

$$Re(z) = x = r \cos \omega t \tag{2.12}$$

Equation (2.12) shows that the motion described by x is harmonic. From the equation (2.11), we infer that a harmonic motion can be taken to correspond to rotation of a vector of constant length. It is a very convenient method of representing the vibrations. This method assumes greater importance since modern computers can automatically handle complex arithmetic.

2.4 SUM OF TWO NONSYNCHRONOUS VIBRATIONS

So far we studied the method of addition of two synchronous vibrations. Now, let us define the sum of two nonsynchronous vibrations. For this, we first define an RMS amplitude of a vibration. It is given by

$$A_{rms}^2 = \frac{1}{T}\int_0^T [x^2]\,dt \tag{2.13}$$

where T is the period of vibrations and Arms is the RMS amplitude.

The RMS value of the sum of two nonsynchronous harmonic vibrations is

$$A_{rms}^2 = \left|\frac{1}{T}\int_0^T [x_1 e^{i\omega_1 t} + x_2 e^{i\omega_2 t}]^2 dt\right| \qquad (2.14)$$

or,

$$A_{rms}^2 = \left|\frac{1}{T}\int_0^T \left[x_1^2 e^{2i\omega_1 t} + x_2^2 e^{2i\omega_2 t} + 2x_1 x_2\, e^{i(\omega_1+\omega_2)t}\right] dt\right|$$

If the integration is carried over a sufficiently long period of time then the third term in the integrand upon integration becomes zero. Thus, the RMS value of the sum of two nonsynchronous vibrations is

$$A_{rms}^2 = (A_1)_{rms}^2 = (A_2)_{rms}^2 \qquad (2.15)$$

It may be observed that the above rule is similar to that for synchronous vibrations.

In case two harmonic vibrations of amplitudes A_1 and A_2 have different frequencies ω_1 and ω_2 respectively, and the vibrations are added. Then the resulting vibrations will be periodic only when one of the frequencies is a multiple of the other.

2.5 BEATS PHENOMENON

It should become clear from the above that the sum of two harmonic motions of different frequencies is not harmonic. An important case arises when two harmonic motions of the same amplitude but of slightly different frequencies are imposed on a vibrating body. Let us consider two such motions given by

$$x_1 = A\cos\omega t \qquad (2.16)$$

$$x_2 = A\cos(\omega + \Delta\omega)t \qquad (2.17)$$

where $\Delta\omega \ll \omega$.

Resultant motion of the body is the sum of two motions. Using the principle of superposition, it is

$$x = x_1 + x_2$$

$$= A\,[\cos\omega t + \cos(\omega + \Delta\omega)t]$$

$$= 2\mathrm{A}\cos(\Delta\omega/2)t\,\cos(\omega + \Delta\omega/2)t \qquad (2.18)$$

The amplitude of the resultant motion(x) may be considered to be a cosine wave having a frequency $(\omega + \Delta\omega/2)$ (or stated approximately as ω) and a variable amplitude. The amplitude varies between 0 and 2A according to the term 2A cos ($\frac{\Delta\omega}{2}$). This special pattern of motion is called the phenomenon of beating. A beat is said to occur whenever the amplitude reaches a maximum. The beat frequency fb is given by

$$f_b = \Delta\omega/2\pi \tag{2.19}$$

The beating phenomena is a useful technique in frequency measurement in which unknown frequency is compared with a known standard frequency. Beats phenomenon is shown fig. 2.5.

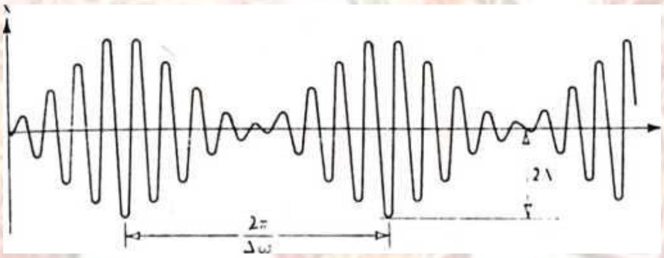


Fig.2.5: Beats phenomenon

2.6 FREQUENCY ANALYSIS

Forces which arise from machinery, in many cases, are not harmonic. They are periodic in nature. Vibrations can also be periodic rather than harmonic. The periodic vibrations or forces, to facilitate analysis, can be converted into a set of harmonic motions. This is accomplished by the use of a Fourier series analysis. This procedure of converting a periodic motion into a series of harmonic components is also called harmonic analysis. Basics of this methodology are outlined below;

Fourier showed that any periodic curve may be expressed in terms of a series of harmonic functions given below:

$$y - f(t) = \frac{A_0}{2} + \sum_{K=1} A_k \cos k\omega t + \sum_{K=1} B_k \sin k\omega t \tag{2.20}$$

Coefficients $A0,\ Ak$ and Bk are determined with the help of the following relations:

$$A_0 = \left(\frac{2}{\tau}\right)\int_0^{\tau} f(t)dt$$

$$A_0 = \left(\frac{2}{\tau}\right)\int_0^{\tau} f(t)\cos n\omega t\, dt \tag{2.22}$$

$$B_n = \left(\frac{2}{\tau}\right)\int_0^{\tau} f(t)\sin n\omega t\, dt$$

The terms of the series within the summation sign are called harmonics or tones.

If f(t) is an even function of time, i.e. f(t)= f(-t), then all the coefficients Bk are zeros.

If f(t) is an odd function of time, i.e. f(t)= -f(-t), then all the coefficients Ak are zeros.

A proper choice of the origin of time may, sometimes, render a function even or odd, and may thus simplify calculations.

A set of representative time functions and their Fourier series equivalents are given below:

A periodic function with finite number of discontinuities can be expressed into a Fourier series. This series has a sum higher than the function f(t) by about 18% (Gibb's phenomenon). If a truncated Fourier series is being used for a discontinuous function f(t) then the swing of the sum is found to increase in the neighborhood of the point of discontinuity. This is usually accompanied by high frequency vibrations which decay with distance from the point of discontinuity. It is important to note that the less smooth is the function f(t), the slower is the convergence of its Fourier series. Hence, the smoother the function, the poorer it is in higher harmonics

Equation (2.20) can also be written in the following form:

$$f(t) = \frac{C_0}{2}\sum_{K=1} C_k \cos(k\omega t - \emptyset_k) \qquad (2.21)$$

where $C = A_0 \,:\, C_k^2 = A_k^2 + B_k^2$ and

$\emptyset_k = tan^{-1}(\frac{B_k}{A_k})$ (k=1,2,….)

The Plot Ck against kω, for all values of k amplitude spectrum of the function f(t) and the plot of phases ∅ against kω , for all values of ω as its phase spectrum.

2.7 TRANSIENT TIME FUNCTIONS

A transient time function is one which is non-zero for a limited interval of time and is zero at all other times. A function of this kind will be obviously non-periodic. An example of one such transient force function is the force of impact produced during a direct collision of two bodies. Such force functions cannot be represented by Fourier series. However, non-periodic functions of this type can be easily analyzed in the frequency domain using Fourier Transforms. It is important to note that the frequency spectrum of a periodic function is a discrete line spectrum whereas a non-periodic function shows a continuous frequency spectrum.

If a periodic function has in its spectrum at least two harmonic components with an irrational frequency ratio (e.g. square root of the number 5) then this function is called an almost *periodic function.* This type of functions, though exhibit a line spectrum, are actually non-periodic.

CHAPTER III

PROBLEM FORMULATION, MODELING AND SOLUTION

3.1 INTRODUCTION

Analysis of machine vibrations is a quite complex task. In order to get meaningful results from an analysis, it is necessary to have careful planning coupled with a logical procedure based on sound physical principles. One such procedure for handling vibration analysis is discussed here. Its essential steps are

1. Definition of the problem
2. Physical modelling
3. Mathematical modelling
4. Solution
5. Interpretation

3.1 DEFINITION OF THE PROBLEM

Definition is the first and the most important step required to be taken for vibration analysis of a given problem. The meaning of the definition of a problem is to explicitly write down the objective(s) of study together with a complete description of the system. It is necessary to highlight the accuracies required to meet the specified objective(s). Once the problem is properly defined then all subsequent steps are primarily dependent on the correct statement of this step.

3.2 PHYSICAL MODELING

Next logical step is the preparation of a physical model of the given problem. For this, it is necessary to identify the mechanisms which make a system susceptible to vibrations. Presence of at least one pair comprising of an inertial element and an element of restoring force mechanism will make the system susceptible for vibrations. Dissipation mechanisms and external excitations also contribute, in a complex way, to the vibration susceptibility. Identification of these elements is of great importance.

The meaning of physical modelling is illustrated by considering the case of a car passing over a road hump (Fig. 3.1). Let us say that the car executes vertical vibrations. The car has many inertial elements and interconnected continuous with many types of dissipation mechanisms. We shall attempt a simple and meaningful simulation of this car and its occupants. It is the simple version of the physical model which is normally of interest. For this purpose it is sufficient to replace this car by an imaginary physical model which gives a response similar to that of a real system within the limits of acceptable accuracy. This imaginary physical model considered adequate another for one purpose may prove to be useless in application. A physical model of higher fidelity will always be required when higher accuracy is demanded. Two models of this car are shown in figs. 3.2 and 3.3.

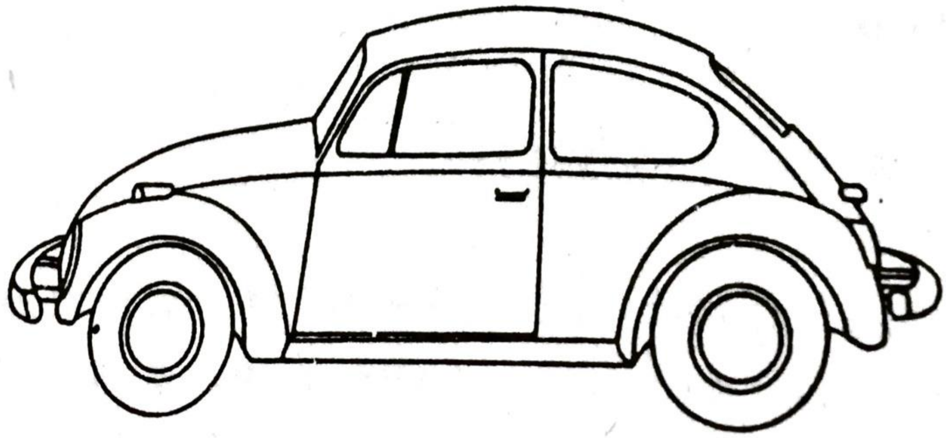


Fig.3.1

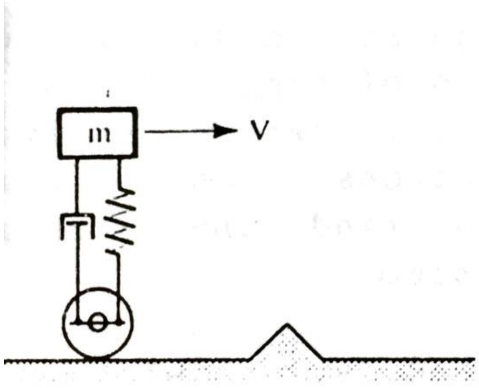


Fig.3.2

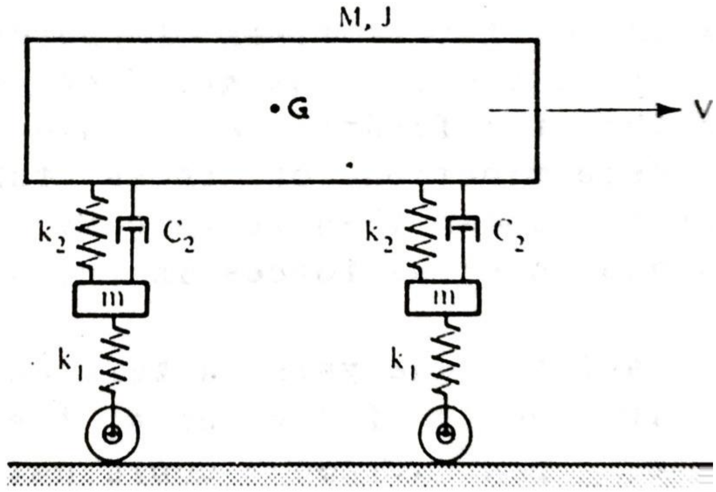


Fig.3.3

There is no general rule for the creation of physical model which adequately describes system. This creation is totally dependent on the ingenuity of the engineer.

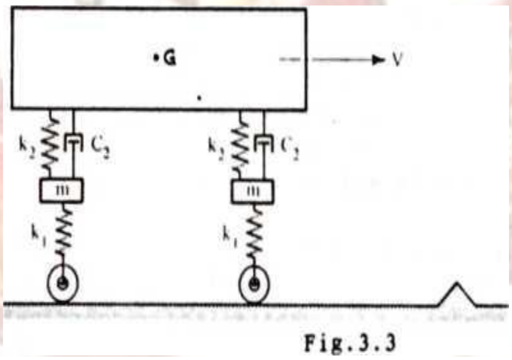


Fig.3.3

3.3. MATHEMATICAL MODELING

The first step needed to solve any vibration problem is to convert it into a suitable mathematical model consisting of inertia, stiffness and damping. Energy sources provide energy to the system.

A physical model developed in accordance with the objectives of the problem is then subjected to the laws of physics i.e. basic laws of nature viz. conservation of mass, conservation of momentum, conservation of energy, and the second and third laws of thermodynamics. Obtain constitutive equations of the materials of which the system is made. Then apply suitable geometric constraints.

This step leads to the development of equations of motion of the physical model. These equations of motion constitute the mathematical model of the system.

Machines and structures are made of elements which undergo deformations under the action of forces. Different elements of a machine also possess relative mot ion between them. These relative motions and/or deformations give rise to forces which develop due to the presence of finite masses of different elements of the system under investigation. The presence of these deformation mechanism(s) together with the inertial effects is the cause of oscillatory behaviour of a given system. It is to be noted that the subject of the dynamics of deformable solids is concerned with this type of oscillatory motion.

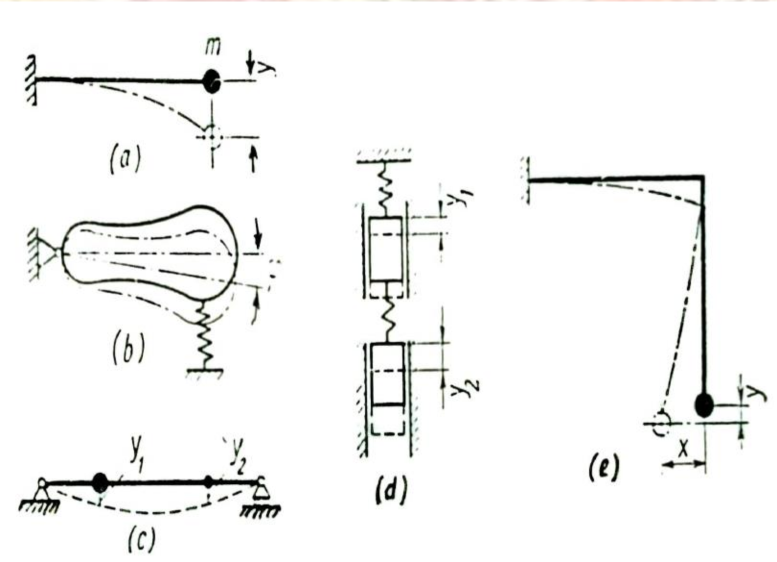
3.3.1 Degrees of Freedom

The configuration of a vibrating system changes with tine. Certain entities are needed to define the system configuration at any instant of time. These entities are the coordinates. Different types of coordinates are used for the purpose. These coordinates are functions of time. It is to be noted that the fundamental problem of vibration analysis is the determination of these functions. Once these functions are known then it is easy to find the strains, stresses and internal forces in the system.

To facilitate the analysis, a tern called degrees of freedom is used. The number of degrees of freedom of a system is the number of generalized coordinates (loosely called independent coordinates) needed to uniquely define its configuration.

Every mechanical system is an assembly of many interconnected continuua (e.g. bean, plate, shell, box, chassis, piston, crank shaft, etc.) along with a number of other complex configurations. Therefore, such a system will consist of infinite number of interconnected particles and its degrees of freedom will be infinite. Practical considerations dictate that such systems be modelled as systems with finite number of degrees of freedom. To get the simplest model, it is usual to assume the lightest parts to be mass-less. Parts for which inertial property is retained are then taken to be particles having concentrated masses with no deformation.

Systems shown in fig. 3.4(a) and fig. 3.4(b) are of one degree of freedom whereas each one of those shown in figs.3.4(c) to fig. 3.4(f) has two degrees of freedom. Fig. 3.4(g) has three degrees of freedom. It is important to note that the figs. 3.4(h) to 3.4(l) show masses which are suspended in different ways but these masses are allowed vibrations along a single fixed straight line. Therefore, these systems are of single degree of freedom. If rolling is not accompanied by slipping then the system in fig. 3.4(m) also possesses one degree of freedom. Fig. 3.4(n) is the case of a rigid beam hinged at one end. It is supported on two nags less springs. The configuration of this bean, at any instant of time can be determined by its angle of rotation about the immovable hinge. Though there are two masses and two springs yet the system possesses only one degree of freedom. Fig. 3.4(p) is the case of a shaft which carries two discs at its ends. Though the vibrations of this system can be defined by just one function, the relative angle of rotation of the disks yet it is a system of two degrees of freedom. So is the case with the system in fig. 3.4 (q).



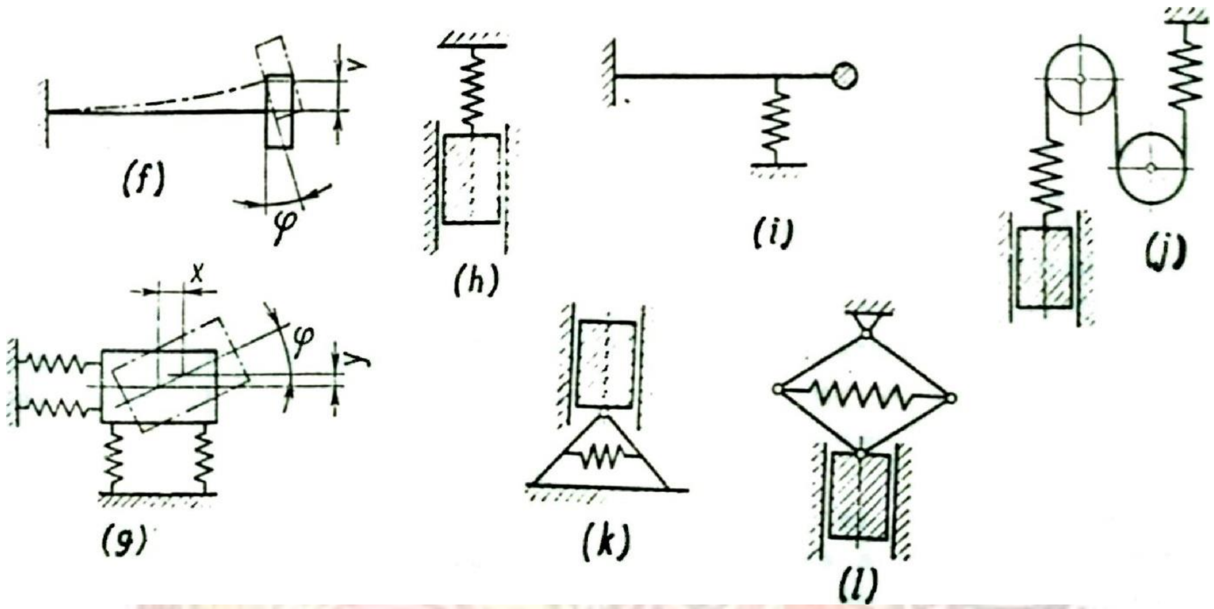


Fig. 3.4(a) to Fig. 3.4 (e) Fig. 3.4(f)

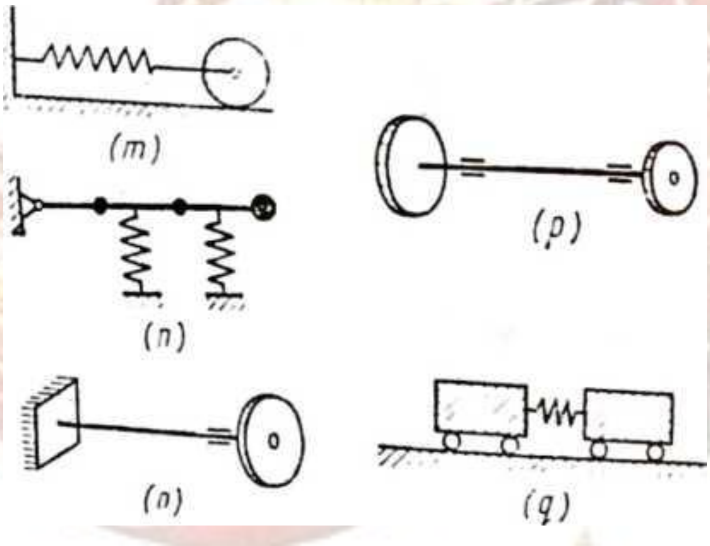


Fig. 3.4(m) to Fig. 3.4(q)

3.3.2 Classification of Forces:

A mechanical system is subjected to different kinds of forces. These forces may be external or internal. Effects of these forces on the vibratory system can also be widely different. Following is the description of different kinds of important forces encountered in vibration analysis:

1. Restoring Forces

2. Dissipative Forces
3. Driving Forces
4. Forces of Mixed Nature

1. **Restoring Forces:**

Once the equilibrium state of a mechanical system is disturbed then certain forces are called into play which try to restore back the equilibrium. These are the restoring forces. The vibratory properties of mechanical systems are the result of the combined act ion of these forces and the inertial forces.

An elastic force is basic type of restoring force. A general expression for the elastic properties of the element is given by the following:

$$P = F(y) \tag{3.1}$$

where F(y) is normally a nonlinear function of displacement. The simplest type of a restoring force is a linear function in which the restoring force P is directly proportional to the deflection, i.e.

$$P = k\ y \tag{3.2}$$

where k is a proportionality constant, called as linear spring constant. It is commonly called as linear stiffness.

Figs. 3.5 (a) to 3.5 (d) show four different vibratory systems. Restoring force (i.e. spring) characteristics for these cases are plotted against linear deflections.

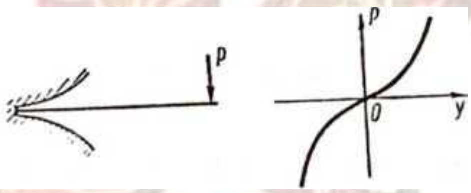
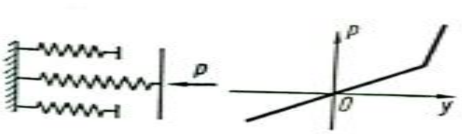


Fig. 3.5 (a) : Soft Spring Fig. 3. 5 (b) : Stiff Spring



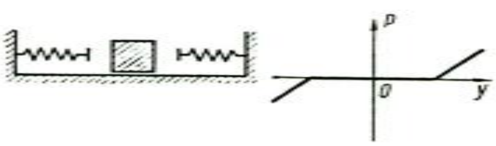
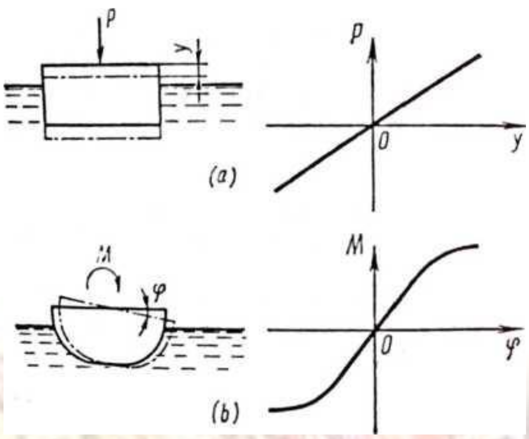


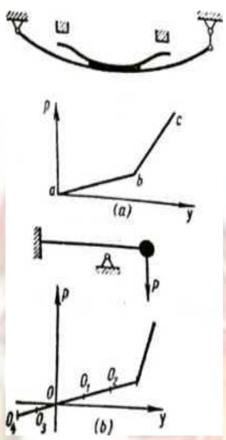
Fig. 3.5 (c) : Bilinear Spring Fig.3. 5(d) : Discontinuous Spring

Figs. 3.6(a) & 3. 6(b) show spring characteristics of systems having buoyant forces.



Figs. 3.6(a) a (b): Buoyant restoring forces

Some typical non-linear restoring force *characteristics are* shown in figs. 3.7(a) to 3.7(f).



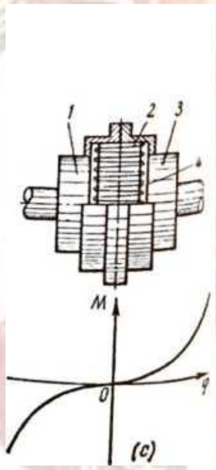


Fig. 3.7(a)&(b) : Automobile suspension

Fig.3.7(c) :Coupling

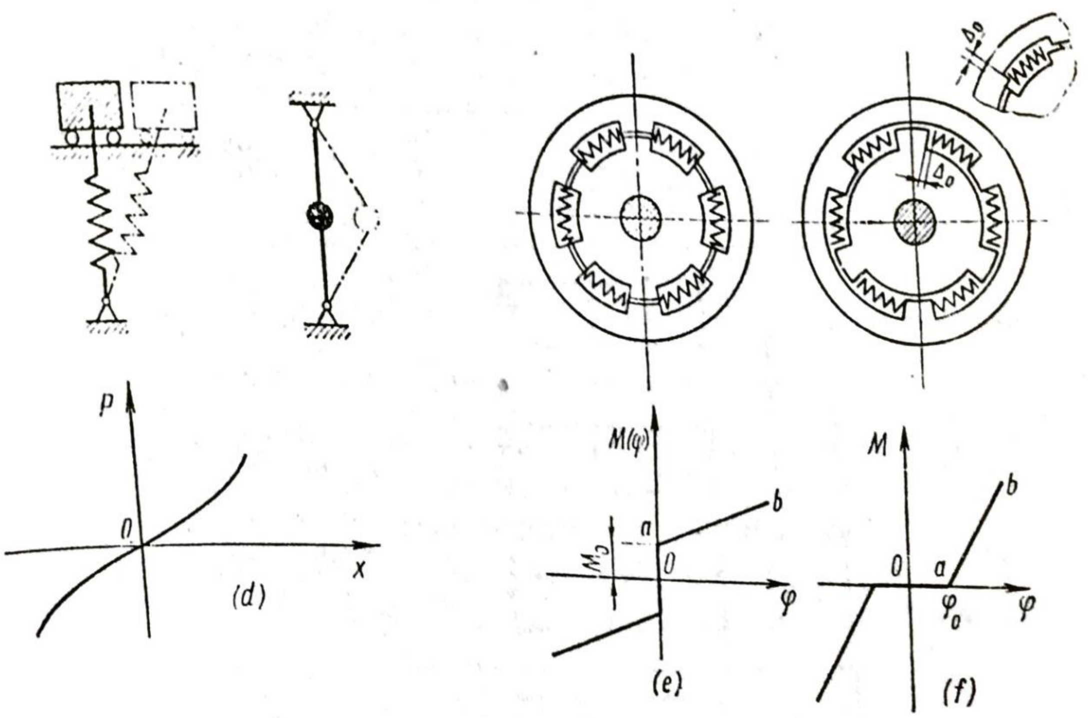


Fig.3.7(d):Stiffening Spring   Figs.3.7(e)&(f):

TABLE I

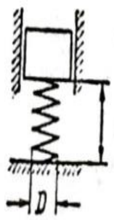
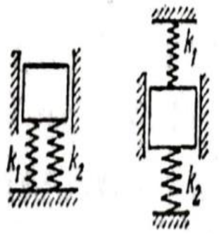
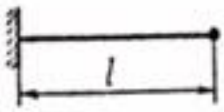
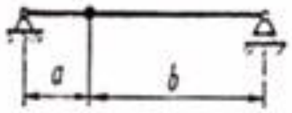
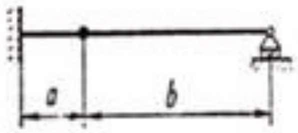
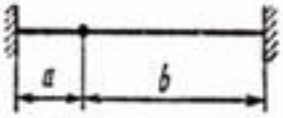
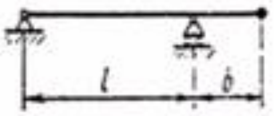
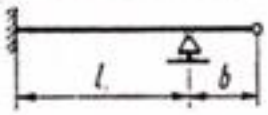
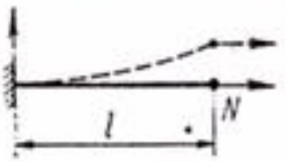
Item No.	Schematic Diagram	Spring Constant, k
1		$\frac{Gd^4}{8nD^3}$ (d = diameter of coil section, D = diameter of spring, G = shear modulus, n = number of coils)
2		$k_1 + k_2$
3		$\frac{k_1 k_2}{k_1 + k_2}$

TABLE I (contd)

Item No.	Schematic Diagram	Spring Constant, k (EI = flexural rigidity)	
4		$\frac{3EI}{l^3}$	
5		$\frac{3EI\,(a+b)}{a^2b^2}$	
6		$\frac{12EI\,(a+b)^3}{a^3b^2\,(3a+4b)}$	
7		$\frac{3EI\,(a+b)^3}{a^3b^3}$	
8		$\frac{3EI}{(b+l)\,b^2}$	
9		$\frac{12EI}{(4b+3l)\,b^2}$	
10		$\frac{\alpha^3EI\cosh\alpha l}{\alpha l\cosh\alpha l-\sinh\alpha l}$	$\alpha=\sqrt{\frac{N}{EI}}$
11		$\frac{\alpha^2EI\sinh\alpha l}{l\,(\alpha l\cosh\alpha l-\sinh\alpha l)}$	

2. Dissipative Forces:

It is a rule rather than exception to have dissipative forces in real vibratory systems. Irrecoverable work is done as these ones are the cause of dissipation of the mechanical energy. Dissipation forces are commonly called as damping forces. These forces arise: (a) at supports and connections of a mechanical system; (2) due to the resistance of the medium in which vibrations take place and (3) forces which are produced in dashpots (energy *absorbers).*

Simplest dissipation device is a viscous damper in which a piston moves in a viscous medium (fig. 3.8a). A graph of the variation of friction with velocity in shown in fig. 3.8b. It a linear graph. The friction force and the velocity are related as per the following expression:

$$F = c\,x \qquad (3.3)$$

In general, dissipative forces are nonlinear functions of velocity. The graph of a continuous non-linear function is shown in the fig.3.8c (case of fluid damping). These forces can also be discontinuous (e.g. dry friction). Two graphs of such dissipative forces are shown in figs. 3.8d and 3.8e.

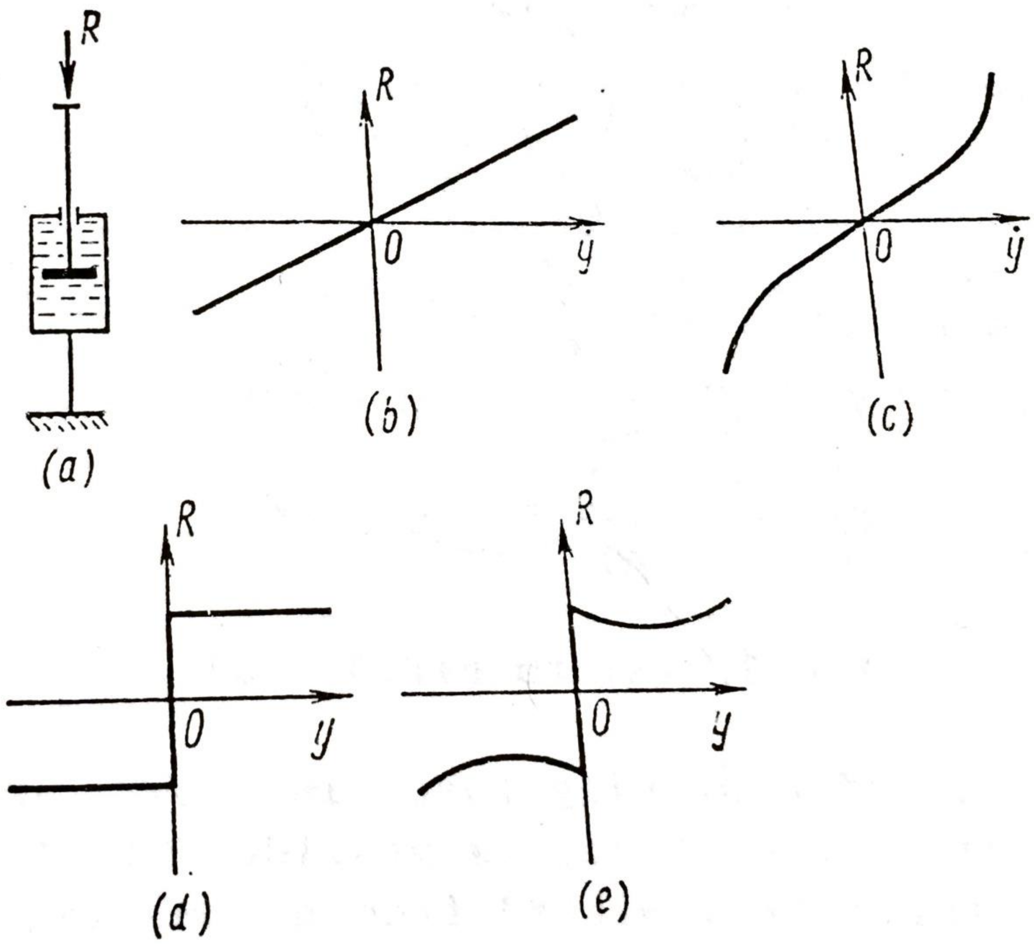


Fig.3.8

3. Driving Forces

Restoring and dissipative forces are exclusively determined by the properties of the system itself. They affect the motion and are themselves governed by this motion. Driving forces are represented by explicit functions of time. These are independent of the motion but significantly affect driving force may be expressed as

$$F = F(t) \tag{3.4}$$

A representative set of various driving forces is shown in fig. 3.9. They are:

a. Harmonic driving forces (fig. 3.9a)
b. Periodic driving forces (fig. 3.9b)
c. Short duration periodic impulses (fig. 3.9c)
d. Non-periodic driving forces (fig. 3.9d)

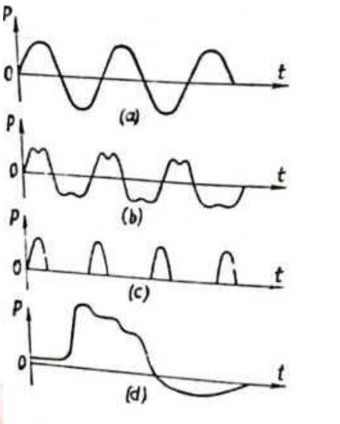


Fig. 3.9 (a) to Fig. 3.9 (d)

A typical example of a driving force is shown in fig.3.10. In this case the driving force is provided by the vertical component of a force transmitted from an unbalanced rotor to the foundation of a machine. In this case shown in fig. 3.10 ω is the angular velocity of the rotor, m its mass and e is its eccentricity. The magnitude of the centrifugal force, which is constant, is equal to $m\omega^2e$. Vertical component of this force is given by the following expression:

$$P_y = m\,\omega^2\, e \sin \omega t \tag{3.5}$$

and the horizontal component is

$$P_h = m\,\omega^2\, e \cos \omega t \tag{3.6}$$

Forces P_y and P_h which may be the causes of vibrations of elastic systems are the examples of driving forces.

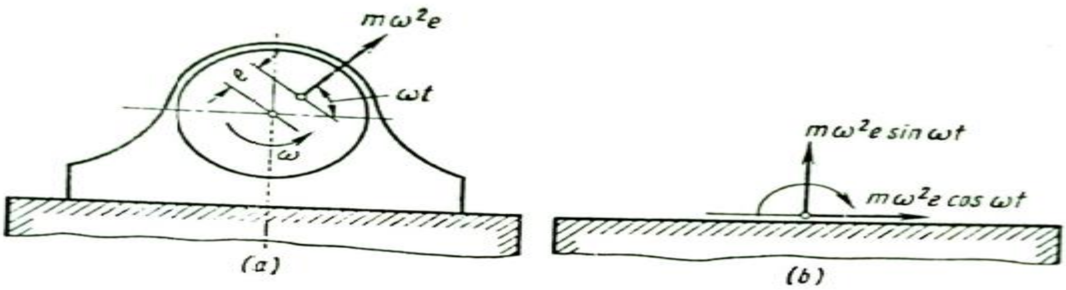


Fig.3.10

3.3.4 Mathematical Modeling of Simple Lumped Parameter Model

The intent of the modeling has got to be specified which includes analysis, design, and synthesis. Analysis is required when all parameters are specified and the vibrations of the system are predicted. Design implication means parametric design which is needed to achieve a certain design objective.

In order to simplify the modeling it is necessary to make certain assumptions in order to avoid making resulting equations complex. It is necessary to take the continuum as the basic concept. All materials are assumed to be linear, isotropic, and homogeneous. In this book it is assumed that plane sections remain plane for beams in bending and circular sections undergoing torsional loads do not warp

A massless spring with a time variable force P(t) applied to one of its ends is shown in fig. 3. 11. The other end of the spring is fixed, as shown. If k is the spring stiffness, then the displacement x of its end is given by

$$x = P(t)/k \tag{3.7}$$

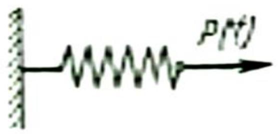


Fig. 3. 11 (a)

Fig. 3.1 i (b)

It may be noted that though the displacement given by the equation (3.7) is a function of time yet it is not dynamic. A true dynamic process in a mechanical system must include the property of inertia in its formulation. Now, consider the case shown in fig, 3.11(b).

Figure shows a simple one degree of freedom system. You will observe that it is no longer possible to work with relationships of the type given in equation (3.7). It is, now, necessary to write the expression for displacement in the form of a differential equation. The differential equation of motion for this case is obtained below with the help of the free body diagram given in fig. 3.11(c).

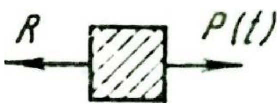


Fig. 3. 11 (c)

Inertia force: $m\ddot{x}$

Restoring force: k x(=R= reaction of spring on mass)

External force: P(t)

Application of Newton's law of motion to the above case gives the following differential equation of motion of this system:

$$m\ddot{x} + kx = P(t) \tag{3.8}$$

Equation (3.8) is a differential equation of second order with constant coefficients. The function x(t) is obtained only after the integration of this equation. The state of strain of the system at any instant of time is completely defined by one coordinate x(t) only. This system possesses one degree of freedom.

3.4. SOLUTION

Various methods of solution of the equations of motion are available. One is required to select the one which is best suited to the system consistent with the requirements of the desired accuracy.

3.5 INTERPRETATION

Results obtained from the solution of equations are to be used to obtain solution to the problem at hand. This will normally be in the form of specific design decisions. This is on the talent of the engineer.

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CHAPTER IV

VIBRATIONS OF SINGLE-DEGREE-OF-FREEDOM SYSTEMS

4.1 INTRODUCTION

A single degree of freedom vibratory system has its mass elements rigidly inter-connected so that only one spatial coordinate is sufficient to completely define its geometric configuration.

Simplest dynamical model of a physical system is that of one degree of freedom. Outward appearance of this type of model is normally quite different from that of the physical model. This descript ion can at best be a first order of approximation. Though the solutions obtained from the analysis of single degree of freedom models may not be directly usable for practical and real life problems yet the information provided is of vital importance, viz. concepts developed are easily extended to multi-degree of freedom analysis, the results can be used as starting values for advanced numerical analysis.

In this chapter, we discuss the mechanics of vibrations of single degree of freedom systems.

4.2 FREE VIBRATION WITHOUT DAMPING

4.21 Derivation of the Equation of Notion:

Consider a single degree of freedom vibratory system shown in fig. 4.1a. The mass m supported on rollers is attached to a fixed wall by means of a linear spring k. If there is no external force acting on the system then the mass will be in its equilibrium position. In this position, the force in the spring will be zero. If the mass m is disturbed from its equilibrium position by an amount x, as shown in fig. 4.1b, then force in the spring will act in a direction opposite to the displacement. This spring force is

$$F = -kx \tag{4.1}$$

Fig. 4. 1

Upon application of Newton's second law of mot ion to the mass, we get the following:

$$m\ddot{x} = F = -kx$$

Above expression is rewritten to give the following equation of motion for this system:

$$m\ddot{x} + kx = 0 \tag{4.1}$$

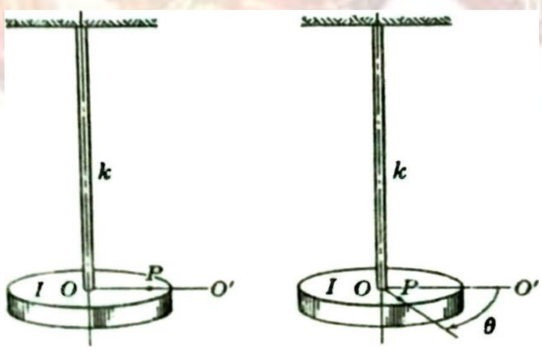
Equation (4.1) is a linear differential equation of second order with constant coefficients, we rewrite this equation as

$$\ddot{x} + \omega_n^2 x = 0 \tag{4.2}$$

where the notation ω_n^2= k/m has been introduced.

Now, consider a torsional vibratory system shown in fig. 4.20. It consists of a disc having a moment of inertia I (about a vertical axis through its centroid). This axis is taken to be coincident with the axis of the vertical shaft supporting it. If the shaft is twisted through an angle θ, then it will exert a torque T acting in a sense opposite to the angular displacement. This torque is given by

where k is the torsional spring constant of the shaft .



(a) (b)

Fig. 4.2

Let equilibrium posit ion of the disc be defined by a line 00. In this position both the torque and the angular displacement are zero. Now, assume that the disc is rotated through an angle θ. This will create a restoring torque T in the shaft (fig. 4.2b). Newton's second law is applied to the disc. We get the following

$$I\ddot{\theta} = T = -k\theta$$

It is rewritten to get the following equation of motion of this torsional system:

$$\ddot{\theta} + \omega_n^2\theta = 0 \tag{4.3}$$

where ω_n^2=k/I.

It is interesting to note that the equations of motion for translational vibrations and torsional vibrations are of the same type (see equations (4.3) and (4.2)). Solution of one equation is the solution of the other with appropriate parameters substituted. Responses of the two types of systems will be similar. Table I gives a comparison of the parameters of rectilinear and rotational (or torsional) systems.

TABLE I : Comparison of units of rectilinear and rotational systems

	RECTILINEAR SYSTEM		TORSIONAL SYSTEM	
QUANTITY	BRITISH UNITS	SI UNITS	BRITISH UNITS	SI UNITS
Time	sec	s	sec	s
Displacement	in.	m	rad	rad
Velocity	in./sec	m/s	rad/sec	rad/s
Acceleration	in./sec^2	m/s^2	rad/sec^2	rad/s^2
M.I. (mass)	lb -sec^2/in.	kg	lb –sec^2	m^2.kg
Damping factor	lb-sec/in	s.N/m	in.lb-sec/rad	msN/rad
Stiffness	1b/ in.	N/m	in.lb./rad	mN/rad
Force/Torque	1b.	N	in.1b.	mN
K.E.	in. lb.	J	in. lb.	J
P.E.	in. lb.	J	in. lb.	J
Frequency	rad/sec Hz	rad/s Hz	rad/sec Hz	rad/s Hz

4.2.2 Solution of the Equation of Motion:

Equations (4.2) and (4.3) are second order ordinary linear differential equations with constant coefficients. Solution of such equations are normally sought in the form given below

$$x(t) = A\, e^{pt} \tag{4.4}$$

where A and p are yet undetermined constants.

A solution of equation (4.2) obtained by the substitution of the equation (4.4) in it is

$$A(p^2 + \omega_n^2) \tag{4.5}$$

It implies that equation (4.4) is a so1ution of equation (4.2) provided the following holds good

$$p^2 = -\omega_n^2 \text{ or } \qquad p = \pm\omega_n$$

Since both the values of p give solution, therefore, the general solution of equation (4.2) is:

$$x(t) = A_1\, e^{i\omega_n t} + A_2\, e^{-i\omega_n t} \tag{4.6}$$

where A_1 and A_2 are arbitrary constants.

Since $e^{i\phi} = \cos\phi + i \sin\phi$, equation (4. 6) can again be rewritten as

$$x(t) = B_1\, \cos\omega_n t + B_2\, \sin\omega_n t \tag{4.7}$$

where B_1 and B_2 is another set of two arbitrary constants.

In order to determine these arbitrary constants (B_1 and B_2) initial conditions are used. For this case these are

$$x(0) = x_0 \text{ and } \dot{x}(0) = v_0 \tag{4.8}$$

where x(0) and $\dot{x}(0)$ are the initial displacement and velocity of the mass m. Either or both of these initial conditions initiate motion.

The prescribed initial conditions (4.8) are used into the equation (4.7). Upon simplification, we get the following:

$$x(t) = x_0\, \cos\omega_n t + (v_0/\omega_n)\, \sin\omega_n t$$

$$= \text{A} \sin(\omega_n t + \alpha) \tag{4.9}$$

Where $A = [x_0^2 + (v_0/\omega_n)^2\,]^{\frac{1}{2}}$ and $\alpha = tan^{-1}(x_0\omega_n/v_0)$

Response given by the equation (4.9) shows a harmonic motion having an angular velocity ω; frequency f = ω/2π and the period $T = 2\pi/\omega_n$.

The angular velocity (ω_n) is called the natural frequency of vibration.

It is important to note that the natural frequency of vibrations depends only on the physical properties of the parameters k and m and it is independent of the initial conditions

4.3 APPLICATIONS OF THE THEORY

(Determination of natural frequency)

EXAMPLE 1:

Determine the natural frequency of vibration of the system shown in fig. 4.3(a)

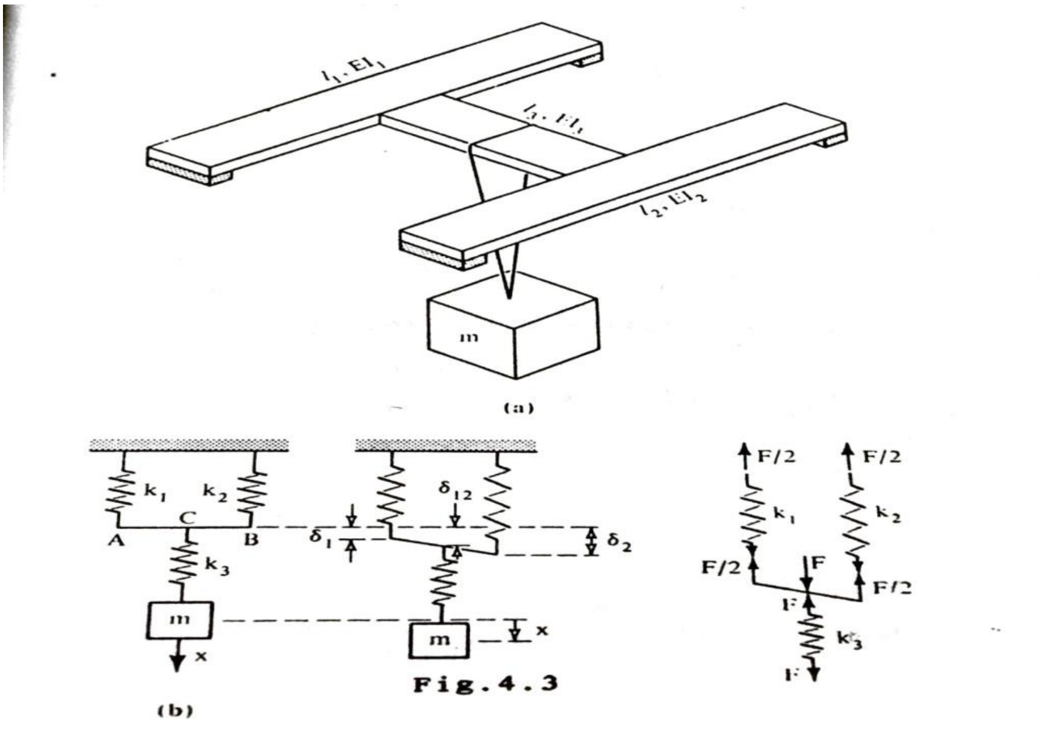


Fig.4.3

Solution:

Physical model shown in fig.4.3a is converted to its equivalent model shown in fig. 4.3(b). Equivalent spring stiffnesses are calculated on the basis of considering the three elastic beams involved as simply supported.

Spring stiffness of a simply supported elastic beam can be easily obtained using principles of strength of materials. In this case it is

$$k = 48\, EI\, /L^3$$

Assume that the mass is displaced by an amount x. Compatibility of displacements demands

$$\delta_1 = (\tfrac{F}{2})/k_1; \qquad \delta_2 = (\tfrac{F}{2})/k_2;\ \delta_3 = F/k_3;$$

$$\delta_{12} = (\delta_1 + \delta_2)/2;$$

$$\delta_{123} = \delta_{12} + \delta_3 = x$$

where F (an unknown force applied to mass m) causes a deflection x.

Thus, $x = \delta_{123} = \frac{F}{4k_1} + \frac{F}{4k_2} + \frac{F}{k_3}$

Thus the equivalent spring stiffness (k_{eq}) of the system is

$$k_{eq} = \delta_{123} = \frac{1}{\frac{1}{4k_1} + \frac{1}{4k_2} + \frac{1}{k_3}}$$

and the equation of motion is

$$m\ddot{x} + k_{eq}x = 0$$

Hence natural frequency is

$$\omega_n = \left(k_{eq}/m\right)^{1/2} = \left[\frac{1}{\left(k + \frac{1}{4k_{12}} + \frac{1}{k_3}\right)m}\right]^{1/2}$$

It may be noted, (k_1+ k_2) cannot be simply added to give the equivalent stiffness of these parallel springs. It is so because AB does not remain horizontal.

EXAMPLE 2:

Fig. 4.4 shows a pipe cutting device consisting of a disc of radius r and mass M. The disc can rotate about its center O. A motor of mass m is attached to a cutting wheel via a light rod of length l. The system can oscillate in the plane of the disc about the point O.

a) Determine the period of vibrations of the system for small angles of rotation and
b) Determine the maximum velocity of the motor, if the arm is displaced by an angle θ_0 and is released.

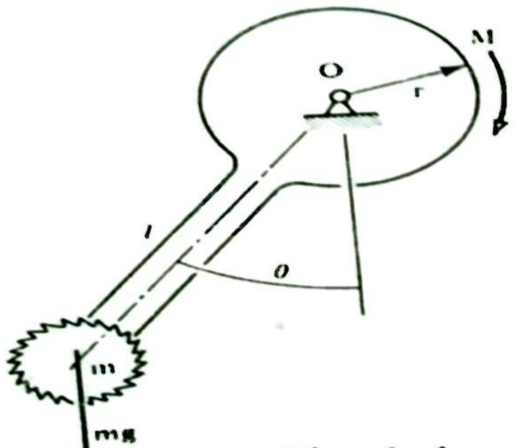


Fig.4.4

SOLUTION:

The system rotates about the center O. Newton's second law of notion is applied about the point O. It yields

$$J\ddot{\theta} = -mg(l+r)\sin\theta$$

where

J = mass moment of inertia of the system about the point.

$$= \left(\frac{1}{2}\right)mr_0^2 + m(l+r)^2 + \left(\frac{1}{2}\right)mr^2$$

a) For small θ

$$J\ddot{\theta} + mg(l+r)\theta = 0$$

The period of natural vibrations is

$$T_n = 2\pi\sqrt{m/k}$$

$$= 2\pi \left[\frac{m\,g(l+r)}{\left(\frac{1}{2}\right)(mr_0^2 + mr^2) + m(1+r^2)} \right]^{1/2}$$

b) The solution of the equation of mot ion is

$$\theta = \theta_0 \cos \omega_n t$$

where $\omega_n = 2\pi/T_n$ and θ_0 is the amplitude of vibrations

The angular velocity is

$$\theta = -\theta_0 \omega_n \sin \omega_n t$$

Maximum angular velocity is

$$(\theta_{\max}) = \theta_0 \omega_n = \theta_0 \left[\frac{\left(\frac{1}{2}\right)(mr_0^2 + mr^2) + m(1+r^2)}{m\,g(l+r)} \right]^{1/2}$$

EXAMPLE 3:

A plate of mass m rests on ground of equivalent spring stiffness k. A mass m falls from a height h on the plate with a perfectly plastic impact. Determine resulting vibration.

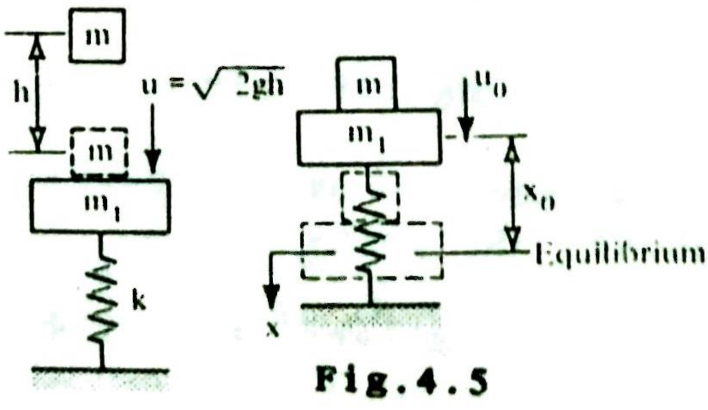


Fig.4.5

SOLUTION:

The velocity u_1 with which the mass m strikes the plate is

$$u_1 = \sqrt{2gh}$$

Conservation of momentum requires that

$$mu_1 = (m_1 + m)u_0$$

where u_0 = velocity of two masses after impact.

For the determination of initial conditions consider the physical situation: After impact the system ceases to be in static equilibrium and the mass m is loaded with additional load mg. This causes lowering down of the static equilibrium position by an amount mg/ k. Thus the initial conditions are

$$x_0 = -mg/k \text{ and } u_0 = \sqrt{2gh}/(\frac{m_1}{m} + 1)$$

The equation of motion of the system after impact is given by the following

$$\left(\frac{m_1}{m}\right)\ddot{x} + kx = 0$$

Solving the above equation, using the initial conditions given above and upon simplifying, one gets the following expression for the ensuing vibration:

$$x(t) = -(mg/k)\cos\omega_n t + \frac{\sqrt{2gh}/(\frac{m_1}{m} + 1)}{\omega_n}\sin\omega_n t$$

Where $\omega_n = \sqrt{(k}/(m + m_n))$.

EXAMPLE 4;

A cylinder of mass m and radius R_1 rolls without slipping on curved surface of radius R. Derive the equation of motion of this system and obtain the natural frequency of vibration (Fig. 4.6).

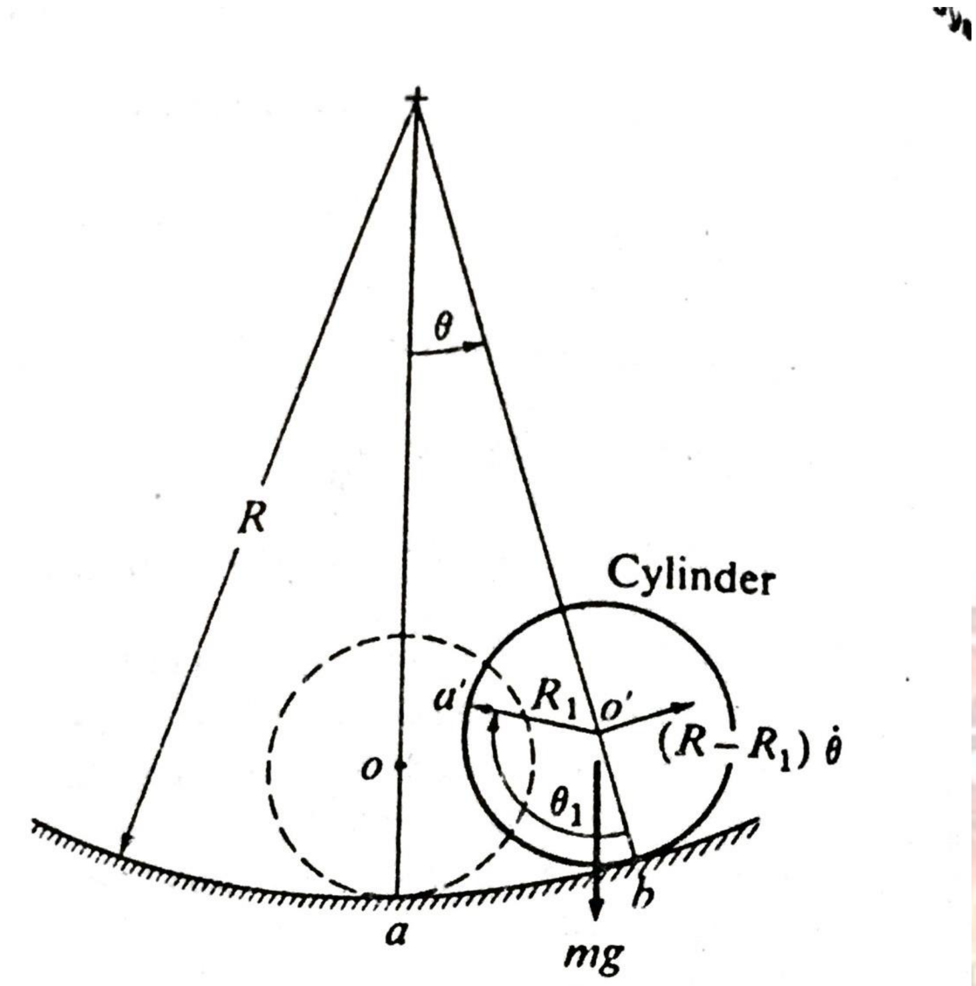


Fig.4.6

SOLUTION:

For no slip $R\theta = r_1\theta_1$. The absolute rotation of the cylinder is $(\theta_1 - \theta)$ (since θ_1 is the rotation of the cylinder relative to the curved surface). Taking moments about the instantaneous center of rotation b, the equation of motion is

$$J_b(\ddot{\theta}_1 - \ddot{\theta}) = -mgR_1 \sin\theta$$

where

$J_b = (J_0 + mR_1^2)$ and $J_0 = \left(\frac{1}{2}\right) mR_1^2$.

For small angels θ:$\theta = \sin\theta$ and $\ddot{\theta}_1 = (\frac{R}{R_1})\,\ddot{\theta}$. Making use of these substitutions in the equation of motion, we obtain

$$\left((3/2)mR_1^2\right)\left((R/R_1)-1\right)\ddot{\theta}+mgR_1\theta=0$$

Or

$$\ddot{\theta}+\frac{2g}{3(R-R_1)}\,\theta=0$$

Therefore, the natural frequency of vibration is

$$\omega_n=\sqrt{\frac{2g}{3(R-R_1)}}$$

EXAMPLE 5:

An engine valve gear system is shown in the fig.4.7. Obtain the natural frequency of vibration of this system.

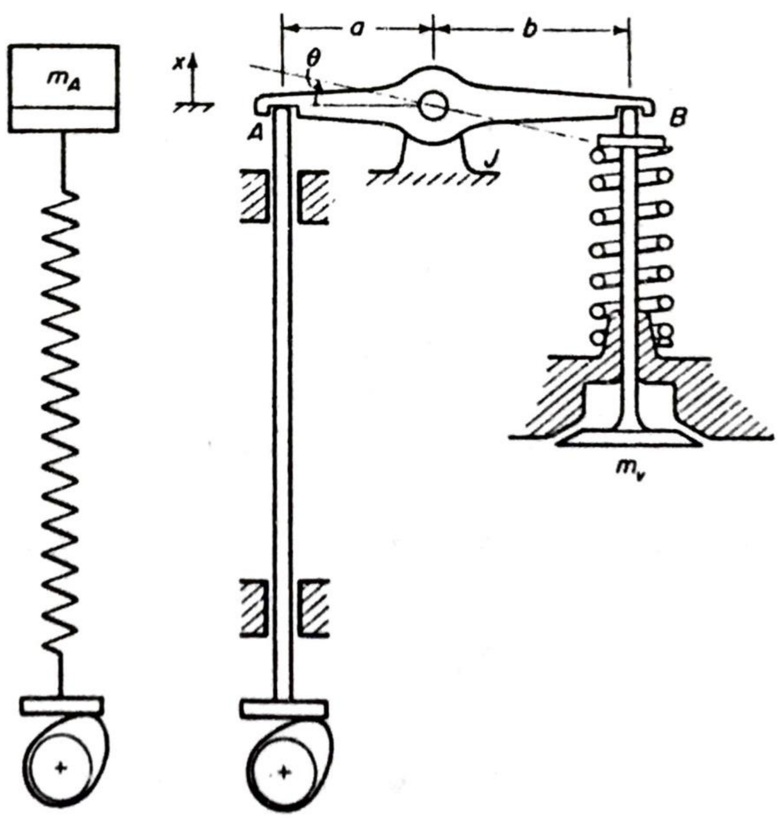


Fig.4.7

SOLUTION:

In order to solve such a system it is desirable to convert this physical system into a simpler equivalent one. The rocker arm of moment of inertia J, the valve of mass m_v, and the spring of mass m_s is reduced to a single mass at A by equating kinetic energy expressions for the given physical system and that for the equivalent system. The kinetic energy expression of the given physical system is

$$\boldsymbol{T = \left(\frac{1}{2}\right)J\dot{\theta}^2 \left(\frac{1}{2}\right)m_v(b\dot{\theta})^2 + \left(\frac{1}{2}\right)\int_0^1 \left(b\dot{\theta}\frac{y}{l}\right)^2 (m_s/l)dy}$$

$$\boldsymbol{= \left(\frac{1}{2}\right)\left(J + m_v b^2 + \left(\frac{1}{3}\right)m_s b^2\right)\dot{\theta}^2}$$

The velocity at A is $\dot{x}$=aθ. Using this information in the above, we get the following expression

$$T = \left(\frac{1}{2}\right)\left[\frac{\boldsymbol{J + m_v b^2 + \left(\frac{1}{3}\right)m_s b^2}}{a^2}\right](\dot{x})^2$$

Thus the effective mass at A is given by

$$m_A = \left[\frac{\boldsymbol{J + m_v b^2 + \left(\frac{1}{3}\right)m_s b^2}}{a^2}\right]$$

The push rod is now reduced to a spring and an additional mass at the end A. The entire system is thus reduced to single mass and a single spring system as shown.

Now, the total mass lumped at A is given by

$$m = m_A + m_P$$

The equivalent spring stiffness for the push rod is given by

$$k_P = \left(\frac{EA}{L}\right)$$

where E = Young's modulus of elasticity of the push rod material, A is its area of cross-section and L is its length. The natural frequency of vibration of the system is

$$\omega_n = (k_P/m)^{1/2}$$

EXAMPLE 6

An aluminium wire having a cross-sectional area of .0625 cm^2 and length 25.0 cm is attached to the ceiling at one end and carries a weight of 0.5 kg at the other end as shown in the figure. The weight can move laterally in a channel and owing to a special design, the friction is so small that it can be ignored. Before the upper end was fixed, the wire was stretched so that a pres tress of 2000 kg/ sq. cm was induced. Assuming that the wire is weight less and its modulus of elasticity is 0.8 x 10^6, calculate the natural frequency

of lateral vibrations and that of the axial vibrations. The wire in its vertical position is shown in fig. 4.8 (a).

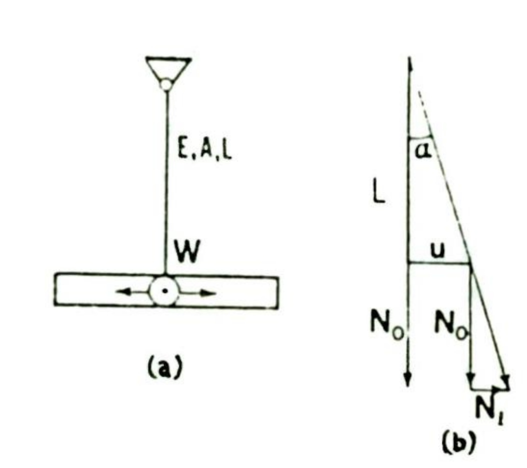


Fig. 4.8

SOLUTION:

The displaced position of the wire from the vertical is shown in fig. 4.8(b). The wire has two kinds of stiffnesses, viz. axial stiffness and transverse stiffness. We calculate these stiffnesses as given below:

Transverse stiffness: The axial stretching force, which is initially denoted by N_o, grows in proportion to length. In the displaced position the force in the wire must be aligned with the direction of the wire and may be treated as the vector sum of N_o and some N_L. From the similarity of triangles we have

$$N_o/N_L = u/L$$

where N_L is the force which is needed to move the end of the prestressed wire in a direction perpendicular direction of the axis of the wire.

Thus lateral stiffness $= k_L = (N_L/u) = N_o/L$

$= A_o/L = (.0625 \times 2000)/ 25$

$= 5.0$ kg/ cm

Natural frequency of lateral vibrations $(gk_1/W)^{1/2}$

$= ((981 \times 5.0)/0.5)^{1/2}$

$= 99.04$ rad/ sec

Axial stiffness:

The axial stiffness $= k_\alpha = EA/L = 0.8 \text{ X } 10^\delta \times 0.0625/25$

$= 2000$ kg/cm

If axial vibrations were allowed then the natural frequency of axial vibrations would be given by

$\omega^2 = 981 \times 2000/0.5$

$= 1980.9$ rad/ sec

It may be noted that in spite of a sizable prestress, the lateral stiffness is considerably smaller than the axial value.

4.4 FORCED VIBRATIONS WITHOUT DAMPING

Forced vibrations are the result of the action of external forces on a system. These external forces can be very complex in nature. Here, we confine ourselves to the case of external forces of the type given below:

$$F = P \cos\omega t$$

4.4.1 Forced Vibration of a spring:

Simplest type of forced vibrations are seen to occur in the case of a massless spring when a time variant force acts it. This type of situation is depicted in fig. 4.9. Equation of motion, for this case, is

$$kx_k = P_k cos\omega t$$

or $$x_k = (P_k/k)cos\omega t \tag{4.10}$$

The motion of the end of the spring is a simple harmonic motion and frequency of vibration is the same as that of the disturbing force. The system can store potential energy only.

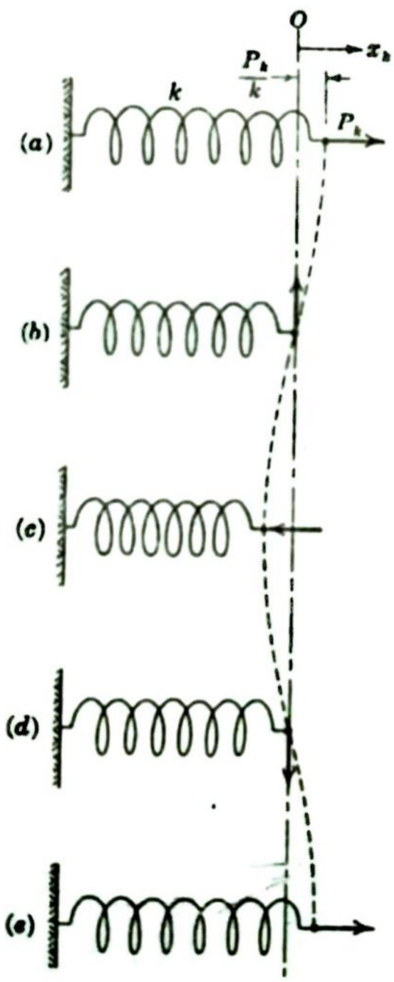


Fig. 4.9

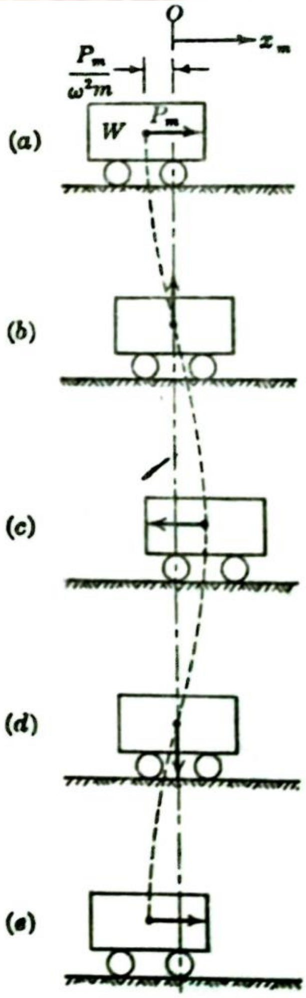


Fig. 4.10

4.4.2 Forced Vibrations of a Mass:

Consider another system consisting only of a movable mass with a vibratory force acting on it (fig. 4.10)

Equation of mot ion written with the help of Newton's second law of motion is

$$(\frac{W}{g})\ddot{x}_m = P_m cos\omega t \tag{4.11}$$

This equation is integrated twice and the following solution is obtained:

$$x_m = -\left(\frac{P_m g}{W\omega^2}\right) cos\omega t + At + B \tag{4.12}$$

Let the mass oscillate about the origin i.e. A = B = 0. This requires initial velocity to be zero, i.e.

$$\dot{x_0} = 0 \text{ and } x_0 = -\left(\frac{P_m g}{W\omega^2}\right)$$

The equation (4. 12), for this case, is: $x_m = -\left(\frac{P_m g}{W\omega^2}\right) cos\omega t$ This is a simple harmonic mot ion.

It is important to note that the two cases discussed above show oscillatory responses even though both mass and spring are not present together in either case. As a matter of fact this requirement is essential for the case of free vibrations but is not necessary for forced vibrations.

4.4.3 Forced Vibration of a Spring-Mass System:

Let us, now, investigate the motion of a system comprising of a mass and a linear spring under the act ion of an oscillatory force of the harmonic type (fig.4.11).

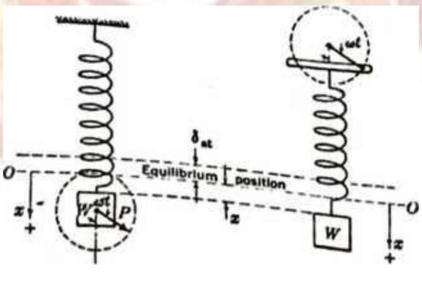


Fig.4.11

Newton's second law of motion is applied to the free body diagram and following is the equation of motion:

$$\left(\frac{W}{g}\right)\ddot{x} = W - k(\delta_{st} + x) + P\,cos\omega t \tag{4.13}$$

$$\text{or} \quad m\ddot{x} + ks = P\ cos\omega t \tag{4.14}$$

(Since $W = mg = k\delta_{st}$)

We introduce $\omega_n = \sqrt{k/m}$ in the above equation. We get

$$\ddot{x} + \omega_n^2 x = \left(\frac{P}{m}\right)\ cos\omega t \tag{4.15}$$

General solution of this equation consists of two parts, viz. the solution of the homogeneous equation called the complementary solution (x_C) and the particular integral (x_P).

Complementary solution is identical with the solution of the equation of notion for free vibration. It is

$$x_c = A\ cos\omega t + B\ sin\omega t \tag{4.16}$$

where A and B are arbitrary constants decided by the initial conditions.

Particular integral is that solution which must satisfy complete differential equation. A particular integral nay be given by

$$x_P = C\ cos\omega t + D\ sin\omega t \tag{4.17}$$

Where C and D depend on the exciting force and the Physical constants of the system. In order to find the values of these constants, let this assumed solution be substituted into the equation of notion of the system. The resulting equation is

$$-C\ \omega^2 cos\omega t - D\ \ \omega^2 sin\omega t + C\ \omega_n^2\ cos\omega t + D\ \ \omega_n^2\ sin\omega t$$

$$= C(\omega_n^2 - \omega^2)cos\omega t + (\omega_n^2 - \omega^2)sin\omega t = P\ cos\omega t$$

This equation can only be satisfied for an arbitrary value of t and ω not equal to ω_n if

$$C = \frac{P/m}{(\omega_n^2 - \omega^2)} \text{ and } D=0 \tag{4.18}$$

Particular integral, in this case, is

$$x_P = \frac{(\frac{P}{m})cos\omega t}{(\omega^2 - \omega^2)} = \frac{(\frac{P}{m})cos\omega t}{(1 - \omega^2/\omega^2)} \tag{4.19}$$

Now, the complete solution of the differential equation of motion is

$$x = A\ cos\omega_n t + B\ sin\omega_n t + \frac{(\frac{P}{k})}{(1 - \omega^2/\omega_n^2)} cos\omega t \tag{4.20}$$

Physical interpretation of the mot ion represented by the equation (4.20) is simple. First two terms of the equation represent the free vibration which is initiated at or before the start of forced motion. The last term in the equation represents a mot ion that has the same frequency as the exciting force. It is this part of the response that remains in existence till the external force acts on the system. This part of the solution is called steady state solution.

RESONANCE

Particular solution of the above equation of notion has an amplitude proportional to a "resonance" factor given below

$$\frac{1}{1-(\omega^2/\omega_n^2)}$$

It is important to note the following:

1. For $\omega/\omega_n < 1$, the resonance factor is positive and the mass is in the same direction as the exciting force.
2. For $\omega/\omega_n > 1$, the resonance factor is negative and the mass is in a direction opposite to that of the exciting force.
3. For $\omega/\omega_n = 1$, For the resonance factor takes on a value equal to infinity. This condition, at which the natural frequency coincides with the forcing frequency, is called RESONANCE. The solution of the steady state motion is not valid here and there is no steady state motion corresponding to $\omega/\omega_n = 1$.

4.5 FORCED VIBRATIONS WITH DAMPING

4.5. I The Nature of Damping

Damping is the energy dissipating mechanism present in any vibratory system. This dissipation takes place, normally, by a conversion of mechanical energy into heat energy. This heat may cause hot-bearings, unequal expansion of machine parts and general impairment of the functioning of machines. In many cases this conversion of mechanical energy into heat is small but for a complete understanding of the vibration phenomena it must be thoroughly analyzed.

Sometimes damping is purposely introduced into the system. This is done to dissipate the unwanted energy accumulations. Most common form of unwanted energy is its temporary storage in a vibrating system. Vibration engineers use a variety of duping devices, e.g. devices based on different types of friction forces (viz. dry, viscous, solid), devices based on the principle of generating electric current and then its dissipation through resistors, etc.

There is a wide variety of effects which are responsible for damping. Some of the causes of damping are qualitatively described below;

1. Internal friction material:

When a deformable member is subjected to vibration, then the stress-strain diagram drawn over a cycle of loading shows hysteresis loop. The area of this loop is the amount of energy dissipated as heat per cycle. This energy loss takes place due to the internal friction of the material. It should be noted that the size of this loop is independent of the stress reversal cycles. It is dependent on the amplitude of vibration besides many other factors.

2. Fluid friction:

It is a common experience that vibrating systems operate in an environment involving air, gas, water, steam, oil etc. In order to move these media some energy is consumed which is drawn from the system itself. There are many situations where the fluids are moved in specialized spaces, e.g. an oil film between sliding surfaces, intermittent flow of fluid through close clearances between a piston and cylinder (as in the case of a dashpot), through the orifices (shock absorber), etc.

3. Dry friction:

Many surfaces having relative not ion show rubbing action. This action nay take place under conditions of dry or imperfect lubrication. This type of situation leading to energy losses is encountered in the cases of pistons sliding in cylinders, ball or roller bearings, clutch slippage, working of a key in its key-way, loose fit between the shaft and hub, bearing friction at pinned joints in mechanisms, etc.

The theory of duped vibrations is simplest in the case of viscous damping (forces are proportional to the relative velocity of the duping elements). Other types of damping usually result in energy dissipations which are functions of displacements as well as velocity. The equation of motion of a system with non-viscous damping is nonlinear and complicated from the mathematical standpoint. For this reason, different types of damping mechanisms are, in the first approximation, replaced by equivalent viscous damping. An equivalent viscous damper is obtained by equating the total energy dissipated per cycle by the viscous damper to that by the nonlinear damper.

4.5.2 THE NATURE OF FORCED VIBRATIONS OF LINEARLY DAMPED SYSTEMS

The forces that excite a dynamic system into mot ion may be quite complex. To get a clear picture of the effect of given damping on vibratory systems it is instructive to consider the case which is excited by a harmonic force P sinωt.

4.5.2.1 Viscous Damper and Spring:

In a viscous damper the damping force is proportional to the relative velocity of its terminals (fig. 4.12) and is always opposed to the direct ion of motion.

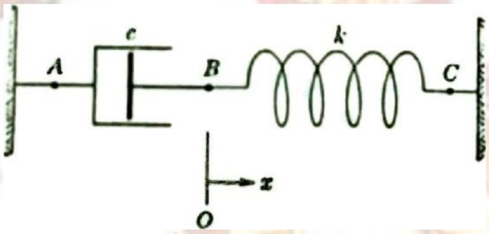


Fig. 4. 12

Thus, the damping force is

$$F_d = -c\left(\frac{dx}{dt}\right) \tag{4.21}$$

The spring force is

$$F_s = -kx \tag{4.22}$$

where x is the displacement of the ends B and C.

The equation of motion of the viscous is obtained force balance at B. The equation of motion so obtained is

$$c\dot{x} + kx = 0 \tag{4.23}$$

integration of equation (4.23) results in the following (4.23)

$$\log x = -qt + C \tag{4.24}$$

Where q=k/c.

Following initial condition is introduced: At t = 0, $x = x_0$ (4.25)

The constant of integration C, determined with the help of the initial condition given above, is

$$C = \log x_0 \qquad (4.26)$$

The value of C, so obtained, is used into the equation (4.24) and following is the resulting equation:

$$\log(x/x_0) = - qt$$

or

$$x = x_0 e^{-q}$$

The nature of the response is shown in the fig. 4.13. If the system is initially displaced through a distance x_0, it approaches the equilibrium position asymptotically.

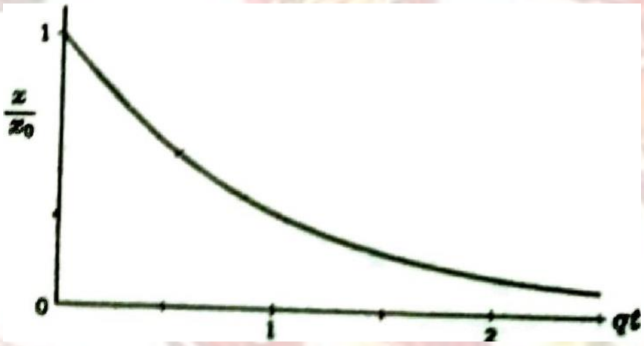


Fig.4. 13

The parameter q has the dimensions of circular frequency (l/q) is referred to as the relaxation time.

Equation of mot ion of a single degree of freedom system with spring, mass and viscous damping is obtained be low using the Newton's second law of motion (fig.4.14).

The damping force is $F1=c(dx/dt)$.

The spring force is $F\,2 = k\,x$

The external force is $P \sin \omega t$

The inertial force is $P3 = m(d^2x/dt^2)$

If the system is given an initial displacement x from its equilibrium position, the forces acting are shown in the fig. 4.14(b). Applying Newton's second law of motion, we get

$$m\ddot{x} = -c\dot{x} - k(\delta_{st} + x) + W + P_0 cos\omega t \qquad (4.28)$$

or $$\ddot{x} + 2\xi\dot{x} + \omega_n^2 x = \left(\frac{P}{m}\right) \sin \omega t \tag{4.29}$$

where

$$2\xi = \frac{c}{m} : \omega_n^2 = \frac{k}{m}.$$

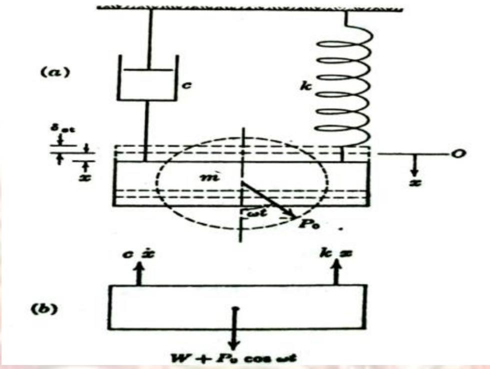


Fig. 4.14

Equation (4. 29) is the differential equation of motion of system under investigation. Solution of this consists of a complementary function and a particular integral. Complementary function is the solution of homogeneous equation whereas the particular integral to satisfy the complete differential equation.

1. Complementary Function:

Homogeneous equation of the system is

$$\ddot{x} + 2\xi\dot{x} + \omega_n^2 x = 0 \tag{4.30}$$

Equation (4.30) is rewritten in the form given below

$$(D^2 + 2\xi D + \omega_n^2)x = 0, \text{ where D = d/dt.} \tag{4.31}$$

The above equation is quadratic in operator D. Roots of its auxiliary equation are

$$M_{1,2} = -\xi \pm \sqrt{\xi^2 - \omega_n^2} \tag{4.32}$$

where M_1 is the root 1 and M_2 is the root 2.

The solution of the equation (4.30) is

$$x = C_1 e^{\left(-\xi + \sqrt{\xi^2 - \omega_n^2}\right)t} + C_2 e^{\left(-\xi - \sqrt{\xi^2 - \omega_n^2}\right)t} \tag{4.33}$$

or

$$x = e^{-\xi t}\left[C_1 e^{t\sqrt{\xi^2-\omega_n^2}} + C_2 e^{-t\sqrt{\xi^2-\omega_n^2}}\right] \tag{4.34}$$

Equation (4.34) is the solution of the differential equation (4.30). The nature of the response given by this solution is dependent on the following three cases:

1. $\xi > \omega_n$
2. $\xi = \omega_n$
3. $\xi < \omega_n$

Case I : $\boldsymbol{\xi > \omega_n}$ (Over Damping)

If $\xi > \omega_n$ then both the roots of the auxiliary equation are real but negative. It means that the response is a sum of two decaying exponentials. The shape of the graph depends on the values of C_1 and C_2 which in turn depend on the initial conditions of motion. The graph of motion is shown in the fig. 4.15.

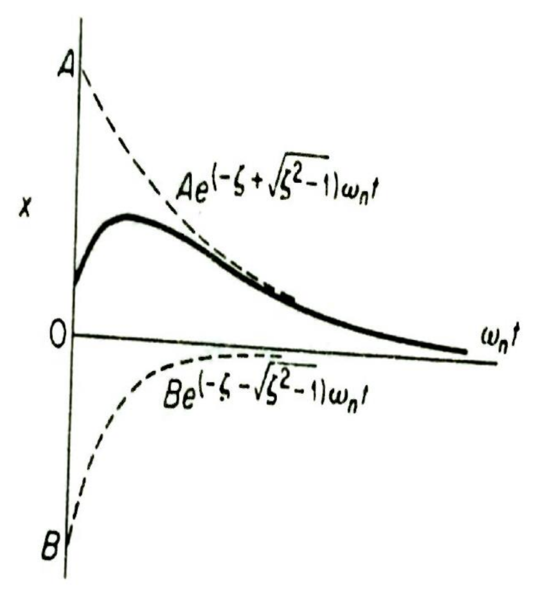


Fig. 4.15

The response (displacement) decreases with time and it tends to become zero when time is increased beyond limit. It is non-oscillatory and non-periodic and is not dead-beat too, since the time taken for the mass to come to rest is very long. This type of damping is called OVER—DAMPING.

Case 2. $\boldsymbol{\xi = \omega_n}$ (CRITICAL DAMPING)

In this case the two roots of the auxiliary equation are equal and hence the solution of the differential equation of mot ion is

$$x = (C_3 t + C_4)\, e^{-\xi t} \tag{4.34}$$

Here the motion just falls short of being periodic and the system is said to be is said to be CRITICALLY DAMPED (transitional condition), is a very convenient basis of comparing damping coefficients. One of the important applications of the use of critical damping is in the indicating type of measuring instruments in which the pointer should reach its ultimate deflection as quickly as possible, without swinging to and fro about this final value before coming to rest (e.g. dead-beat galvanometers). The graph of response is shown in the figure given below

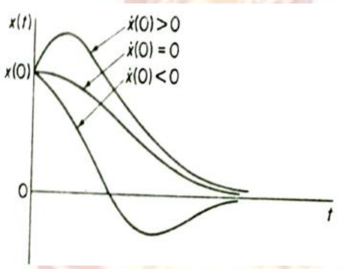


Fig. 4.16

Case 3. $\xi < \omega_n$ (UNDER-DAMPING):

For $\xi < \omega_n$ the roots of the auxiliary equation become imaginary, Let us write

$$\omega_d = \sqrt{(\omega_n^2 - \xi^2)}$$

For this case, equation (4.32) is written as

$$x = e^{-\xi t}\left[C_1 e^{-i\omega_d t} + C_2 e^{-i\omega_d t}\right] \tag{4.35}$$

Using De Moivre's theorem in equation (4.35) and then rearranging, we get

$$x = A e^{-\xi t} \sin(\omega_d t + \emptyset) \tag{4.36}$$

A and $\emptyset$ appearing in equation (4.36) are determined with the help of initial conditions.

Response in this case is seen to be oscillatory. But the amplitudes of vibrations decrease with tine. This decrease is exponential and the mot ion never repeats itself. The kind of damping which gives rise to this type of response is called UNDER-DAMPING. It is a class of periodic notion for which ξ is very small in comparison ω_n.

LOGARITHMIC DECREMENT:

Consider the fig. 4.17 given below. The response curve of an underdamped system is shown.

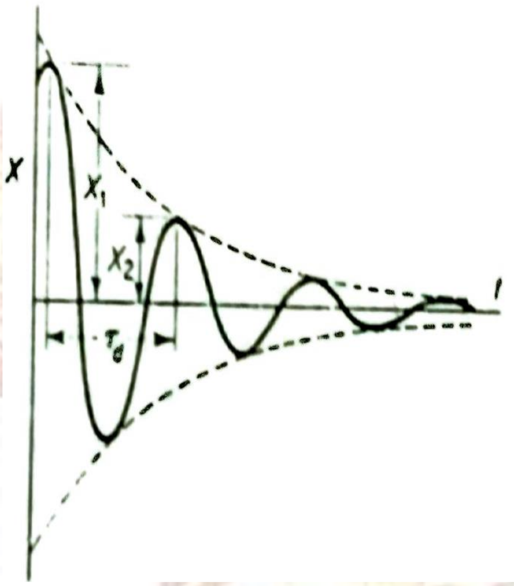


Fig. 4.17

Consecutive points located at maxima on the same side of the tine-axis are located at a distance of one complete period of value) (2π/ω). Hence if a maxima occurs at P at time t_1 having a value x_1, a maxima at Q would occur at a time

$$t_2 = t_1 + 2\pi/\omega_d$$

These maxima will have the following values:

$$x_1 = A\, e^{-\xi t_1}$$

$$x_2 = A\, e^{-\xi(t_1 + \frac{2\pi}{\omega_d})}$$

The ratio of these two maxima is

$$x_1/x_2 = \; e^{\xi(\frac{2\pi}{\omega_d})} = \; e^{\xi t}$$

where $T = \frac{2\pi}{\omega_d}$.

Taking natural logarithm of both sides, we get

$$log_e\left(\frac{x_n}{x_{n+1}}\right) = \xi t$$

This quantity is called logarithmic decrement of motion (δ). Thus,

$$\delta = \frac{2\pi\xi}{\omega_d} = \frac{2\pi c}{(m\omega_d)} \qquad (4.37)$$

Since logarithmic decrement remains constant for a given system therefore it can be readily used for the determination of damping coefficient of a vibrating system. In order to determine the damping coefficient, x_1 and x_2 are needed which can be determined experimentally. These maxima can also be determined displacement curves of under-damped vibrations. Since all the quantities except c are thus known, the damping constant (c) can be evaluated.

2. PARTICULAR INTEGRAL:

Let us reproduce equation (4.29) below:

$$\ddot{x} + 2\xi\dot{x} + \omega_n^2 x = \left(\frac{P}{m}\right) sin\omega t$$

Particular integral of this equation can be easily evaluated and it is given below:

$$x = \frac{(\frac{P}{k})}{\left(1-(\omega^2-\omega_n^2)\right)^2+4(\frac{c}{c_c})(\omega^2/\omega_n^2)^2}\sin(\omega t-\varphi) \qquad (4.38)$$

where $c_c = 2\sqrt{km}$ = critical damping, $\xi = (c_c)$

and $\varphi = tan^{-1}\left(2\xi\omega\omega_n/(\omega_n^2-\omega^2)\right)$.

The complete solution of the differential equation is the sum of the complementary function and the particular integral. It is

$$x = \underbrace{Ae^{-\xi t}\sin(\omega_d t + \emptyset)}_{I} + \underbrace{B(\xi,\omega_n.\omega)\sin(\omega_d t - \emptyset)}_{II}$$

Where $B(\xi,\omega_n,\omega) = \frac{(\frac{P}{k})}{\left\{(1-(\omega/\omega_n)^2)^2+4(\frac{c}{c_c})^2(\omega/\omega_n)^2\right\}}$

Part I of the solution is called transient and part II is called steady state.

RESONANCE:

Mechanical resonance is a phenomenon which occurs when a system undergoing forced, steady-state vibrations displays a displacement, velocity or acceleration peak.

Thus we have amplitude resonance, velocity resonance and acceleration resonance.

To determine the condition of amplitude resonance, i.e. the condition under which the amplitude of the forced vibrations grows maximum, let us take

$$B_1(\xi, \omega_n, \omega) = \frac{(\frac{P}{m})}{\left[(\omega_n^2 - \omega^2)^2 + 4\ \xi^2 \omega^2\right]^{1/2}}$$

Displacement amplitude would become maximum when the denominator of the above equation is minimum.

Let us rewrite the denominator of the above as an equation

$$\alpha = (\omega_n^2 - \omega^2)^2 + 4\ \xi^2 \omega^2$$

Differentiating α w.r.t. ω and equating the result to zero, we get

$$\frac{d}{d\omega}\left[(\omega_n^2 - \omega^2)^2 + 4\ \xi^2 \omega^2\right] = 0$$

Solving the above equation, we get

$$\omega = \sqrt{\omega_n^2 - 2\ \xi^2} \tag{4.39}$$

This is the condition for the existence of maxima of the amplitude of vibrations.

For the determination of the condition of velocity resonance, let us reproduce again the displacement (x) of the forced vibrations below:

$$x = \frac{\mathrm{P}\sin(\omega t - \varphi)}{m\left[(\omega_n^2 - \omega^2)^2 + 4\ \xi^2 \omega^2\right]^{1/2}}$$

Differentiating the above w.r.t. t and simplifying, we get

$$\dot{x} = \frac{\left(\frac{P}{m}\right) cos(\omega t - \varphi)}{\left[\omega^2(\omega_n^2/\omega^2 - 1)^2 + 4\ \xi^2\right]^{1/2}}$$

It is evident that the velocity amplitude will grow maximum when the denominator of the above expression is minimum. This denominator becomes maximum when$\omega_n = \omega$.

Thus, when the frequency of the external excitation equals the natural frequency of vibration of the system, the system, the system picks up the maximum velocity amplitude. Under this condition velocity resonance is said to occur.

Comparison of the conditions for velocity resonance and amplitude resonance would make it clear that the frequency of the excitation at which velocity resonance comes into existence is higher than that at which amplitude resonance occurs.

Condition for the existence of acceleration resonance can be similarly obtained.

Relations between the resonant frequencies are:

Displacement resonant frequency : $\omega_n\,(1-2\xi^2)^{1/2}$

Velocity resonant frequency : ω_n

Acceleration resonant frequency : $\omega_n/\omega_n\,(1-2\xi^2)^{1/2}$

Damped natural frequency : $\omega_n\,(1-\xi^2)^{1/2}$

It is to be borne in mind that for the degree of damping usually present in physical mechanical systems the difference among the three resonant frequencies is small.

Graphical representation of amplitude resonance and phase angle

Now, we rewrite the displacement amplitude contained in the equation (4.38) with the definitions:

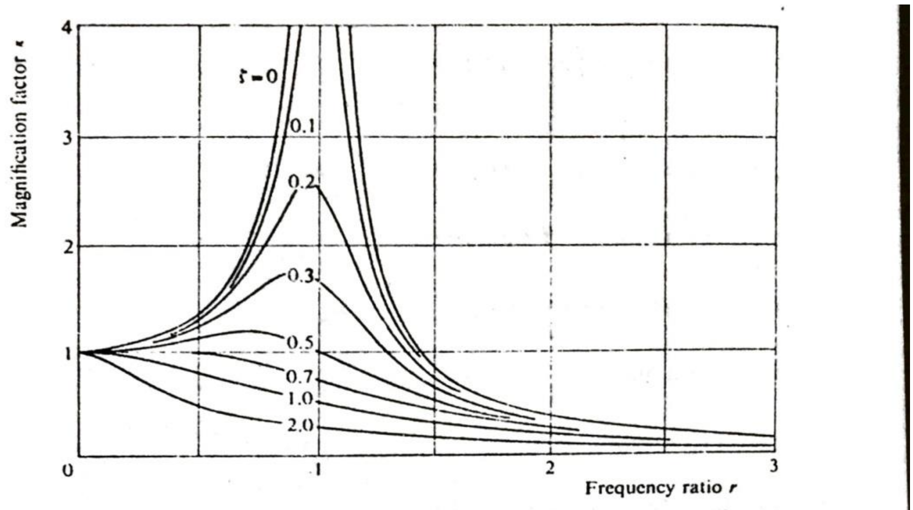
$$x/x_c = \left(\frac{P}{k}\right) \;\&\; r = \omega/\omega_n$$

The first ratio may be regarded as the amplitude ratio of the steady-state response to the static response of the system. This ratio is called as the magnification factor. The second ratio is the frequency ratio.

$$x/x_c = \frac{1}{\{(1-r^2)^2+(2\xi r)^2\}} = \text{Magnification Factor}$$

and the phase angle $\varphi = tan^{-1}\{2\xi r/(1-r^2)\}$

Figure 4.18 shows the variation of the magnification factor with frequency ratio for different values of damping ratios. It can be seen that the magnification factor can be greater than or less than unity. Figure 4.19 shows the variation of the phase angle with frequency ratio for different values of damping ratios.



.18:Variation of magnification factor with freq. ratio

Figure 4.18: Variation of magnification factor with freq. Ratio

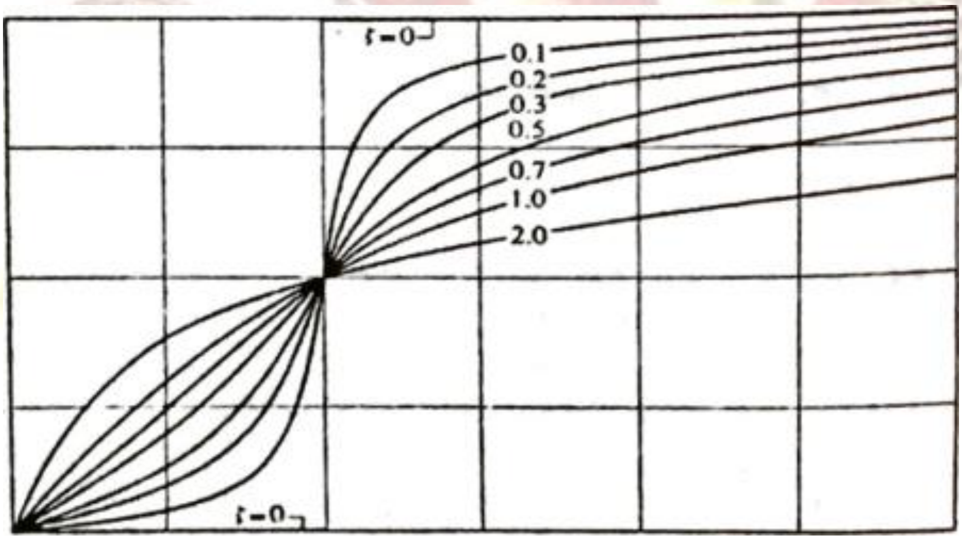


Fig.4.19: Variation of phase angle with frequency ratio

CHAPTER V

VIBRATIONS OF MULTI-DEGREE OF FREEDOM DISCRETE SYSTEMS

5. 1 INTRODUCTION

Selection of a particular mathematical model for a given physical system sets the limits of accuracy of predict ion of dynamic behaviour. Simplest mathematical model of a physical system is a Single degree of freedom (SDF) model the analyses of which were presented in chapter IV. This chapter is concerned with the vibration analyses of multi-degree of freedom system models.

A multi-degree of freedom model is essentially a step above that of the single degree of freedom model but a step below that of a continuum model. Choice of the selection of suitable number of degrees of freedom for a particular case is governed by the cost and the objective of the analysis. As a matter of fact handling of complex system problems requires numerical analysis for solution on a computer necessitating discretization of the continuun into a multi-degree of freedom system.

5.2. General Approach to Setting up of Differential Equations of Mot ion

Conversion of a continuun model into a discrete model is a process of lumping of parameters which essentially involves the separation of inertial elements from elastic elements. This procedure is not unique since it requires Personal judgement, experience as well as common sense.

Once the system has been discretized by the lumping process, Newton's second law of motion is applied to each inertial element and the interact ion of these elements is expressed with the help of Newton's third law of motion. This process yields a system of simultaneous differential equations. Integration of these equations of motion is done after their scalar components have been referred to a set of coordinates. The extent of the case of integration depends on the choice of particular coordinate system. It is, therefore, desirable to have formulation of the basic laws of motion such that these are independent of a particular choice of coordinates used. Two methods free from this kind of limitation and in extensive use are:

(1) Hamilton's principle

(2) Lagrange's equation.

Hamilton's principle is an integral equation representation whereas Lagrange's equation is given in the form of a differential equation in generalized coordinates. Both of these methods make use of various kinds of energies of the system. These equations remain invariant under a coordinate transformation.

Following paragraphs contain an outline of these methods with a view to apply them to practical problems. No attempt is made to give a theoretical exposition of these methods. A prerequisite for proper understanding of these methods is the concept of generalized coordinates and constraints.

5.3. GENERALIZED COORDINATES & CONSTRAINTS

Consideration of the inertia of accelerating masses is very important in vibration analysis. For quantitative expression of inertia of accelerating masses and the related equations of motion, an appropriate selection of coordinates is to be made. Though Cartesian coordinate system serves as a useful tool in the derivation of the equations of motion, yet this approach is not convenient for handling complex problems. Later class of problems are handled by using generalized coordinates.

Concept of generalized coordinates can be grasped once we understand that it is a set of coordinates comprising of quantities which are independent and unrestricted by the constraints. In case the given constraints are holonomic and scleronomous, then it is possible to find a set of these generalized coordinates which are equal (in number) to the number of degrees of freedom of the system. In order to make use of these ideas in actual problems, it is necessary to know the meaning of the constraints. Equations of these constraints represent relationship(s) between all those coordinates which are required to specify a given system configuration in the selected frame of reference. This type of relationship is given by

$$N1 = N2 - N3$$

where N1 = generalized coordinates

N2 = number of selected coordinates for a problem

N3 = number of equations of constraint

If N3=0, then the selected coordinates are generalized coordinates.

5.3.1. CONSTRAINTS

It appears that the mathematical formulation of a physical system is complete once the differential equations of motion are set up. This is not so. It is necessary to define the constraints which limit the motion of the system. A dynamic system would show only rigid-body motion, if the constraints were absent. Constraints are classified as holonomic and non-holonomic. Holonomic constraints are expressible as

$$C(q_1, q_2, q_3, \ldots\ldots q_n, t) \tag{5. 1}$$

Constraints not expressible in this form are nonholononic. These constraints are expressible only as relationships between differentials, e.g.

$$a_1dq_1 + a_2dq_2 + \ldots\ldots + a_ndt = 0 \tag{5. 2}$$

In case the constraints are independent of time then they are called scleronomous. They are rheonomic if time appears explicitely. An example of rheonomic constraint is the case of a bead sliding on a vibrating wire.

Constraints introduce two kinds of difficulties in solving problems, viz. (1) all the coordinates are no longer independent since they are connected by the equations of constraint; (2) the forces of constraint are not furnished a priori. They are among the unknowns of the problem and must be obtained from the solution we seek. In the case of holonomic constraints, the difficulty regarding the coordinates is solved by the introduction of generalized coordinates. To circumvent the second difficulty (related to the forces of constraint), we would like to so formulate the equations of mot ion that the forces of constraint disappear.

5.4. HAMILTON'S PRINCIPLE

Hamilton's principle is as an integral equation in which the energy is integrated over an interval in time. Statement of the Hamilton's principle is

$$\delta \int_{t_1}^{t_2} L\, dt = -\int_{t_1}^{t_2} \delta W\, dt \tag{5.3}$$

where δ is the variation, L is the Lagrangian and W represents non-conservative forces.

Above equation is the statement of Hamilton's principle for a discrete mechanical system subjected to non-conservative forces. If there are no non-conservative forces acting on the system then $\delta w = 0$ and the above equation reduces to

$$\delta I=0$$

where

$$\delta I = \delta \int_{t_1}^{t_2} L\, dt = 0 \tag{5.4}$$

This equation means that the mot ion of a discrete, conservative mechanical system from time t_1 to time t_2 is such that the line integral I is an extremum for the path of motion. This means that amongst all the paths through which a system point in configuration space could travel from its position at time t_1 to its position at time t_2, it will actually traverse that path for which I has a stationary value. The form of Hamilton's principle shows that the actual motion is distinguished from all other mot ions by having the property that I, for a given interval of time, assumes a stationary value. Since L and δW are scalar quantities, therefore, they remain invariant under any coordinate transformation.

5.5. LAGRANGE'S EQUATIONS

It is a method of great power and versatility for the formulation of equations of mot ion for any dynamic system. Unlike Hamilton's principle, which is stated in the form of integral equation, Lagrange equations are differential equations in which one considers different forms of the energies of the system instantaneously in time. Lagrange Equations of mot ion are formulated in terms of generalized coordinates.

Statement of Lagrange's equations

Consider a discretized mechanical system with holonomic and scleronomous constraints. Let us select a set of generalized coordinates q_1, q_2, q_3,........ q_n.

For this system Lagrange's equations of motion are

$$d/dt(\partial T/\partial \dot{q}_i) - (\partial T/\partial q_i) + (\partial V/\partial q_i) = Q_i,\ i=1, 2, \dots n \quad (5.5)$$

The kinetic energy function T is

$$T = T(q_1, q_2, q_3, \dots\dots q_n; \dot{q}_1, \dot{q}_2, \dot{q}_3, \dot{q}_4, \dots\dots \dot{q}_n) \quad (5.6)$$

and the potential energy function V is

$$V = V(q_1, q_2, q_3, \dots\dots q_n) \quad (5.7)$$

Q_i represent generalized forces such that

$$Q_i = Q_i^{(c)} + Q_i^{(n)} + Q_{Ai} \quad (5.8)$$

where $Q_i^{(c)}$= generalized conservative force derivable from a potential function

$Q_i^{(n)}$= general i zed non-conservat i ve force

and Q_{Ai}= generalized externally applied force

Consider a discrete holonomous and sclerononous system defined by generalized coordinates q_i (i=1,2,….., n). The system is such that its potential energy vanishes when all the generalized coordinates have zero values. It is further assumed that the system is conservative and that it is in state of stable, static equilibrium[1] when the generalized forces and generalized velocities vanish. Following is the condition given for static equilibrium:

$$\left(\frac{\partial V}{\partial q_i}\right)_0 = 0,\ i=1, 2, \dots, n \quad (5.9)$$

The static equilibrium position defined by equation (5.9) is stable if and only if V has a minimum at q_i=0 (It is a necessary condition)

A sufficiency condition for one such minimum is given by Sylbester's criteria, according to which the coefficients k_{ij} satisfy the following:

[1] There are three kinds of equilibrium: stable, unstable and neutral. A system displaced, not too far away from the position of its stable equilibrium returns to it; when it displaced from its position of unstable equilibrium, it deviates from it still further. When the system is displaced from the position of neutral equilibrium position it remains in the new position. Free vibrations are possible only in the position of stable equilibrium.

$$k_{11} > 0, \begin{vmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{vmatrix} > 0, \ldots . \begin{vmatrix} k_{11} & k_{12} \ldots \ldots \ldots k_{1n} \\ k_{21} & k_{22} \ldots \ldots \ldots k_{2n} \\ . & . \qquad\qquad . \\ . & . \qquad\qquad . \\ . & . \qquad\qquad . \\ . & . \qquad\qquad . \\ k_{n1} & k_{n2} \ldots \ldots \ldots k_{nn} \end{vmatrix} > 0 \qquad (5.10)$$

where

$$k_{ij} = k_{ji} = \left(\frac{\partial^2 V}{\partial q_i \, \partial q_j}\right)_0 \qquad (5.11)$$

The potential energy function (V), when expanded in a Taylor series about the position of stable equilibrium, is

$$V = V_0 + \sum_1^n \left(\frac{\partial V}{\partial q_i}\right) q_1 + \left(\frac{1}{2}\right) \sum_{i,j}^n \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 q_i q_j + 0(q_i q_j q_k) \qquad (5.12)$$

Neglecting $0(q_i q_j q_k)$, we obtain the following expression for the potential energy:

$$V = \left(\frac{1}{2}\right) \sum_{i,j}^n k_{ij} \, q_i q_j \qquad (5.13)$$

This is positive, definite quadratic form of the potential energy expression.

A kinetic energy function is similarly obtained and it is given below:

$$T = \left(\frac{1}{2}\right) \sum_{i,j}^n m_{ij} \, \dot{q}_i \dot{q}_j \qquad (5.14)$$

This is positive definite quadratic form in the generalized velocities $\dot{q}_i$ & $\dot{q}_j$.

We substitute equations (5.13) and (5.14) into Lagrange's equation (5.8) and obtain the linearized equations of motion given below:

$$\sum_i^n \left(m_{ij} \, \dot{q}_i + \, k_{ij} q_i\right) = Q_i \quad \text{(j=1,2,........n)} \qquad (5.15)$$

5.5.1. APPLICATIONS

EXAMPLE 1 (Hamilton's Principle):

A mass m is constrained by friction less guides to move along the x-axis and a linear spring of spring constant k acts to provide the restoring force as shown below. A force F acts on the mass. Write the equation of motion.

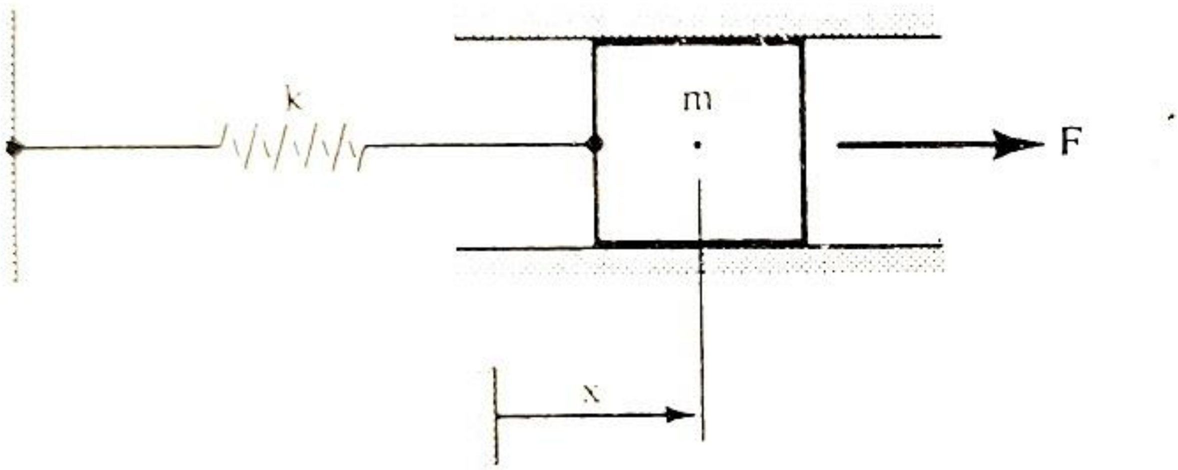


Fig. 5.1

Let x =0 be the position of the mass for which spring force vanishes. Then

$$T = \left(\frac{1}{2}\right) m\dot{x}^2 \text{ and } V = \left(\frac{1}{2}\right) kx^2$$

$$\delta W = F\delta x$$

Therefore, $L = T - V = \left(\frac{1}{2}\right)(m\dot{x}^2 - kx^2)$

$$\delta L = m\dot{x}\,\delta\dot{x} - kx\,\delta x$$

Hamilton's principle, for this case, gives the following:

$$\int_{t_1}^{t_2} (m\dot{x}\,\delta\dot{x} - kx\,\delta x)dt = -\int_{t_1}^{t_2} F\,\delta x\,dt$$

$$\int_{t_1}^{t_2} m\dot{x}\,\delta\dot{x}\,dt = (m\dot{x}\,\delta\dot{x})_{t_1}^{t_2} - \int_{t_1}^{t_2} m\dot{x}\,\delta\dot{x}\,dt \text{ and } (m\dot{x}\,\delta\dot{x})_{t_1}^{t_2} = 0$$

Therefore

$$\int_{t_1}^{t_2} (m\dot{x} + kx - F)\,\delta x\,dt = 0$$

Since (t_2-t_1) and are arbitrary, we obtain the following equation of motion:

$$m\dot{x} + kx = F$$

EXAMPLE II

(Hamilton's Principle) :

Two masses m and m are connected by springs k_1 and k_2 as shown. External forces F_1 and F_2 are acting on the masses. Write the equations of motion.

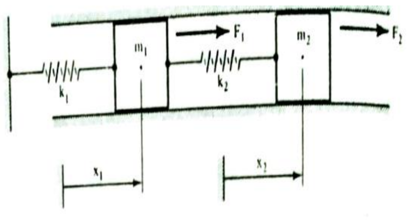


Fig. 5. 2

Then Take spring forces to be zero at $x_1 = x_2 = 0$. Then

$$T = \left(\frac{1}{2}\right) m_1 \dot{x}_1^2 + \left(\frac{1}{2}\right) m_2 \dot{x}_2^2$$

$$V = \left(\frac{1}{2}\right) k_1 x_1^2 + \left(\frac{1}{2}\right) k_2 (x_2 - x_1)^2$$

$$\delta W = F_1 \delta x_1 + F_2 \delta x_2$$

Since L =T-V

$$\delta L = m_1 \dot{x}_1\, \delta\dot{x}_1 + m_2 \dot{x}_2\, \delta\dot{x}_2 - (k_1 x_1 + k_2 x_1 - k_2 x_2)\delta x_1 - -(k_2 x_2 - k_2 x_1)\delta x_2$$

δL obtained above is substituted in equation (5.3). Finally, we get the following

$$\int_{t_1}^{t_2} \{(-m_1 \ddot{x}_1 - k_1 x_1 - k_2 x_1 + k_2 x_2 + F_1)\delta x_1 + (-m_2 \ddot{x}_2 - k_2 x_2 + k_2 x_1 + F_2)\delta x_2\} dt = 0$$

Since (t_2- t_1) is arbitrary and δx_1 & δx_2 are independent variations, therefore, we obtain the following equations of motion:

$$m_1 \ddot{x}_1 + k_1 x_1 - k_2 (x_2 - x_1) = F_1$$

$$m_2 \ddot{x}_2 + k_2 (x_2 - x_1) = F_2$$

EXAMPLE III (LAGRANGE'S EQUATIONS):

Figure (5.3) shows a fly-bal l governor. It is assumed that the entire mechanism is mass-less except two balls. All hinges and the vertical slides are frictionless. Write the equations of motion of the system.

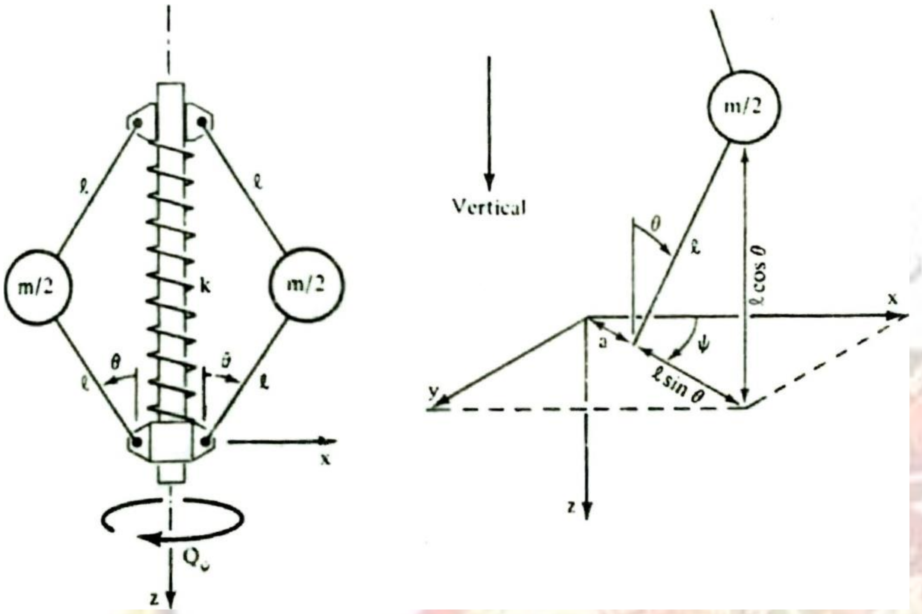


Fig.5.3

Elastic spring force is taken to be zero for θ=0. The torque Q_ψ causes an angular acceleration $\ddot{\psi}$ by rotating the entire system about z-axis. The x, y, z system is stationary.

For the case when ψ=0,

x = (a+1 sin θ) cos ψ

y = (a+1 sin θ) sin ψ

z=-1 cos θ

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = l^2\dot{\theta}^2 + (a + l\sin\theta)^2\dot{\varphi}^2$$

Thus

$T = \left(\frac{1}{2}\right) m[l^2\dot{\theta}^2 + (a + l\sin\theta)^2\dot{\varphi}^2]$

$V = -mgl(1 - \cos\theta) + 2k\,l^2 + (1 - \cos\theta)^2$

Using the above in Lagrange's equation (5.8), we get

$$ml^2\dot{\theta} - ml\dot{\varphi}^2(a + l\sin\theta)\ \cos\theta + 4Kl^2\sin\theta\,(1 - \cos\theta) - mgl\sin\theta = 0$$

$$m\frac{d}{dt}[\dot{\varphi}(a + l\sin\theta)]^2 = Q_\varphi$$

5.6 FREE VIBRATIONS

Oscillations of a conservative system about a mean position of its stable equilibrium are called vibrations when no external exciting forces are present. Linearized equations of mot ion for free vibrations are

$$\sum_i^n \left(m_{ij}\ddot{q}_i + k_{ij}q_i\right) = 0 \; (j = 1, 2, \dots . n) \qquad (5.16)$$

Solution of the above set (of linear, ordinary differential equations with constant coefficients) is sought in the form given below-

$$q_i(t) = a_i \left| \begin{matrix} \sin \omega t \\ \cos \omega t \end{matrix} \right. \qquad (i = 1,2, \dots n) \qquad (5.17)$$

where a_1 =amplitudes of vibration; ω=circular frequencies. Using equation (5. 17) into equation (5.16) , we get

$$\sum_i^n \left(k_{ij} - \omega^2 m_{ij}\right) n_i = 0 \; (j = 1, 2, \dots . n) \qquad (5.18)$$

Equation (5.18) is the statement of the classical eigenvalue problem. Expressed in words, this eigenvalue problem is:

Determine the eigenvalues for which non-trivial solution exists. The solution vector (a_i) corresponding to an eigenvalue is an eigenvector or a MODAL VECTOR and each eigenvector characterizes a normal mode of vibration.

For a non-trivial solution of the equation (5.18) it is required that the determinant of the coefficient matrix in this equation vanish, i.e.

$$\left|k_{ij} - \omega^2 m_{ij}\right| = 0 \; (i = 1,2, \dots n, j = 1, 2, \dots . n) \qquad (5.19)$$

This equation is a polynomial of degree n in ω^2, a solution of which will result in n distinct roots for a nondegenerate case. For each eigenvalue there exists a modal vector (a_{ir} : i= 1,2 ,….n).

Equation (5.18) being homogeneous, its modal vectors cannot be uniquely determined if other relations are not available. We can only fix the ratios of the modal vectors. Expressed in other words: Only the orientations of these modal vectors in the n-dimensional space is defined In order to fix the length of each modal vector it is required to adjoin the following normalization conditions to equation (5.18) :

$$\sum_{i,j}^{n} m_{ij} a_{ir} a_{jr} = 1 \ (r = 1, 2, \ldots . n) \tag{5.20}$$

Divide both sides of this equation by the square of the k tn component of the r th modal vector a_{kr} and then solve for a_{kr}. This will give

$$a_{kr} = \pm\left[\sum_{i,j}^{n} m_{ij} (a_{ir}/a_{kr})(a_{jr}/a_{kr})\right]^{-1/2} \tag{5.21}$$

Consistent use of + or - sign will avoid the confusion about the sense of the modal vector.

5.7 ORTHOGONALITY OF NORMAL MODES

Equation (5.18) is rewritten below:

$$\omega_r^2 \sum_{i}^{n} m_{ij} a_{ir} = \sum_{i}^{n} k_{ij} a_{ir} \tag{5.22}$$

Replace r by s in the above equation and interchange i & j. These operations will yield the following:

$$\omega_s^2 \sum_{j}^{n} m_{ji} a_{js} = \sum_{j}^{n} k_{ji} a_{js} \tag{5.23}$$

We, now, subtract eq. (5.23) from eq. (5.22) . Since the following holds

$$k_{ii} = k_{ji} \text{ and that } m_{ij} = m_{ji},$$

we obtain,

$$(\omega_r^2 - \omega_s^2) \sum_{i,j}^{n} m_{ij} a_{ir} a_{js} = 0 \tag{5.24}$$

For ω_r^2 not equal to ω_s^2, we obtain the following

$$\sum_{i,j}^{n} m_{ij} a_{ir} a_{js} = 0 \tag{5.25}$$

Equation (5.25) is rewritten as given below:

$$\sum_{i,j}^{n} m_{ij} a_{ir} a_{js} = \delta_{rs} \tag{5.26}$$

where δ_{rs} is the Kronecker delta.

Equation (5.26) is the statement of orthogonality of normal nodes of vibrations.

Now, the complete solution of equation (5.16) is given by

$$q_i(t) = \sum_{r=1}^{n} a_{ir} \left(A_r \cos \omega_r t + B_r \sin \omega_r t\right) \ \text{(i=1,2,....n)} \tag{5.27}$$

A_r and B_r are 2n constants which are determined with the help of 2n initial conditions, viz.

$$q_i(t) = q_{i0} : \dot{q}_i(t) = \dot{q}_{i0} \tag{5.28}$$

Using eq. (5.27) into eq. (5.28), we get

$$q_{i0} = \sum_r^n a_{ir} A_r \tag{5.29}$$

$$\dot{q}_{i0} = \sum_r^n \omega_r a_{ir} B_r \tag{5.30}$$

Nov, we determine A_r and B_r.

Multiply both sides of equation (5.29) by $m_{ij}a_{js}$ and sum over i and j. We get

$$\sum_{i,j}^n q_{i0} m_{ij} a_{js} = \sum_r^n A_r \sum_{i,j}^n m_{ij} a_{ir} a_{js} = \sum_r^n A_r \delta_{rs} = A_s$$

Or

$$A_r = \sum_{i,j}^n q_{i0} m_{ij} a_{jr} \tag{5.31a}$$

Similarly

$$\omega_r B_r = \sum_{i,j}^n \dot{q}_{i0} m_{ij} a_{jr} \tag{5.31b}$$

This completely determines the problem of free vibrations

Inspect ion of equations (5.27) and (5.28) reveals that phase of a particular mode of vibration is dependent on the initial conditions but the ratios of the amplitudes of modes of vibrations for a particular frequency are determined by the physical characteristics of the system (i.e. the inertia and stiffness characteristics) . If $\omega_r/\omega_i = K_r$ (r=2,3,...n) for positive integers K_r, then the motion described by the equation (5.27) is periodic but this is usually not so. If we multiply equation (5.22) by a_{jr}, sum over j and solve for ω_r^2 we obtain the following:

$$\omega_r^2 = \sum_{i,j}^n k_{ij} a_{ir} a_{jr} / \sum_{i,j}^n m_{ij} a_{ir} a_{jr} \tag{5.32}$$

When we read the above equation in conjunct ion with the equation (5.20) then it is noted that the denominator bf this equation is equal to unity. Moreover, the numerator is also equal to twice the potential energy which is in the positive definite form. This shows that ω_r^2 is a positive quantity.

5.8 PRINCIPAL COORDINATES:

A set of generalized coordinates which enables to decouple the equations of motion of the system is called a set of principal coordinates or normal coordinates. It means that each one of the generalized coordinates of this set represents a simple harmonic motion with a single natural frequency.

Now, we define a linear transformation between generalized coordinates q_I and principal coordinates p_i. It is

$$q_i = \sum_r^n a_{ir} p_i \qquad (r=1,2,....n) \tag{5.33}$$

where a_{ir} are the components of the modal vectors.

It is simple to show that an inverse transformation exists. To invert equation (5.33) multiply it by ($m_{ij}a_{js}$) and carry out summation over i and j. We get

$$\sum_{i,j}^{n} q_{ij}\, m_{ij}\, a_{js} = \sum_{r}^{n} P_r \sum_{i,j}^{n} m_{ij}\, a_{ir}\, a_{js} = \sum_{r}^{n} P_r \delta_{rs} = p_s$$

Thus it is clear that principal coordinates can be expressed as a linear function of the original generalized coordinates. It is given by

$$P_r = \sum_i^n q_i \sum_j^n m_{ij}\, a_{jr} \tag{5.34}$$

The use of the principal coordinates in mechanical vibration analysis is of great significance. With the help of these coordinates it is possible to describe a system as if it consists of a number of independent linear vibrators. The responses of these vibrators can be easily determined. Once the responses are known in terms of these principal Co ordinates then an application of an inverse transformation converts them back to generalized coordinates.

5.9 FORCED VIBRATIONS:

The equation (5.15) is transformed to principal coordinates using transformation equation (5.33). Then, we obtain

$$\sum_{i,r}^n m_{ij}\, a_{ir}\, \ddot{p}_r = \sum_r^n p_r \sum_i^n k_{ij}\, a_{ir} = Q_j, \tag{5.34}$$

Since

$$\omega_r^2 \sum_i^n m_{ij}\, a_{ir} = \sum_i^n k_{ij}\, a_{ir}$$

therefore, the equation (5.34) is rewritten as follows:

$$\sum_{i,r}^n m_{ij}\, a_{ir}(\,\ddot{p}_r + \omega_r^2 p_r) = Q_j \quad (5.35)$$

Multiply equation (5.35) by a and sum over j, we get

$$\sum_r^n(\,\ddot{p}_r + \omega_r^2 p_r)\, \sum_{i,j}^n m_{ij}\, a_{ir}\, a_{js} = \sum_j^n\, Q_j a_{js} \quad (5.36)$$

Since

$$\sum_{i,j}^n m_{ij}\, a_{ir}\, a_{js} = \delta_{rs}$$

therefore, equation (5.36) reduces to the following:

$$\ddot{p}_r + \omega_r^2 p_r = F_r \quad \text{(r=1,2,....n)} \quad (5.37)$$

Where

$$F_r = \sum_j^n Q_j\, a_{jr} \quad (5.38)$$

The solution of equation (5.37) is

$$p_r(t) = A_r \cos\omega_r t + B_r \sin\omega_r t \int_0^t F_r(t) \sin\omega_r(t-\tau)d\tau \quad \text{(r=1,2,....n)} \quad (5.39)$$

We write initial conditions in terms of principal coordinates. For this purpose equation (5.33) is used and we get the following

$$q_i(0) = \sum_r^n a_{ir} p_r(0) = \sum_r^n a_{ir} A_r \quad (5.40)$$

Equation (5.40) is multiplied by $m_{ij}a_{js}$ and the resulting equation is then summed over i and j, we get

$$\sum_{i,j}^n q_i(0)\, m_{ij}\, a_{js} = \sum_r^n A_r \sum_{i,j}^n m_{ij}\, a_{ir}\, a_{js} = \sum_r^n A_r \delta_{rs} = A_s$$

Therefore

$$A_r = \sum_j^n a_{jr} \sum_i^n m_{ij} q_i(0) \quad (5.41)$$

Similarly

$$\omega_r B_r = \sum_j^n a_{jr} \sum_i^n m_{ij} \dot{q}_i(0) \quad (5.42)$$

Equations (5.39) and (5.38) is the complete solution of the equation of forced vibrations in terms of principal coordinates. One can easily revert back to the original generalized coordinates using equations (5.33), (5.41) and (5.42). This solution is obtained using a_{ir} and ω_r from free vibration analysis.

5.10 RAYLEIGH'S PRINCIPLE

(Stationary property of natural frequencies)

We rewrite expression for ω_r^2 , given in equation (5.32) , as

$$\omega_r^2 = \frac{\sum_{i,j}^{n} k_{ij}\, a_{ir}\, a_{jr}}{\sum_{i,j}^{n} m_{ij}\, a_{ir}\, a_{jr}} \tag{5.43}$$

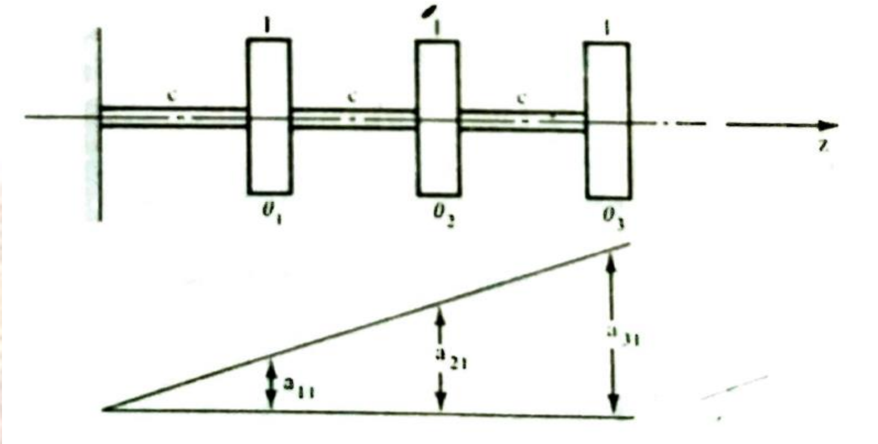
Upon differentiating the equation (5.43) and simplifying it, we get

$$\frac{\partial(\omega_r^2)}{\partial a_{ir}} = 2\sum_j^n k_{ji} a_{jr} - 2\omega_r^2 \sum_j^n m_{ji}\, a_{ir} = 0 \tag{5.44}$$

A statement of the Lagrange's Theorem is contained in equation (5.44). According to this theorem the square of the circular frequency expressed in terms of modal vector components is stationary w.r .t . the components of the modal vector .

EXAMPLE:

A torsional vibrations system is shown in fig. 5.4. Estimate its fundamental natural frequency of vibrations using Rayleigh's principle.



The rotation angles of the rotors are chosen as generalized coordinates. The kinetic energy and potential energy expressions are given by

$$T = \left(\frac{1}{2}\right) I\, \dot{\theta}_1^2 + \left(\frac{1}{2}\right) I\, \dot{\theta}_2^2 + \left(\frac{1}{2}\right) I\, \dot{\theta}_3^2$$

$$V = \left(\frac{1}{2}\right) k\, (\theta_3 - \theta_2)^2 + \left(\frac{1}{2}\right) k\, (\theta_2 - \theta_1)^2 + \left(\frac{1}{2}\right) I\, k\, \theta_1^2$$

where k = torsional spring constant of each shaft segment,

I = moment of inertia of each rotor w.r.t. shaft axis

Now, m_{ij} =0, for i not equal to j

=1, for i=j

$k_{13}= k_{31}=0$; $k_{33}=k$; $k_{22}=k_{11}=2k$;

$k_{12}= k_{21}=-k$; $k_{23}=k_{32}=-k$,

Using equations (5.43) and (5.20)

$$\omega_r^2 = \sum_{i,j}^{3} k_{ij}\, a_{ir}\, a_{jr} \qquad (5.45)$$

Once the mode shape is guessed then equation (5.45) is used to obtain the fundamental frequency of vibrations. It is easiest to guess a shape to be linear. For this guess let us assume the following:

$$a_{21}/a_{11} = 2 \text{ and } a_{31}/a_{11} = 3 \tag{5.46}$$

Since

$$\sum_{i,j}^{n} m_{ij}\, a_{ir}\, a_{jr} = 1$$

therefore, for this case

$$I\, a_{11}^2 + I\, a_{21}^2 + I\, a_{31}^2 = 1 \tag{5.47}$$

or, $1 + (a_{21}/a_{11})^2 + (a_{31}/a_{11})^2 = I\,/(a_{11}^2)$

Assumed values of (a_{21}/a_{11}) and (a_{31}/a_{11}) are put into this equation. Following is obtained:

$$I\, a_{11}^2 = 1/14$$

Substitute eq. (5.46) into eq. (5.47), divide by k a_{11}^2 and get

$$\frac{\omega_1^2}{k\, a_{11}^2} = 2 + 2(a_{21}/a_{11})^2 + (a_{31}/a_{11})^2 - 2(a_{21}/a_{11}) - 2(a_{21}/a_{11})(a_{31}/a_{11}) \tag{5.48}$$

Upon simplification and substitution of numerical values, we obtain

$\omega_1^2 = \left(\frac{3}{14}\right) K/I$ or $\omega_1 = 0.462\sqrt{K/I}$

Compare the above result with the exact value of

$$\omega_1 = 0.445\sqrt{K/I}$$

It is seen that the calculated value is 3.82% higher than the exact value. It may be noted that though the assumed modal vector components were crude approximations yet the solution given by Rayleigh's principle turns out to be a very good approximation.

5.11. SPECIAL FEATURES OF MULTI-DEGREE-OF-FREEDOM SYSTEMS

There are special characteristics which are unique to multi degree of freedom systems and not available with single degree of freedom systems. To understand these features we consider the case of a two degree of freedom solved in example II under the application of Hamilton's principle. These equations are reproduced below with the difference that forces F_1 and F_2 are taken as $F_1 sin\omega t$ and F2 $sin\omega t$,

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = F_1 sin\omega t) \qquad (5.49)$$

$$m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = F_2 sin\omega t) \qquad (5.50)$$

We seek steady-state solution of this set of equations. For this to obtain, we select particular integrals to have the form given below:

$$x_1 = X_1 sin\omega t \text{ and } x_2 = X_2 sin\omega t \qquad (5.51)$$

Inserting these solutions into differential equations (5.49) and (5.50), we obtain

$$(k_1 + k_2 - m_1\omega^2)X_1 - k_2x_2 = F_1 \qquad (5.52)$$

$$-k_2x_1 + (k_2 - m_2\omega^2)X_2 = F_2 \qquad (5.53)$$

Solving above equations for X_1 and X_2, we get

$$X_1 = \frac{\begin{vmatrix} F_1 & -k_2 \\ F_2 & (k_2 - m_2\omega^2) \end{vmatrix}}{\Delta(\omega)} \qquad (5.54a)$$

$$X_2 = \frac{\begin{vmatrix} k_1 + k_2 - m_1\omega^2 & F_1 \\ -k_2 & F_2 \end{vmatrix}}{\Delta(\omega)} \qquad (5.54b)$$

where

$$\Delta(\omega) = \begin{vmatrix} k_1 + k_2 - m_1\omega^2 & -k_{2_1} \\ -k_2 & k_2 - m_2\omega^2 \end{vmatrix} \qquad (5.54c)$$

Following significant inferences can be deuced from the above equations:

I. It is obvious from the equations (5.54a) and (5.54b) that the amplitudes of vibrations of the masses tend to infinity when the denominators of these equations are zero, i.e. the resonance sets in. The denominators of these equations, when equated to zero, give the characteristic equation of the system. This means that the resonance, in a conservative system, can occur when the excitation frequency happens to coincide with one of the natural frequencies of vibration of the system. Since the roots of the characteristic equation are two, thus there are two finite natural frequencies and so system has two resonances.

II. It is also possible that both the denominator and numerator of one of the expressions in equations (5.54a) and (5.54b) simultaneously vanish. The numerator of the other expression must then of necessity also be zero. In this case the resonance does not occur at a natural frequency though the denominator is zero. Let us examine this situation. For this system we can identify two natural frequencies which are ω_{n1} and ω_{n2} Denominators of equations (5.54a) & (5.54b), when set to zero, will give the following relations:

1. at $\omega = \omega_{n1}$;

$$\frac{F_1}{F_2} = \frac{-k_2}{k_2 - m_2\omega_{n1}^2} = \frac{k_1 + k_2 - m_2\omega_{n1}^2}{-k_2} \tag{5.55}$$

$$\frac{F_1}{F_2} = \frac{-k_2}{k_2 - m_2\omega_{n2}^2} = \frac{k_1 + k_2 - m_2\omega_{n2}^2}{-k_2} \tag{5.56}$$

Let the amplitudes of vibrations of the two masses corresponding to the two natural frequencies be designated by X_{ij} (i , j =1,2), where i refers to the mass number (m_1, m_2) and j to the frequency number(ω_{n1} & ω_{n2}).

Now, consider free vibrations of this system. The equations of mot ion, for this case, are

$$k_1 + k_2 - m_1\omega^2 + k_2 x_2 = 0 \tag{5.57}$$

$$-k_2 x_1 + (k_2 - m_2\omega^2)x_2 = 0 \tag{5.58}$$

Using equations (5.57) & (5.58) and (5.59) & (5.60), we get the following ratios :

$$\frac{x_{11}}{x_{21}} = \frac{-k_2}{k_1 + k_2 - m_2\omega_{n1}^2} \tag{5.59}$$

$$\frac{x_{22}}{x_{12}} = \frac{-k_2}{k_2 - m_2\omega_{n2}^2} \tag{5.60}$$

Using equations (5.55) & (5. 56) and (5.59) & (5.60), we get

$$F_1 X_{11} + F_2 X_{21} = 0 \tag{5.61}$$

$$F_1 X_{12} + F_2 X_{22} = 0 \tag{5.62}$$

Let us consider following two vector pairs:

(F_1 . F_2) and (X_{11} . X_{21}) as one set ;

(F_1 . F_2) and (X_{12} . X_{22}) as second set.

The equations (5.61) & (5.62) can be treated as orthogonality conditions. These conditions indicate that the works done by exciting forces on the displacements of free vibrations at one of the natural frequencies is zero. It means to say that the system energy remains unchanged even though the external excitation forces are present. Thus the vibration amplitudes are limited and no resonance sets in.

III. Vibration absorption is, perhaps, the most unusual features of multi-degree of freedom systems. To understand this feature, let us consider again the two degree of freedom system whose solution is given by equations (5.54a) to (5. 54c).

Expanding the expressions contained in equations (5.54a) and (5.54b), we get the following:

$$X_1 = \frac{F_1(k_2 - m_2\omega^2) + F_2K_2}{\Delta(\omega)} \tag{5.63}$$

$$X_2 = \frac{F_2(k_1 + k_2 - m_1\omega^2) + F_1K_2}{\Delta(\omega)} \tag{5.64}$$

$$\text{where } \Delta(\omega) = (k_1 + k_2 - m_1\omega^2)\ (k_2 - m_2\omega^2)\text{-}k_2^2 \tag{5.65}$$

Let us assume that $F_1 \neq 0$ and $F_2=0$ in equations (5.64) & (5.65). Also consider that the following holds good:

$$k_2 - m_2\omega^2 = 0 \tag{5.66}$$

It can be readily seen that $F_1=0$ and $F_2=- F_1/k_2$.

This situation reveals that the mass m_1 remains stationary, even though it is the one to which the external excitation is app1ied. It is called the vibration absorption of the concerned mass. This curious result is extensively used in the design of dynamic vibration absorbers.

To illustrate the application of this concept, let us take a system (fig. 5.5a) subjected to an external excitation P sinωt of which vibrations are to be suppressed. For the suppression of the vibrations, it is sufficient to attach to it an additional mass mounted elastically (fig. 5.5b) as per the condition given by equation (5.66). Then the vibrations of the main mass will vanish. The attachment of an additional mass to the main mass changes the number degrees of freedom of the system and accordingly the number of natural frequencies of vibrations. Fig. 5.5a shows system of one degree of freedom. It has one natural frequency of vibration given by

$$P_0 = \sqrt{(k_1/m_1)} \tag{5.67}$$

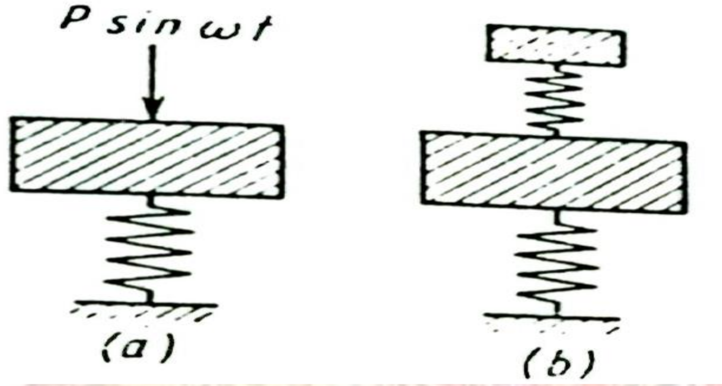


Fig. 5.5

Attaching an additional mass (absorber mass) to it changes its number of degrees of freedom from one to two. Now, it has two natural frequencies. The characteristic equation for this new system is

$$\omega^4 = [(k_1 + k_2)/m_1 + k_2/m_2]\omega^2 + (k_1 k_2/m_1 m_2) = 0 \tag{5.68}$$

Let us take the tuning frequency to be

$$P_{t0} = (k_2/m_2)^{1/2} \tag{5.69}$$

Using the equation (5.69) into the frequency equation (5.68), we get

$$(\omega/P_{t0})^4 - (\omega/P_{t0})^2 - [(P_0/P_{t0})^2 + 1 + \mu] + (P_0/P_{t0})^2 = 0 \tag{5.70}$$

where $\mu = m_2/m_1$

For a tuned dynamic vibration absorber, P_0/P_{t0}. Let us take

$[\omega^2/(k_1/m_1) + \omega^2/(k_2/m_2)] = r^2$. Then the above equation becomes

$$r^4 - r^2(2 + \mu) = 0 \tag{5.71}$$

Resonance frequencies of the tuned system are determined from the roots of this equation with as a parameter This equation is shown plotted in fig. 5.6.

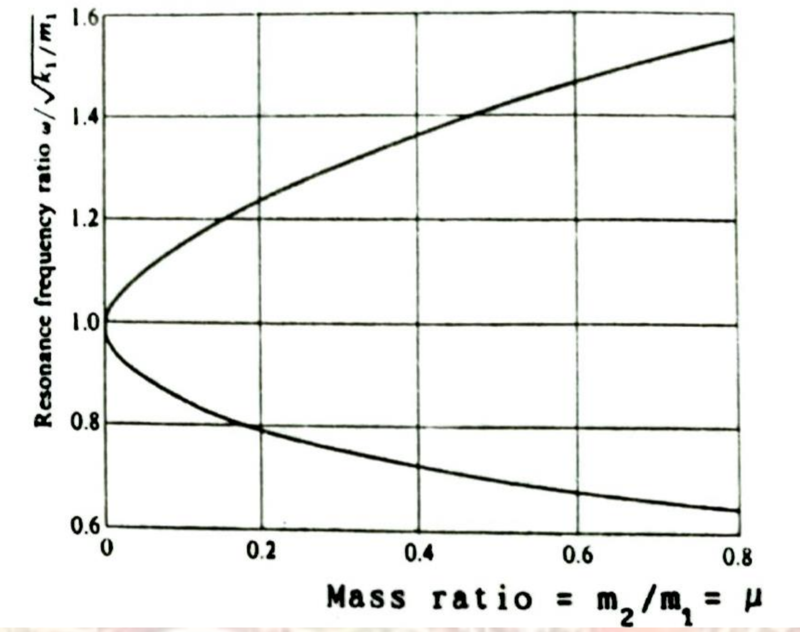


Fig. 5.6

The installation of a dynamic absorber may be recommended only in cases where the external excitation has a stable frequency. The performance of absorber may be improved by the incorporation of damping as shown in fig. 5.5c. It is not recommended to install a vibration absorber for the foundation of piston compressors and internal combust ion engines.

The principle of absorbers can be applied to shafts undergoing torsional vibrations. A small additional disk can act as absorber when tuned with the main system. This absorber will not be suitable for shafts of internal combust ion engines. For this pendulum absorbers are used. For details, see Timoshenko[5].

5.12 DAMPING

In an ideal conservative system there is a continuous exchange between the potential energy and the kinetic energy with the total energy of the system held at a constant level . It means that the system executes vibrations without a decrease in the amplitude. This is never realized. The reason is the inevitable presence of energy dissipation mechanism, An energy dissipation mechanism is called damping. Damped systems are non-conservative. There are many types of damping mechanisms. Exact mechanisms of damping phenomena are not known. Some of the important types of dampings are out lined below:

Types of Damping

Damping forces present in a vibrating system may originate from different sources. These damping forces depend upon the vibrating system as well as on elements exterior to it. Formulation of expressions for damping forces is a difficult task. Different approximations are made in order to simulate different damping situations. As a matter of fact it is an area of ongoing research.

Damping forces have been classified into different categories. However, the most important situations are covered under the following heads:

1. Viscous damping
2. Structural damping
3. Coulomb damping

Most mechanical systems exhibit damping characteristics a rising out of a combination of damping mechanisms mentioned above

VISCOUS DAMPING: This type of damping arises when a system shock Vibrates in a fluid medium (viz. air, oil, etc.), e.g. shock absorbers, hydraulic dashpots and sliding of a body on a lubricated surface. In the case of a multi-degree of freedom System, the jth damping force is expressed as

$$F_{Dj} = c_j u_j$$

in which the constant c_j characterizes the jth damping mechanism.

STRUCTURAL DAMPING: This type of damping arises in that case in which there is internal friction within the material itself. It also arises at connections between elements of structural type system. This damping force may be a function of strain, deflection. In the case of an elastic system the jth structural damping force (F_{Dj}) is proportional in magnitude to the internal elastic force (F_E) and opposite in direction to the velocity vector (u_j). The damping force is

$$F_{Dj} = ig\, F_{Ej}$$

where g is a constant and i is the unit imaginary number. Structural damping force and the internal elastic forces expressed in constrained coordinates (u) are transformed in terms of generalized coordinates (q) through a coordinate transformation

$$u_j = u_j\,(q_1, q_2, q_3, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ q_n) \qquad (j = 1,2,\ldots.m)$$

Once the damping forces are expressed in terms of generalized coordinates then Lagranges equations can be used to write the equations of motion.

COULOMB DAMPING:

It is also called dry friction. It occurs whenever two dry surfaces are in relative sliding motion. The resulting damping force is taken to be nearly constant. This damping force is dependent on the normal force N between the moving bodies and the coefficient of friction between them. It is given by

$$F_D = \mu N$$

Difficulties are experienced in mathematical representations of various types of damping mechanisms. In many cases it is found convenient to approximate the system damping matrix by an equivalent viscous damping matrix . For this the energy dissipation per cycle is taken to be the same in both the cases (i.e. actual damping and the equivalent damping).

CHAPTER VI

VIBRATIONS OF NON-LINEAR SYSTEMS

6.1 INTRODUCTION

Modeling of dynamic systems using linear elements is very popular amongst vibration engineers. It is primarily so because of the availability of a fully developed apparatus of linear mathematical analysis and non-availability of unified general solutions of non linear cases. Most of the mechanical systems are basically non linear in character and modeling these as linear systems is not free from risk. As a matter of fact, in many cases, the elements employed are at best linear to a first degree of approximation only and some cannot be approximated by ideal linear elements without an essential alteration in character. The presence or absence of nonlinearities in a system model can lead to important changes in the predict ion of dynamic behaviour, e.g. an unstable nonlinear system may be analyzed as stable using linear elements ; a nonlinear system having several equilibrium positions will be shown to have only one such equilibrium position, etc. The adverse effects of such conclusions cannot be overemphasized. It is not an easy task to assess the effects of neglecting nonlinearities without extensive experience. Now, the last few decades have seen a rapid growth in the knowledge of nonlinear systems, the theory of nonlinear mathematical analysis and ease of the availability of powerful computer facilities. This has led to a spurt in the area of nonlinear dynamic analysis. We shall discuss, in this chapter, basic concepts of nonlinear mechanical vibrations analysis with the objective for direct use in applications. Following must be noted at the outset:

Distinct Attributes of Non linear Systems

1. Superposition principle is not valid for nonlinear systems. (linear systems obey this principle)
2. A nonlinear system can have more than one equilibrium position. (linear systems have only one equilibrium position)

Some Examples of Nonlinear Systems

Vibrations of bridges and buildings, aircraft and missile structures, engines and machinery, earthquakes, oscillatory flow of fluids in pipelines etc. are primarily nonlinear

Causes of existence of Nonlinearities in Systems

Some of the causes of the presence of nonlinearities in a given system are listed below:

1. Geometrical nonlinearities: Nonlinearities due to large displacements .
2. Physical nonlinearities: Nonlinearities due to physical properties of materials.
3. Energy dissipation mechanisms.
4. Geometrical imperfections.
5. Nonstationary characteristics of the system parameters or
6. External excitations.
7. Dynamic nonconservative forces.
8. Gyroscopic forces.

General form of Differential Equation of Motion Nonliner System

$$\ddot{x} + f(\dot{x}, x, t) = 0 \qquad (6.\,1)$$

6.2. SPECIAL CONCEPTS RELATED TO NONLINEAR SYSTEMS

Phase Portrait:

Investigation of various types of equilibria and the identification of singular points of differential systems with positions of equilibrium are of great importance in nonlinear vibration analysis. These aspects are best studied through integral curves drawn on a phase space. The integral curves are 3-dimensional and depict well defined patterns called phase patterns or phase portraits. A phase pattern or phase portrait is a graphic representation of the movement of a representative point on a phase space. These integral curves drawn on a 2-dimensional plane are known as phase curves and the corresponding plane is the phase plane

In order to understand the meaning of integral curves, let us consider an autonomous system consisting of two first order equations of the form

$$dx/dt = P(x,y) \text{ and } dy/dt = Q(x,y)$$

In this case it is possible to eliminate dt between the two equations and therefore, we write

$$dy/dx = Q(x,y) / P(x,y), \qquad P(x,y) \neq 0$$

This equation is that of integral curves. The solution of this equation defines integral curves of the system.

As an illustration of this aspect, consider a linear single degree of freedom oscillator for which the equation of motion is

$$m\ddot{x} + c\dot{x} + k\,x = 0 \tag{6.2}$$

We define a new variable given below

$$y = \dot{x} \tag{6.3}$$

Using the above variable, equation (6.2) is represented as a set of following two equations

$$y = \dot{x} \tag{6.4a}$$

$$\dot{y} = -\left(\frac{b}{m}\right)y - \left(\frac{k}{m}\right)x \tag{6.4b}$$

Equation (6.4a & 6.4b) is an equivalent set of equation (6.2). To obtain the phase portrait, we divide equation (6.4b) by equation (6.4a). Upon carrying out this process, we get

$$\frac{dy}{dx} = -(2hy + \omega_n^2 x/y) \tag{6.5}$$

$$\frac{dy}{dx} = -(2hy + \omega_n^2 x/y) \tag{6.5}$$

where 2h = b/m and ω_n^2=k/m

Equation (6.5) is the differential equation of the integral curves. This equation gives a set of geometrical curves without any reference to what happens in time. It defines a vector field on the phase plane (x,y). The origin is a singular point.

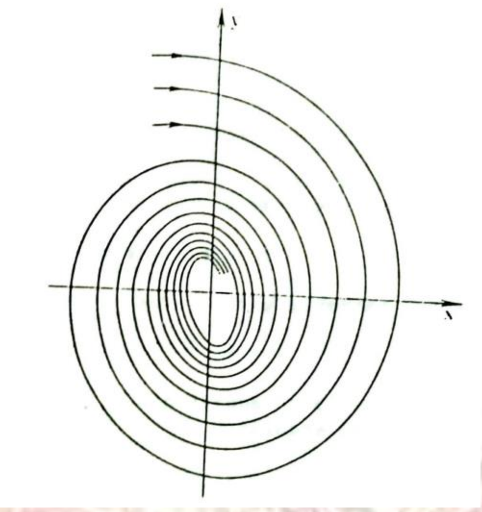


Fig.6. 1: Integral curves on the phase plane

This vector field is studied through isoclines which are geometrical loci of points at which the tangents to al l the Integral curves have the same slope.

To understand the concept of isoclines, let us consider the case of an oscillator defined by the equation of an isocline with slope has the form

$$dy/dx = \theta$$

$$\text{or } y = -\frac{\omega_n^2}{\theta + 2h} x = -\alpha x \qquad (6.6)$$

Equation (6.6) represents a set of straight lines which pass through the origin. This origin is a singular point.

Solution of equation (6.5) has the following form

$$y^2 + 2hxy + \omega^2 x^2 = c \exp[2\left(\frac{h}{\omega}\right) \arctan(y + hx)/\omega x] \qquad (6.5a)$$

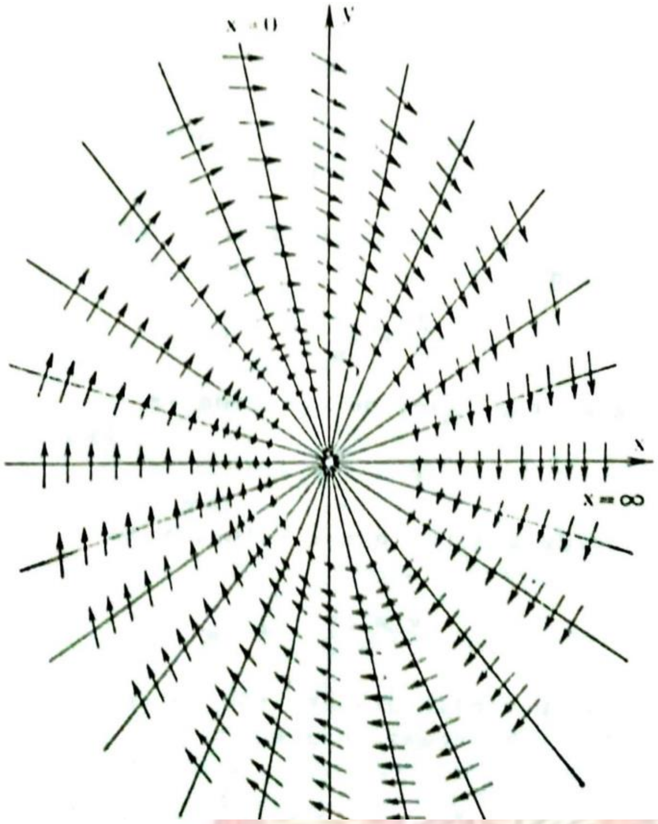


Fig. 6.2 : Vector field of isoclines

Above solution is obtained for constant to be determined from initial conditions. This solution is represented by a family of logarithmic spirals shown in fig. 6.1. The vector field of isoclines is shown in fig. 6.2.

Following equation is used to find out the phase velocity:

$$v = i\,\dot{x} + j\,\dot{y} \tag{6.7}$$

where i and j are unit vectors and v is the phase velocity. For the present case under study, equation (6.7) would become

$$v = i\,y + j(2hy - \omega_n^2 x) \tag{6.8}$$

and

$$|v|^2 = \dot{x}^2 + \dot{y}^2 = \omega_n^4 x^2 + 4h\omega_n^2 xy + (1 + 4h^2)y^2$$

The phase velocity decreases upon approach to the origin of coordinates and vanishes there.

For an undamped harmonic oscillator h=0 and the integral curves represent a family of ellipses given by

$$y^2+\omega_n^2 x^2 = const \tag{6.9}$$

For this case, the isocline equation is

$$y = -\omega_n^2\, x/\theta \tag{6.10}$$

where θ=dy/dx and the phase velocity is

$$v = i\, y + j\omega_n^2 x \tag{6.11}$$

For a damped aperiodic process for which $h^2 > \omega^2$, the roots of the characteristic equation are

$$\lambda_{1,2} = -h \pm (h^2 - \omega_n^2)^{1/2} \tag{6.12}$$

The phase pattern of this system is shown in the fig. 6.3.

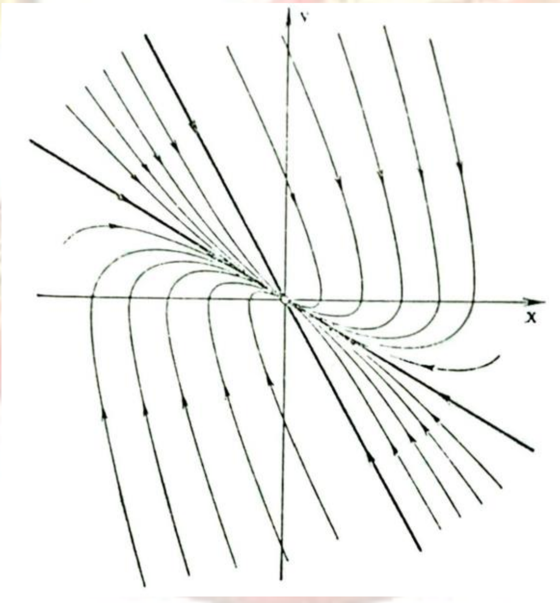


Fig. 6.3

6.3 SINGULAR POINTS

Consider an autonomous system given by

dx/dt =P(x,y) and dy/dt=Q(x,y)

A point (x_0, y_0) for which P(x_0, y_0)=Q(x_0, y_0)=0 is a singular point, otherwise it is an ordinary point.

A "flow" analogy can be advanced to give a physical meaning to singular point and phase plane. The phase plane may be considered as a "flow" regime. According to the theorem of Liouville the phase space volume is invariant. The phase space does change during motion, but not in volume. This Situation is like that of a real incompressible fluid. The fluid volume is defined by the velocity vector field and the direction of motion at each point is given by this velocity vector. Cauchey-Lipschitz theorem applied to autonomous systems regarding the integral curves leads to this : through ever v point of the phase plane there passes one and only one integral curve. Thus the integral curves are tangent at every point to the velocity vector. A singular point may be considered to be a stationary point of the flow and the integral curves passing through it consist of the point itself.

For the case of a second order differential equation of a vibrating system, we obtain the autonomous system (6.12) by the substitution dx/dt=y. In this case P(x, y) = y, so that $y_0 = 0$. Since x is position and y is velocity, a solution

$$x(t) \equiv x_0 \ ; \ y(t) \equiv 0 \ \text{ at } (x_0, y_0)$$

tells that the singular point is a position of equilibrium. Thus we are led to the conclusion that, for dynamic systems, the singular points identify the posit ions of equilibrium.

6.31 TYPES OF SINGULAR POINTS

For a harmonic oscillator without friction all the phase curves are closed ellipses. They surround a singular point called CENTER. For a damped oscillator the singular point is an asymptotic point of all the curves that have the form of spirals inserted one into another. This singular point is called FOCUS. For the case of damping with aperiodic motion the integral curves pass through a singular point called NODE. Different kinds of singular points are discussed below:

6.4. GENERAL INVESTIGATION OF SINGULAR POINTS OF A SECOND ORDER SYSTEM

Consider a general second order system given by the set of two first order equations:

$$\dot{x} = a_1 x + a_2 y + X(x, y)$$

$$\dot{y} = b_1 x + b_2 y + Y(x, y) \qquad (6.13)$$

where X(x,y) and Y(x,y) are polynomials which contain terms of order higher than the first in x and y.

Now, we consider only the neighbourhood of the origin and restrict the treatment to a linear approximation. In this situation X(x,y)≅0 and Y(x,y)≅0. Then equation (6.13) is

$$\dot{x} = a_1 x + a_2 y$$

$$\dot{y} = b_1 x + b_2 y \qquad (6.14)$$

The differential equation of the integral curves of this set is

$$\frac{dy}{dx} = \frac{b_1 x + b_2 y}{a_1 x + a_2 y} \qquad (6.15)$$

Now. we seek a solution to equation (6.14) in the form given below:

$$x = A \exp(\lambda t) \text{ and } y = B \exp(\lambda t)$$

Upon substitution of the above solution into equation (6.14), two simultaneous algebraic equations are obtained. For a non-trivial solution of these equations it is necessary that

$$\begin{vmatrix} a_1 - \lambda & a_2 \\ b_1 & b_2 - \lambda \end{vmatrix} = 0$$

or

$$\lambda^2 - (a_1 + b_2)\lambda + a_1 b_2 - a_2 b_1 = 0 \qquad (6.16)$$

Equation(6. 16) is a quadratic equation which has two roots, viz. λ_1 and λ_2. Complete solution of the equation is

$$x = A_1 \exp(\lambda_1 t) + A_2 \exp(\lambda_2 t)$$

$$y = B_1 \exp(\lambda_1 t) + B_2 \exp(\lambda_2 t) \qquad (6.17)$$

The values of the constants A_1, A_2, B_1, B_2 are determined by initial conditions.

Equation (6.17) is used for the classification of singular points. We give below the classification of singular points according to Poincare which is based on the behaviour of the integral curves in their neighbourhood .

1. The discriminant (D) of the equation (6.16) is

$$D = -(a_1 - b_2)^2 - 4a_2 b_1$$

If D≤0, then both the roots are real.

a) If $a_1b_2 - a_2b_1 = 0$, signs of the roots are identical following cases arise

1. $\lambda_1, \lambda_2 < 0$. The solution represents exponentials which decrease with time it means that the system once disturbed from this point returns to it. This singular point is a stable node (curve 1, fig. 6.4).
2. $\lambda_1, \lambda_2 > 0$. The solution represents exponentials that increase with time. It means that the system recedes from the singular point. The singular point is a unstable node (curve 2, fig. 6.4).

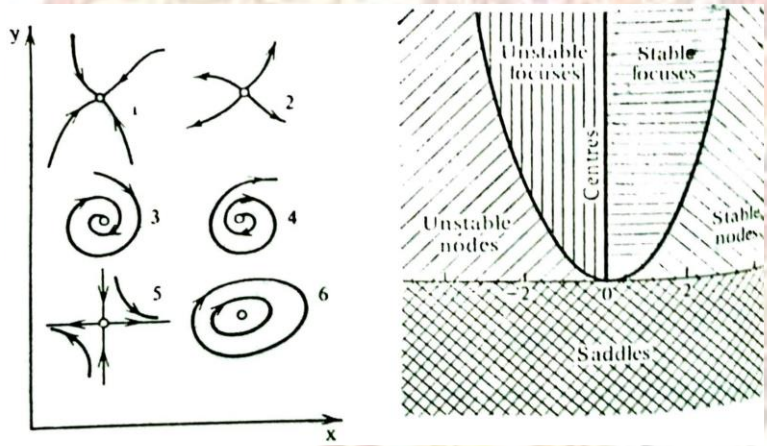


Fig. 6.4 : Types of singular points

b) If the discriminant D≤0 and $a_1b_2 - a_2b_1 < 0$, then the

the roots λ_1, λ_2 have opposite signs, the singular point is unstable and is called point (curve 5, fig. 6.4). Only two integral curves pass through it. These curves are called separatrices. Remaining other trajectories go to infinity bypassing the singular point.

II. If the discriminant D>0, then the roots are complex conjugate. Following cases are obtained:

1. Real parts of the roots are negative, i.e. $a_1 + b_2 < 0$. Damped oscillations occur in the system; the singular point about which the spiral phase trajectories wind up is a stable focus (curve 3, fig. 6.4).
2. Real parts of the roots are positive, i.e. $a_1 + b_2 > 0$. The singular point is unstable focus (curve 4, fig. 6.4) from which the spiral phase trajectories unwind. This represents a case in which the amplitudes of vibrations increase with time.

3. The roots $\lambda_1 = -\lambda_2$ are imaginary (i.e. $a_1+b_2=0$). The oscillations take place and the singular point is a center (curve 6, fig. 6.4). The phase plane trajectories are concentric ellipses.

6.5 SUBHARMONICS AND ULTRAHARMONICS

Nonlinear systems possess a wide variety of periodic motions besides those which have the same period as the external period.

Consider the differential equation

$$\ddot{x} + \omega_n^2 x = p\cos\omega t \qquad (6.18)$$

Since we are concerned with periodic mot ions, we may take the time t = 0 such that $\ddot{x}(0)$=A, the solution of equat ion (6. 18) is

$$x(t) = B\sin\omega t - B_1\cos\omega t$$

where $B = A + P/(\omega^2 - \omega_n^2)]$ and $B_1 = A + P/(\omega_n^2 - \omega^2)]$

It is only in the following cases that x(t) is periodic:

1. B=0, free vibrations do not occur.
2. $\omega = n\omega_n$, n is any integer except n=1. B_z is not zero. This condition defines subharmonic vibrations of order n.
3. $\omega = \omega_n/m$, m is any integer except m=1. B is not zero. Ultraharmonic vibrations are defined by this condition.
4. $\omega = n\omega_n/m$, n, m are relatively prime numbers. B is not zero.

Ultrasubharmonic vibrations are defined by this condition.

Different types of responses mentioned above are given in the fig. 6.5.

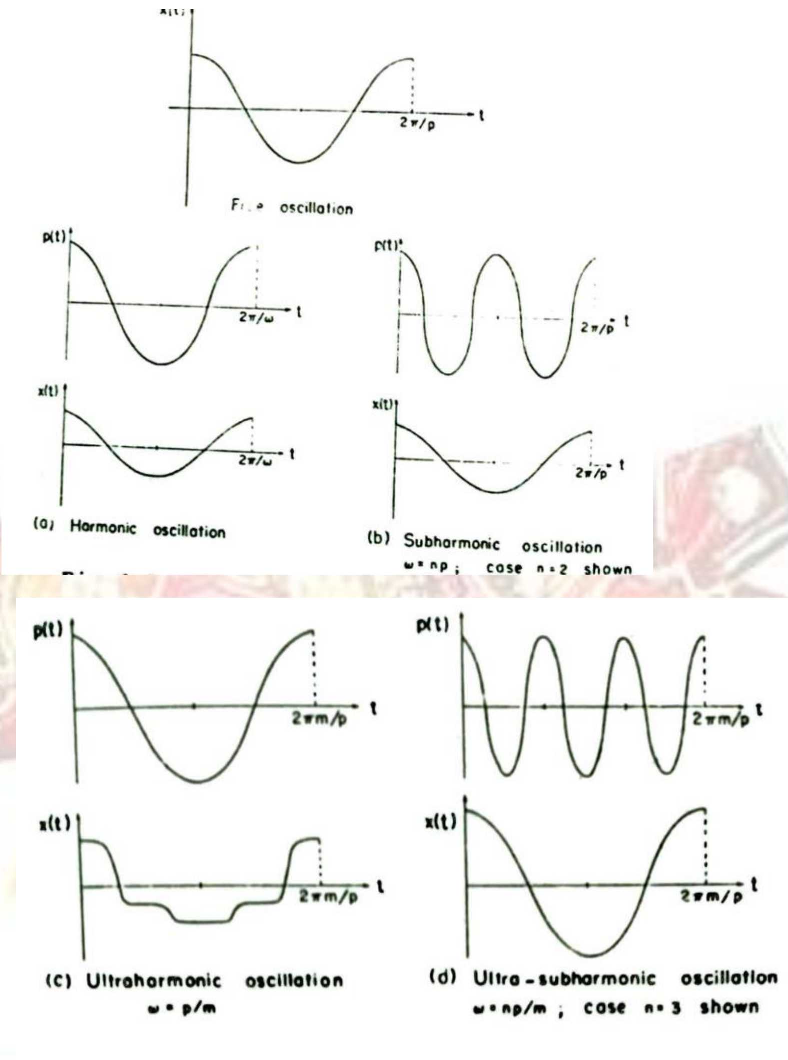


Fig. 6. 5: Different types of responses

6.6. GEOMETRICAL DISCUSSIONS OF THE INTEGRAL CURVES IN THE PHASE PLANE:

Integral curves are also called energy curves. Closed integral curves correspond to periodic motions. i.e. if an integral curve is closed it means that the displacement and velocity at any time reach the same values again after a periodic time T. This is stated at x(t+T)=x(t) & v(t+T)=v(t) where v(t) is the velocity at a point on the integral curve. The period T can be calculated by the line integral

$$T = \phi\ (dx/v) \tag{6.20}$$

Integral is to be taken along a closed integral curve in the direction of increasing t. Calculation of periodic time T is shown below for the case of a S.D. F. spring—mass system. The equation of motion for this case is

$$\ddot{x} + f(x) = 0$$

For representation on the (x,y) plane, this equation is rewritten as

$$v\frac{dv}{dx} + f(x) = 0 \qquad (6.21)$$

We consider following cases :

1 . Linear case : f(x) = kx

Integrate equation (6.21) with f(x) as given above. We get

$$v^2 + k\,x^2 = v_0^2 + k\,x_0^2 \qquad (6.22)$$

where x_0 and v_0 are the initial displacement and velocity.

It is obvious that all the integral curves will be ellipses for k>0 and hence every mot ion will be periodic. For this case, we know, from chapter IV that the motion is a SHM and it is given by

$$x = a\cos\omega_n t \qquad (6.23)$$

where $\omega_n^2 = K$.

The amplitude of vibrations (a) is evidently given by

$$a = (v_0^2 + k\,x_0^2)^{1/2}/k \qquad (6.24)$$

The period T of the motion is

$$T = \oint\left(\frac{dx}{v}\right) = 4\int_0^a [1/(a^2 - x^2)^{\frac{1}{2}}]dx = 2\pi/\omega_n \quad (6.25)$$

This expression for T tells that the period of vibration is independent of the amplitude. It gives a very important conclusion that the closed integral curves in the phase plane are in same time.

If k<0 then the integral curves would be hyperbolas and no periodic motion would exist .

II. Nonlinear case: $f(x) = \alpha x + \beta x^3$, α>0,

For this case, the equation of motion in (x, v) plane is

$$v\frac{dv}{dx} = -(\alpha x + \beta x^3) \qquad (6.26)$$

Integration of this equation yields

$$v^2 + \alpha x^2 + \frac{\beta x^4}{4} = h = constant \qquad (6.27)$$

constant h in the above equation represents twice the total energy in the system. In the neighbourhood of x=0, v=0 the curves given by equation (6.27) are ellipse like. It is readily seen that the maximum displacement (a) satisfies

$$a = (-\alpha + (\alpha^2 + 2.\beta h)^{\frac{1}{2}})/\beta \qquad (6.28)$$

since equation (6.27) shows symmetrical phase curves, the period T of motion can be obtained in the form

$$T = 4\int_0^a \frac{dx}{\left((h-(\alpha x^2+\frac{\beta x^4}{2}\right)^{1/2}} \qquad (6.29)$$

The integral in equation (6.29) is transformed into another form by the substitution x =a sin θ. The resulting form is

$$T = 4(2)^{1/2}\int_0^{\pi/2} \frac{d\theta}{((-2\alpha+\beta a^2+\beta a^2 sin^2\theta)^{1/2}} \qquad (6.30)$$

Equation (6.30) shows that unlike linear case, the periodic tine is dependent on the amplitude of vibrations. We have, following cases:

1. β=0, we get linear case and the period is independent of amplitude.
2. β>0, the period is seen to decrease (or frequency is seen to increase) with amplitude. This case is called the case of hard spring.
3. β<0, the period is seen to increase (or frequency is seen decrease) with amplitude. This case is called the case of soft spring

It is to be observed that the phase curves for the case of a hard spring (β=0) are always closed, i.e. the mot ion is periodic. However, for the case of a soft spring

where β<0, the curves are closed in a certain region only Fig. 6.6(a) shows the relationships between amplitude frequency in the three cases of linear springs, and soft springs. Phase trajectories for hard springs and soft springs are drawn in fig. 6.6(b).

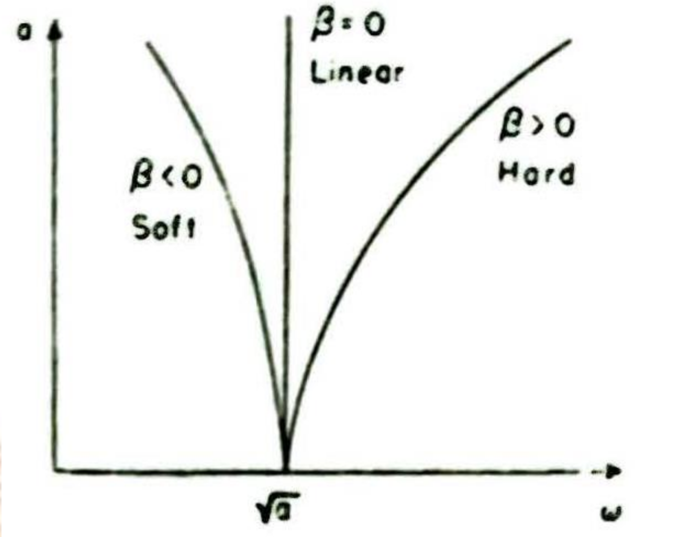


Fig . 6.6(a)

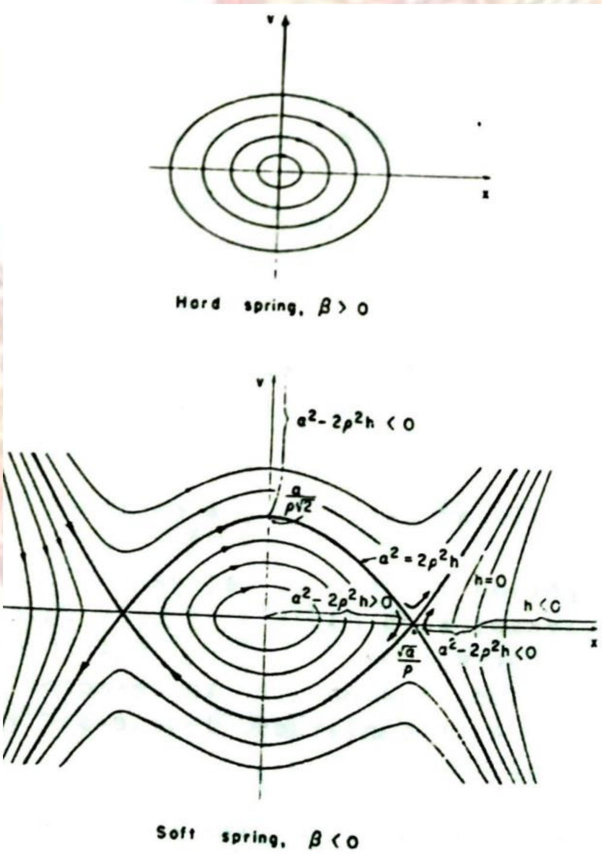


Fig.6.6(b)

6.7. FORCED VIBRATIONS OF SYSTEMS WITH NONLINEAR RESTORING FORCES:

Consider the differential equation

$$\ddot{x} + c\dot{x} + f(x) = p\cos\omega t \qquad (6.31)$$

in which f(x) is non linear. This equation is found to represent many physical situations.

If we take f(x) = $\alpha x \pm \beta x^3$ the resulting equation is called Duffing equation. We investigate periodic motions of this equation for the case when damping is zero. Motions are sought using the method of iteration which is essentiality a process of successive approximations. Under this scheme a solution is assumed and it is substituted into the differential equation, which is integrated to obtain a solution of improved accuracy. This process is repeated a number of times till the desired accuracy is obtained.

Let the first assumed solution be

$$x_0 = A\cos\omega t \qquad (6.32)$$

Substitute this solution into the differential equation

$$\ddot{x} = -\omega_n^2 A\cos\omega t \mp \beta A^3(\frac{3}{4}\cos\omega t + \frac{1}{4}\cos 3\omega t) + P\ \cos\omega t$$

$$= (-\omega_n^2 A \mp \frac{3}{4}\beta A^3 + P)\cos\omega t) \mp \frac{1}{4}\beta A^3\ \cos 3\omega t$$

We integrate this equation and set constants of integration to zero for the solution to be harmonic with period $\tau = 2\pi/\omega$. Thus, an improved solution so obtained is

$$x_1 = \left(\frac{1}{\omega^2}\right)(\omega_n^2 A \pm \frac{3}{4}\beta A^3 - P)\ \cos\omega t \mp \cdots \dots .. \qquad (6.33)$$

where the higher harmonic term is ignored.

As per the procedure of the method if the first and second approximations are reasonable solutions to the problem. then the coefficients of cos ωy the two equations (6.32) and (6.33) must not differ greatly. For this case, we equate these coefficients on the assumption that the two approximations are the reasonable solutions. We obtain

$$A = \left(\frac{1}{\omega^2}\right)(\omega_n^2\, A \pm \tfrac{3}{4}\beta A^3 - P) \qquad (6.34)$$

Equation (6.34) is solved for ω^2 and we get

$$\omega^2 = \omega_n^2 \pm \tfrac{3}{4}\beta A^2 - P/A \qquad (6.35)$$

It is seen from the equation (6.35) that in the event of nonlinear parameter (β) being zero, we get the exact result for the linear case, i.e.

$$A = P/(\omega_n^2 - \omega^2)$$

For getting frequency of free vibrations of the nonlinear case, we equate P to zero. This frequency is

$$\omega^2 = \omega_n^2(1 \pm \tfrac{3}{4}\beta(A^2/\omega_n^2)) \qquad (6.36)$$

We now plot $|A|$ vs (ω/ω_n). We get the curves shown in fig. 6.7(a) for a softening spring and fig. 6.7(b) for the case of a hardening spring.

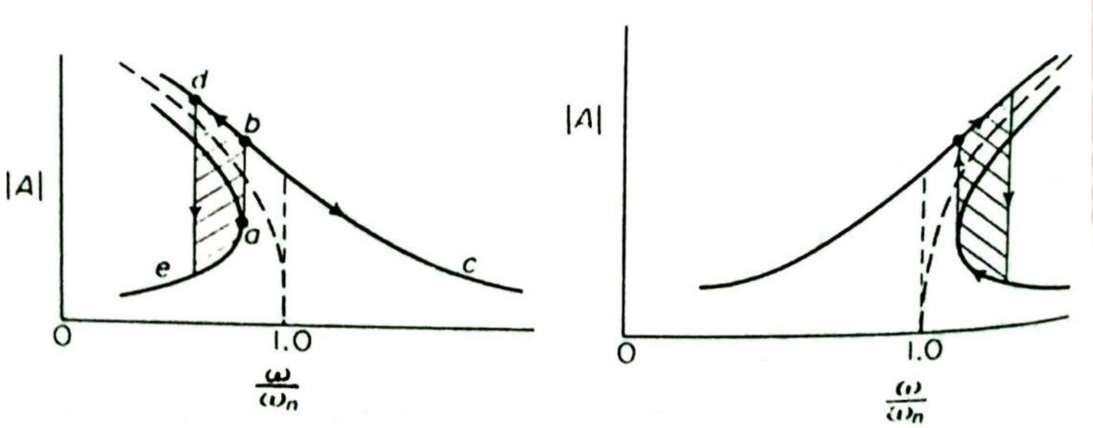


Fig.6.7 (a) : Softening Spring Fig.6.7(b) Hardening spring

JUMP PHENOMENON:

Figs. 6.7(a) and 6.7(b) show that the amplitudes 'A' undergo sudden discontinuous jump near resonance. This is the famous jump phenomenon of nonlinear systems. It is important to note from the figures that the shaded regions correspond to unstable aptitude frequency plot. Effect of damping is to limit the size of the unstable region and the upper end of the curve will cross over in a continuous curve. It is shown in fig. 6.8.

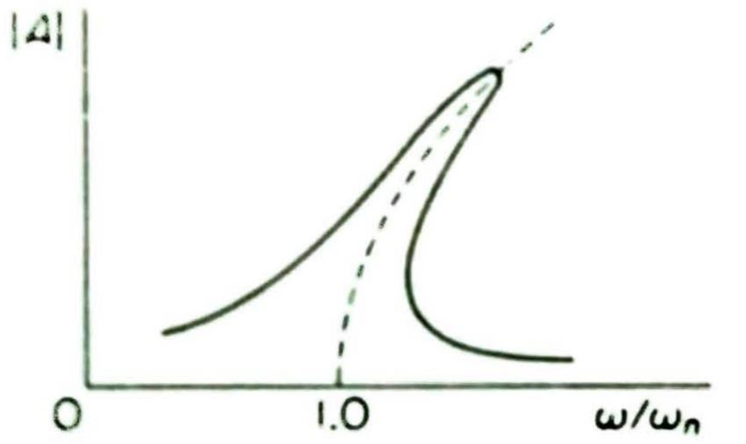


Fig .6.8

6.8. PARAMETRICALLY EXCITED SYSTEMS

This is a special class of systems in which the excitation is not caused by an external agency but is due to the presence of internal mechanism in the parameters of the system itself. The governing differential equation of motion is homogeneous with rapidly varying coefficients which are periodic. Such vibrations are called parametric vibrations. Parametric cases show an extraordinary property: They tend to develop large amplitude responses even if the parametric excitation is snail and the frequency of parametric excitation is far away from the natural frequency of the linear system.

Simplest case of parametric excitation is shown by Mathieu equation

$$\ddot{x} + (\delta + \varepsilon \cos 2t)\, x = 0 \tag{6.37}$$

Using Floquet theory it can be shown that equation possesses normal solutions of the form

$$x(t) - \exp(\gamma t)\emptyset(t) \tag{6.38}$$

Where γ is called a characteristic exponent and $\emptyset(t) = \emptyset(t + \pi)$. When the real part of one of these characteristic exponent; is positive definite. x is unbounded (i.e. unstable) with time. The vanishing of the real parts of the γ's separates the stable motions from the unstable ones. The loci of the corresponding values of ε and δ are called transition curves. These loci divide ε-δ plane into regions of unstable and stable motions. This diagram is shown in fig. 6.9.

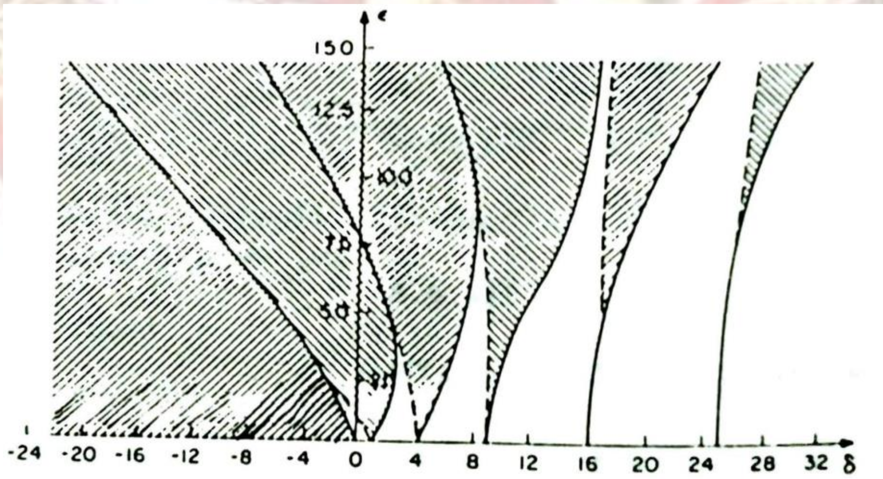


Fig. 6.9: Stable & Unstable Regions for Mathieu equation

6.9. METHODS OF SOLVING NONLINEAR VIBRATION PROBLEMS

Following are some of the methods used to solve nonlinear vibration problems:

1. Qualitative analysis or the phase portrait method
2. Quantitative analysis :
   - a. Perturbation method
   - b. Lindstedt-Poincare method
   - c. Method of multiple scales
   - d. Method of harmonic balance
   - e. Methods of averaging
     1. Krylov-Bogoliubov method.
     2. Krylov-Bogoliubov-Mitropolsky method
     3. Generalized method of averaging
     4. Averaging method using canonical variables
     5. Averaging method using Lie series and transforms
     6. Averaging method using Lagrangians

For a discussion of these methods, the reader is advised to consult the following reference:

Non-linear Oscillations
by
A.H. Nayfeh and D.T. Mook

NUMERICAL SOLUTION OF NONLINEAR PROBLEMS:

Nonlinear problems can be solved on a digital computer using numerical methods. Here, we shall consider a powerful method suited to handle systems of ordinary differential equations which are expressed in normal form. Consider the following:

$$\dot{x} = f(x, t) \tag{6.39}$$

where

$x = [x_1 \; x_2 \; x_3 \dots \dots x_n]^T$ and $f = [f_1 \; f_2 \; f_3 \dots \dots f_n]^T$

As an illustration of the use of the normal form, let us consider the following single degree of freedom system:

$$\ddot{x} + f(x, \dot{x}, t) = 0 \tag{6.40}$$

where $f(x, \dot{x}, t)$ is a nonlinear function .

Let $x_1 = x$ and $x_2 = \dot{x}$, use this in the above equation and get the following:

$$\dot{x}_1 = f_1(x_1. x_2. t) \qquad \text{and} \qquad \dot{x}_2 = f_2(x_1. x_2. t) \qquad (6.40)$$

The system represented by the equation (6.41) can also be written in the following matrix form :

$$\dot{y} = g(y, t), \; y = [x_1 \; x_2]^T, \quad f = [f_1 \; f_2]^T \tag{6.42}$$

Solutions of systems of this type can be easily obtained with the help of standard numerical methods available in most computer systems. It is necessary to specify the functions $f_{1,}$ $f_{2,}$ $f_{3,}$ … … f_n and the initial conditions y_0.

The return is the values of the functions $x_1, x_2, \dots \dots x_n$ at incremental values of time , Δt, 2Δt, where Δt is a specified time interval.

An illustrative example of nonlinear parametric vibrations was programmed. The source listing and the results are given at the end of the chapter VII.

CHAPTER VII

SYNTHESIS OF COMPLEX SYSTEMS

7.1 INTRODUCTION

Methods of vibration analysis of simple systems, discussed in previous chapters, prove to be inefficient and uneconomical when applied to very complex systems having large number of subsystems (e.g. artificial satellites, space shuttle, ocean liners, aircraft carriers, train of railway coaches, etc.). Vibration synthesis is used for handling such types of systems. Discussed in this chapter are the salient features of this methodology. It should be borne in mind at the outset that the concepts developed in previous chapters are required here in this chapter.

Basic to all vibration synthesis is the subdivision of the given system into subsystems. These sub-systems could be simple components or complex assemblage of elements. The analysis of these sub-systems could be handled by different organizations or by different groups belonging to the same organization. All the results of different sub-systems so obtained are grouped together in a manner so that the required functionality of the system is achieved, i.e. the system is synthesized. It should be borne in mind that this synthesis is not the same as the synthesis phase of a design cycle. Following are the essential steps which are required for implementation of vibration synthesis:

1. Identification of components of the system
2. Order of magnitude analysis
3. Assembling of system equations
4. Solution of system equations

Identification of components was discussed in chapter VIII and will not be pursued here except when needed. We should note in the passing that the techniques used to describe the dynamic characteristics of a highly complex component are based on the concept of normal modes and frequency range of interest.

The starting point for any meaningful vibration synthesis is the study of order of magnitude analysis. This is usually done with the help of a simplified version of the system model. Various methodologies of analysis were used for simplified versions of models. Results of those analyses can be used as order of magnitude analysis in the advanced phase of system modeling. Determination of appropriate model is always a problem to deal with. The system model cannot be overly simple because something important might be missed .It cannot be very complex too. A complex model may demand enormous expenditure of money and time. In order to arrive at a proper selection of an appropriate model 'befitting cost and time, sometimes an experiment, given below, is done:

Derive a model and observe its dynamic behaviour. Redesign it with increased complexity. Observe if there is marked difference in the dynamic behaviour of two models. If yes, repeat the process of improvement of the model. If no, stop at this stage. This experiment, in some cases, may yield good economic results. But it may fail in other situations.

OVERVIEW OF MODELINO PROBLEMS

An ideal model is one which closely resembles physical model with respect to all elastic and inertial features. Due to cost considerations and time limitations it is normally not possible to be very close to the physical model. To understand it let us take the case of some machine. It can be easily visualized for a machine to be a complex aggregate of continua joined together to accomplish a specified task.

Thus the inertial and elastic elements can be treated to be continuously distributed along its physical dimensions. In contrast to this, a discrete model may be selected which employs lumped masses connected with weight less springs. Use of this model will give rise to discretization error. This error will show up in different ways. If the model is aimed at preserving natural frequency(ies) it is quite likely that it gives inaccurate results in response calculation, and so on. Proper selection of flexibilities, is also an important step. Consider a simply supported beam like element loaded at the center. The most obvious deflection component is the flexure. But others (viz., shear deformation of the cross-section under the load and deflection of the supports) nay be equally important. A disregard to these flexibilities nay lead to overestimation of natural frequencies as compared to tested ones. In complex cases it may not be possible to understand such a situation so simply.

OBJECTIVE OF VIBRATION SYNTHESIS

In essence the objective of complex system synthesis is the creation of lumped inertia and lumped flexibility node is of sub-systems of a given complex system such that the dynamic response of their assembly is consistent with its functional requirements without tine delays and without prohibitive money requirements.

Loads acting on a system will invariably dictate the model properties. Loads can be continuously distributed or concentrated. Discrete models allow only concentrated loads. Therefore, a serious thought must be given to the method of discretization of continuun and loading.

7.2 Mass matrix generation of complex models

Discrete generation of a physical system means its replacement by a series of nodes (or joints) connected with elastic elements. In certain situations, placement at points of nodes of can be done without much thought, application of concentrated loads, at points of it change needs of a cross-section, etc. However, at other places let us careful planning. In order to understand this point, let us consider the case of a beam given below:

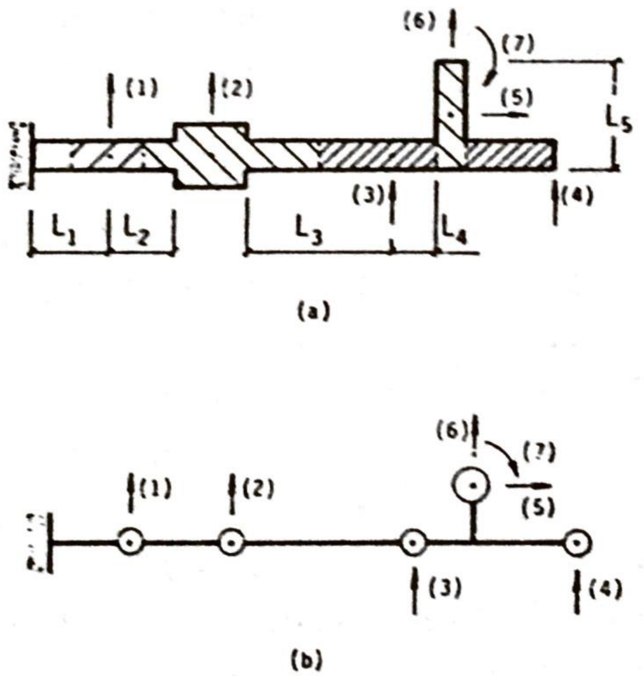


Fig.7.1

The beam is taken to have seven degrees of freedom, designated by arrows at the nodes marked by dots. We associate a portion of beam inertia as mass concentrated at each of the nodes. Mass at coordinate (1) is taken to consist of half the mass of L_1 and half that of L_2. The short segment $L_1/2$ which is adjacent to the fixed end is assumed not to contribute to the moving inertia elements.

Now, we take up another case of a beam for the creation of a lumped parameter model. The physical model is shown in fig. 7.2 (a) and the lumped parameter model is shown in fig. 7.2(b).

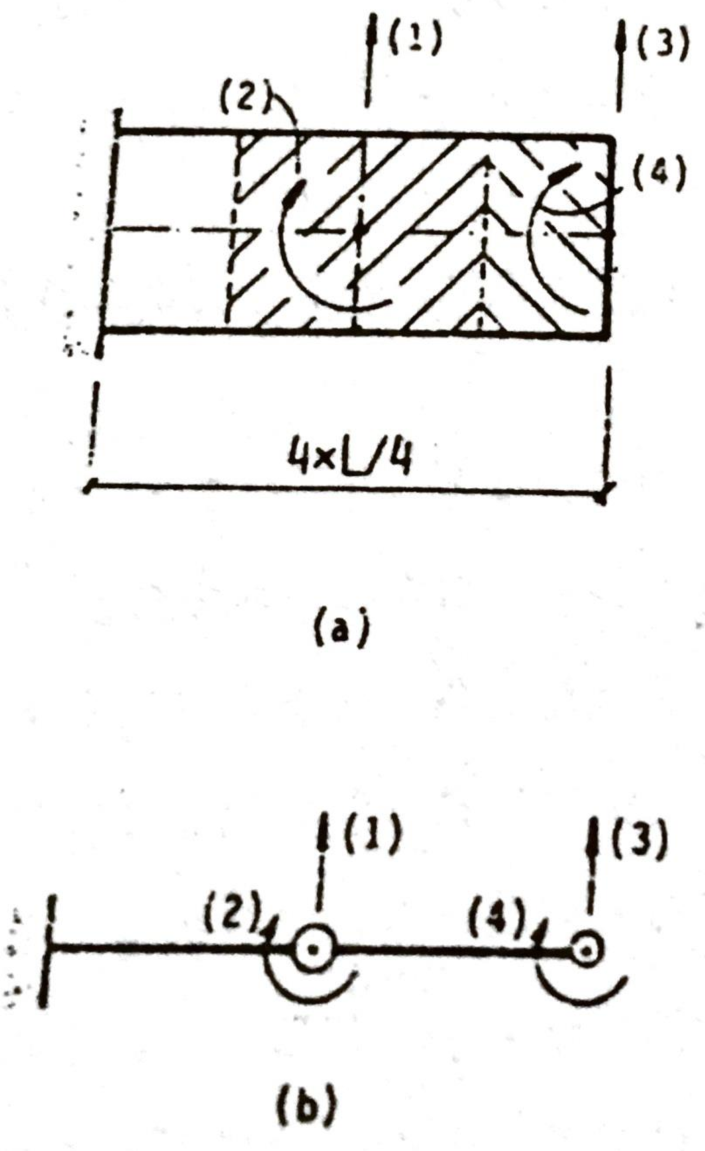


Fig.7.2

This case relates to a deep beam. Rotary inertia is important here. Nodal points are located at the tip and the aid-length. Note that two coordinates are attached with each node, one for translation and the other for rotation. It can also be seen that the mass attached to the center node is twice as big as that at the tip (assuming a uniform mass distribution along the length). The inertia terms associated with the degrees of freedom are:

$$\boldsymbol{M_1 = \left(\frac{1}{2}\right) A\rho L} \qquad \boldsymbol{M_3 = \left(\frac{1}{4}\right) A\rho L}$$

$$\boldsymbol{M_2 \equiv J_2 = \left(\frac{1}{2}\right) I\rho L} \qquad \boldsymbol{M_4 \equiv J_4 = \left(\frac{1}{4}\right) I\rho L}$$

where A: cross-sectional area; I=second area moment

and I = mass density of the material of the beam.

It is to be observed that the rotary inertia associated here is not the mass moment of inertia of a respective segment but only a part of it that is associated with the dimension normal to the axis of the beam.

The analyst selects some preliminary network of nodes, attaches some inertia element and associates a few dynamic degrees of freedom with each of them. This results in a lumped mass matrix. It is important to understand that it is not necessary to assign mass to each and every coordinate and the final decision in this regard is based on the judgement of the analyst.

The mass matrix can also be generated by yet another technique using Rayleigh's principle. According to this scheme the kinetic energy of the deformed pattern is preserved. Suppose, we have a bean for which stiffness matrix has been generated. We can calculate the kinetic energy of the beam element on the assumption that dynamic and static deflect ions patterns remain the same so long as end displacements are equal. Now, the elements of the mass matrix can be calculated using the above principle.

Discrete Models of Continuous Systems

The concept of half-wave length of a vibratory pattern is extensively used for the generation of discrete models. Suppose, we have a system built of identical segments. This system when excited will show a deformation pattern which will repeat itself in space. If x(x) is the deflect ion, wave-length 2λ is defined by

$$X(x+2\lambda) = X(x) \quad \text{and} \quad X(x+\lambda) = -X(x)$$

In other words, the change of x by λ shifts the point to a place where the sign of def lect ion is opposite. If we increase x by another λ, the point reaches a location where X is again the same.

The above concept is explained with the help of two examples given below:

1. A fixed-fixed axially vibrating bar:

The equation of motion, for this case, is the wave equation. It is given below:

$\ddot{u} - c^2u'' = 0$ where $c^2 = (\frac{E}{\rho})$

The solution for the nth vibratory mode may be represented as

$x_n = \sin(\frac{\pi x}{\lambda_n})$ where $\lambda_n = \frac{\pi c}{\omega_n} = L/n$

Lateral projection of the axial deflection of the bar for n = 4 is shown in the figure.

It is logical to discretize this bar into four segments, vibrating independently from one another and each segment behaving like a bar with fixed ends.

2. A flexural beam:

Equation of motion of a flexural beam is

$$EIu'''' + m\ddot{u} = f(x,t)$$

For a simply supported beam, n th vibratory mode is

$x_n = \sin(\frac{\pi x}{\lambda_n})$ where $\lambda_n = (L/n)^2 = \left(\frac{\pi}{\omega_n}\right)^2 (\frac{EI}{m})^{1/2}$

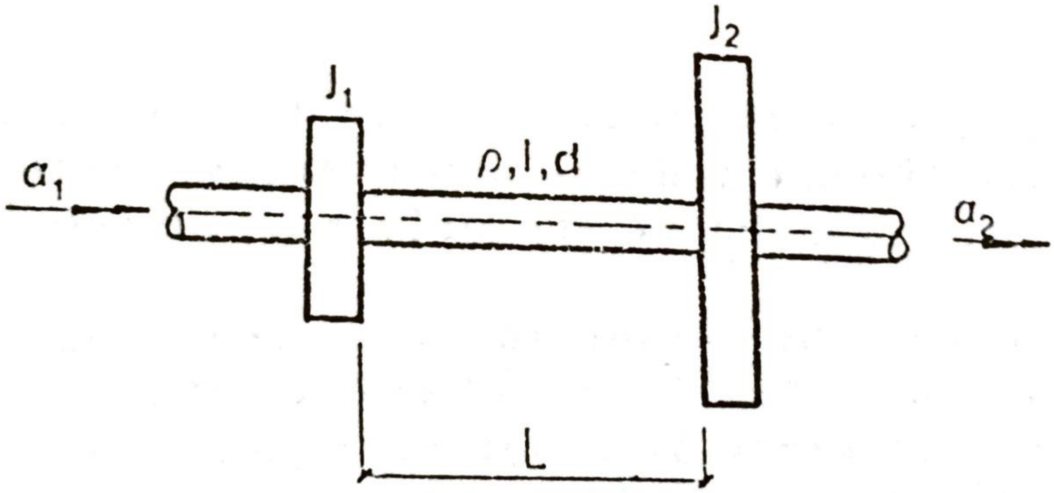
The wave pattern is sinusoidal. In this case each half-wave nay be treated separately as a simply supported beam with length λ_n.

Both the cases treated above show ideal wave patterns using which we were able to discretize successfully. In case of complicated end conditions, the wave patterns will get disturbed and discretization will not be so straightforward. Sound judgement on the part of the analyst will then rule the decision. It is not possible to discuss all that is necessary for successful discretization. However, following are some of the important points (besides what has been said above) which must be borne in mind for carrying out discretization:

1. We must recognize the highest significant frequency and highest significant mode of a vibratory shape which is to remain representative of the original system. The higher the mode number the lesser is its contribution to the overall response. The group of modes above the highest significant frequency need not be considered.
2. When a maximum forcing frequency is specified, it is necessary that we place a sufficient number of lumped masses to make the response at that frequency possible. For taking care of higher frequencies, it is necessary to have a denser placement of nodes and masses.
3. When it is not possible to specify the maximum forcing frequency, say, due to the nature of loading, then we must use the concept of having maximum calculated response as close as possible to that experienced by the original system.
4. Since modeling is affected by the distribution of load in space and time, therefore, the most favourable spatial distribution is in the shape of a natural mode. The system will then respond in that mode only.
5. When the excitation is in the form of base mot ion then it must be noted that the forces are imparted to the system by the attachment points, In case we are interested in relative motion with respect to the moving base then the base motion can be replaced by a kinetically equivalent inertia load.
6. When the base motion is complex then the response spectrum is used as an external loading. The spectrum enables the analyst to calculate the indivimode responses but it does not specify the summation proc

ILLUSTRATIVE EXAMPLE:

A shaft carries two discs as shown in the figure given below. The discs vibrate with equal amplitudes. Assume that the deformation of the shaft between discs varies linearly. Calculate what port ion of the distributed inertia be assigned to each disc to preserve the kinetic energy of the system when the discs vibrate in the (1) same direct ion and (2) opposite direction.



Solution:

The motion of the discs is taken to a harmonic function of time. The kinetic energy of the shaft is

$$T = (\frac{\omega^2}{2})\rho I_0 \int \alpha^2(x)dx$$

where I_0 : angular inertia per unit length;

$\alpha(x)$ = displacement function assumed linear w.r.t. x

when both the discs rotate in the same direction then the entire shaft segment can be taken to rotate like a rigid body, i.e. $\alpha_1 = \alpha_2$=constant = C. C can be taken as unity.

We can assign following fraction of inertia to each disc:

$$\Delta J = (\frac{1}{2})\rho I_0 L$$

When the discs arc out of phase then the displacement function can be taken as

$$\alpha=(1-2x/L)$$

For this case: $T = \left(\frac{\omega^2}{2}\right) * \left(\frac{1}{3}\right) \rho I_0 L = \left(\frac{\omega^2}{2}\right)(2\Delta J)$

Thus $\Delta J = \left(\frac{1}{6}\right) \rho I_0 L$ should be added to each disc to preserve the kinetic energy.

7.4 SYNTHESIS OF COMPLEX SYSTEMS

For synthesis of a complex systems it is necessary to subdivide it into appropriate number of subsystems. Dynamic models of all of these subsystems are generated as per the methodology outlined above. Then extensive dynamic analyses of all these sub-systems are carried out. Now, the entire system model is assembled and then it is analyzed. One of the important methods of vibration synthesis used for the handling of such complex systems is the method of component mode synthesis credited to Craig and Banpton.

Let us understand the fundamentals of this method. We begin with the analysis of an undamped free vibration of a system consisting of two components (α and β). The set of physical Coordinates of the components is divided into a set of interface or juncture coordinates ($[u]_j$) and a set of interior coordinates ($[u]_i$).

The undamped equation of mot ion of a component of this system is written be low in partitioned form:

$$\begin{bmatrix} m_{ii} & m_{ij} \\ m_{ji} & m_{jj} \end{bmatrix} \begin{Bmatrix} \ddot{u}_i \\ \ddot{u}_j \end{Bmatrix} + \begin{bmatrix} k_{ii} & k_{ij} \\ k_{ji} & k_{jj} \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} f_i \\ f_j \end{Bmatrix} \qquad (7.1)$$

Now, the physical coordinates, $[\mathbf{u}]$ are represented in terms of the component generalized coordinates, [p], by the following transformation equation:

$$[u]=[\psi][p] \qquad (7.2)$$

$[\psi]$ is the matrix of one of the component modes selected in advance. These could be rigid body modes, normal modes of free vibration, constraint modes, attachment modes and inertia relief attachment modes. These are described below:

7.4.1 NORMAL MODES:

Consider an eigenvalue problem given below:

$$([k] - \omega^2[m][\phi] = [0] \qquad (7..3)$$

Obtaining component normal modes of this eigenvalue problem is dependent on whether all, none or some of the juncture coordinates are constrained. These component normal modes are defined as fixed-interface normal modes, free-interface normal modes and hybrid-interface normal modes respectively. It is assumed that the nodes are normalized with respect to [m]. That is

$$[\phi]_n^T[m][\phi]_n = I_{nn} \qquad (7.4)$$

and

$$[\phi]_n^T[k][\phi]_n = [A_{nn}] \equiv diag(\omega_r^2) \qquad (7.5)$$

where $[\phi]_n$ is a matrix whose columns are the component normal modes.

7.4.2 CONSTRAINT MODES

In order to understand the const rained modes let us partition the physical coordinates [u] into a set C relative to which constraint modes are to be defined. Let F be the complement of C. A constraint mode is defined by imposing a unit displacement on one of the physical coordinates of the set C and zero displacement on the remaining coordinates of the set C. This is given below in the form of a matrix equation :

$$\begin{bmatrix} k_{ff} & k_{fc} \\ k_{cf} & k_{cc} \end{bmatrix} \begin{bmatrix} \varphi_{fc} \\ I_{cc} \end{bmatrix} = \begin{bmatrix} 0_{fc} \\ R_{cc} \end{bmatrix} \qquad (7.6)$$

where R_{cc} is the set of reactions at the C coordinates.

From equation (7.6) :

$$[\varphi_{fc}] = -[k_{ff}]^{-1}[k_{fc}] \qquad (7.7)$$

The constraint mode matrix is thus given by the following:

$$[\varphi_c] = \begin{bmatrix} \varphi_{fc} \\ I_{cc} \end{bmatrix} = \begin{bmatrix} -k_{ff}^{-1} \quad k_{fc} \\ I_{cc} \end{bmatrix} \qquad (7.8)$$

Rigid body modes are obtained in the process of solving the eigenvalue problem for component normal modes as the ones with zero frequency. These are also a special case of constraint modes. Consider a case in which a component has N rigid-body degrees of freedom. We use a set

R consisting of coordinates to be used to restrain the component against rigid body motion. The rigid-body modes corresponding to this set R is obtained by setting c = r in eq. (11.6) and noting that $R_{rr} = 0$. Thus rigid body modes are

$$[\varphi_r] = \begin{bmatrix} \varphi_{fr} \\ I_{rr} \end{bmatrix} = \begin{bmatrix} -k_{ff}^{-1} & k_{fr} \\ & I_{rr} \end{bmatrix} \tag{7.9}$$

ATTACHMENT MODES

Consider a set P. We would like to define attachment modes relative to this set. Let P be divided into three subsets: R, A, W. R is a statically determinate constraint get providing restraint against rigid body. A is a set of those physical coordinates where unit forces are to be applied to define attachment modes. Then the attachment modes relative to the constraint set R are given by

$$\begin{bmatrix} k_{ww} & k_{wa} & \vdots\, k_{wr} \\ k_{aw} & k_{aa} & \vdots\, k_{ar} \\ k_{rw} & k_{ra} & \vdots\, k_{rr} \end{bmatrix} \begin{bmatrix} \varphi_{wa} \\ \varphi_{aa} \\ 0_{ra} \end{bmatrix} = \begin{bmatrix} 0_{wa} \\ I_{aa} \\ R_{ra} \end{bmatrix} \tag{7.10}$$

It should be noted that the attachment modes are essentially columns of a flexibility matrix. Let g_{wa} and g_{aa} be obtained from the inverse of the upper left of [k] in eq. (11.10). Then

$$\varphi_a = \begin{bmatrix} \varphi_{wa} \\ \varphi_{aa} \\ 0_{ra} \end{bmatrix} = \begin{bmatrix} g_{wa} \\ g_{aa} \\ 0_{ra} \end{bmatrix} \tag{7.11}$$

7.4.5 ATTACHMENT MODES FOR CONSTRAINED COMPONENTS

An attachment mode is defined as the static displacement of the component when a unit force is applied to one of the coordinates of the set A (subset of P relative to which attachment modes are to be defined) while other coordinates of this get remain force free. For a restrained component let W be the complement of A in P, Now, the set Of attachment modes is

$$\begin{bmatrix} k_{ww} & k_{wa} \\ k_{aw} & k_{aa} \end{bmatrix} \begin{bmatrix} \varphi_{wa} \\ \varphi_{aa} \end{bmatrix} = \begin{bmatrix} 0_{wa} \\ I_{aa} \end{bmatrix} \tag{7.12}$$

If [gl is the flexibility matrix i.e. $[g]=[k]^{-1}$, then the attachment modes for the restrained component are columns of the flexibility matrix. Hence

$$\varphi_a = \begin{bmatrix} \varphi_{wa} \\ \varphi_{aa} \end{bmatrix} = \begin{bmatrix} g_{wa} \\ g_{aa} \end{bmatrix} \tag{7.13}$$

7.5 SYSTEM SYNTHESIS FOR UNDAMPED FREE VIBRATION

Once the components of a system are identified and their structures are established then it is necessary to couple them to obtain the system equations. In the following sections is described a generalized substructure coupling procedure as applied to undamped free vibrations. We must not forget that this coupling is carried out within the definition of the component mode synthesis.

Consider a system consisting of two components α and β having a common interface.
Interface constraints of the physical displacements are given by the following

$$[u]_j^{\alpha} = [u]_j^{\beta} \tag{7.14}$$

The interface forces are related by

$$[f]_j^{\alpha} = [f]_j^{\beta} = [0] \tag{7.15}$$

The system kinetic energy is given by

$$T = \frac{1}{2}[\dot{p}]^T[\mu][\dot{p}]$$

$$= \frac{1}{2}[\dot{p}]^{\alpha T}[\mu]^{\alpha}[\dot{p}]^{\alpha} + \frac{1}{2}[\dot{p}]^{\beta T}[\mu]^{\beta}[\dot{p}]^{\beta} \tag{7.16}$$

and system potential energy is

$$= \frac{1}{2}[p]^{\alpha T}[k]^{\alpha}[p]^{\alpha} + \frac{1}{2}[p]^{\beta T}[k]^{\beta}[p]^{\beta} \tag{7.17}$$

$$\text{where } [\mu] = \begin{bmatrix} \mu & 0 \\ 0 & \mu^{\beta} \end{bmatrix}; \quad [k] = \begin{bmatrix} k & 0 \\ 0 & k^{\beta} \end{bmatrix} \tag{7.18}$$

$$[\mu]^{\alpha} = [\varphi]^{\alpha T}[m]^{\alpha}[\varphi]^{\alpha} \text{ and } [k]^{\alpha} = [\varphi]^{\alpha T}[k]^{\alpha}[\varphi]^{\alpha}, \text{ etc. ..} \tag{7.19}$$

$$\text{and } [p] \equiv [p^{\alpha} p^{\beta}]^T \tag{7.20}$$

The equation of physical displacement constraints at the interface can also be written in terms of the generalized coordinates (p) as

$$[c]\,[p]=[0] \tag{7.21}$$

The Lagrangian for the system is, now, written with the help of [λ] vector of Lagrange multipliers, Kinetic energy and potential energy expressions as

$$L=T-V+[\lambda]^T\,[c]\,[p] \qquad (7.22)$$

To obtain system equations, we use Lagrange's equations given below:

$$d/dt(\partial L/\partial \xi_s)-(\partial L/\partial \xi_s) = Q_s \qquad (11.23)$$

where ξ_s refers to either p_s or λ_s and Q_s is the generalized force. It may be noted that the generalized force arises due to forces at the component interfaces. These are reactive forces and the virtual work done due to these will be zero.

Using equations (7.16), (7.17) & (7.22) in equation (7.23), we get the following equation of motion of the system;

$$[\mu][\dot{p}] + [k][p] = [c]^T[\lambda] \qquad (7.24)$$

In order to solve this set of coupled equations, we introduce a linear transformation defined below:

$$[p] = [s][q] \qquad (7.25)$$

Equation (11.25) is now partitioned into dependent coordinates [p] and independent coordinates (pl

and independent coordinates $[p]_i$.

The partitioned equation (7.25) is written as

$$[c_{dd} \quad c_{di}]\begin{Bmatrix} p_d \\ p_i \end{Bmatrix} = [0] \qquad (7.26)$$

where $[c]_{dd}$ is a nonsingular square matrix.

From equation (7.26) the following is obtained:

$$[p]\equiv\begin{Bmatrix} p_d \\ p_i \end{Bmatrix} = \begin{bmatrix} -c_{dd}^{-1} & c_{di} \\ I_{ii} & \end{bmatrix} [p]_1 \equiv [s][q] \qquad (7.27)$$

Equations (7..26) and (7.27) show that [c] [S] =[0].

Now, equations (7.24) and (7.25) are combined to give

$$[M][\ddot{q}] + [K][q] = [s]^T[c]^T[\lambda] \qquad (7.28)$$

where

$$[M] = [s]^T[\mu][S] \text{ and } [K] = [s]^T[k][S]$$

Using [c] [S] = [0] in eq. (7.28), we get

$$[M][\ddot{q}] + [K][q] = [0] \qquad (7.29)$$

Eq. (7.29) is the desired system equation of motion.

STEPS IN CARRYING OUT COMPONENT MODE SYNTHESIS:

Consider a system consisting of two components α and β. In order to carry out component mode synthesis for this system, following steps are to be taken:

1. Select the component modes for inclusion in $[\varphi]^\alpha$ and $[\varphi]^\beta$.
2. Obtain $[\mu]^\alpha$, $[\mu]^\beta$, $[k]^\alpha$, $[k]^\beta$.
3. Establish dependent coordinates ($[p]_d$) in [p].
4. Remainder coordinates are $[p]_i$ adn $[p]_i \equiv [q]$.
5. Write the constraint equations and solve for [S].
6. Determine [M] and [K].

7.6 APPLICATION: Consider a clamp-clamp beam given below in the fig. 11.5(a):

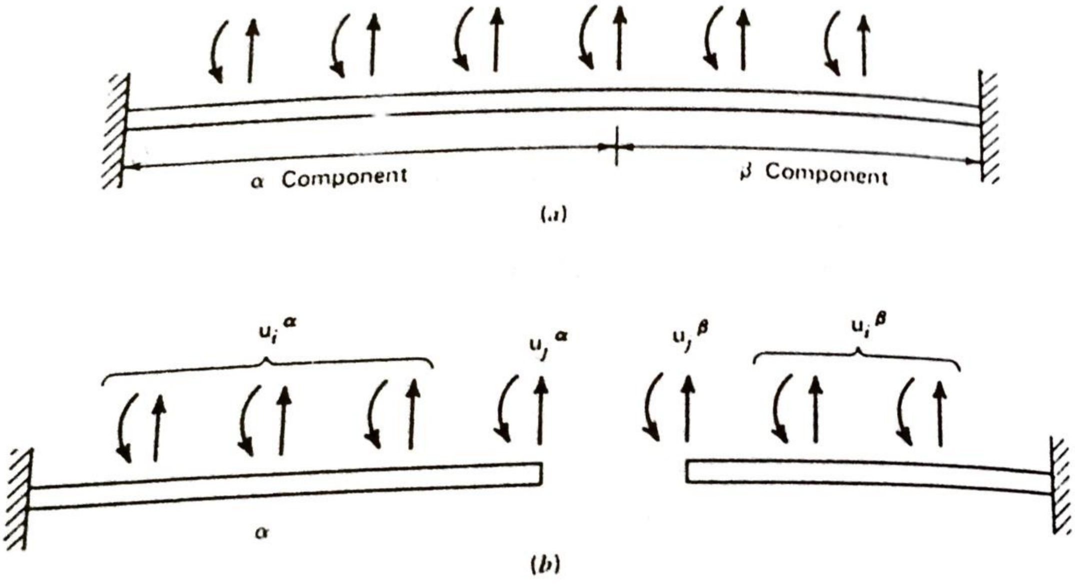


Fig.7.5

This beam is taken to be a coupled structure and it is considered to consist of two components α and β as shown in the fig. 7.5(b).

Let us assume that the set of physical coordinates [u] of the system components is divided into the following two sets.

1. a set of juncture coordinates $[u]_j$

(juncture coordinates are those coordinates where components are joined together.)

2. a set of interior coordinates $[u]_j$.

Let each of the two components be represented by a set of constraint nodes defined for the interface coordinates plus a truncated set of fixed interface normal modes. Let us assume that the fig. 7.6 shows two constraint modes for the component α and β the lowest frequency fixed interface normal node for this component.

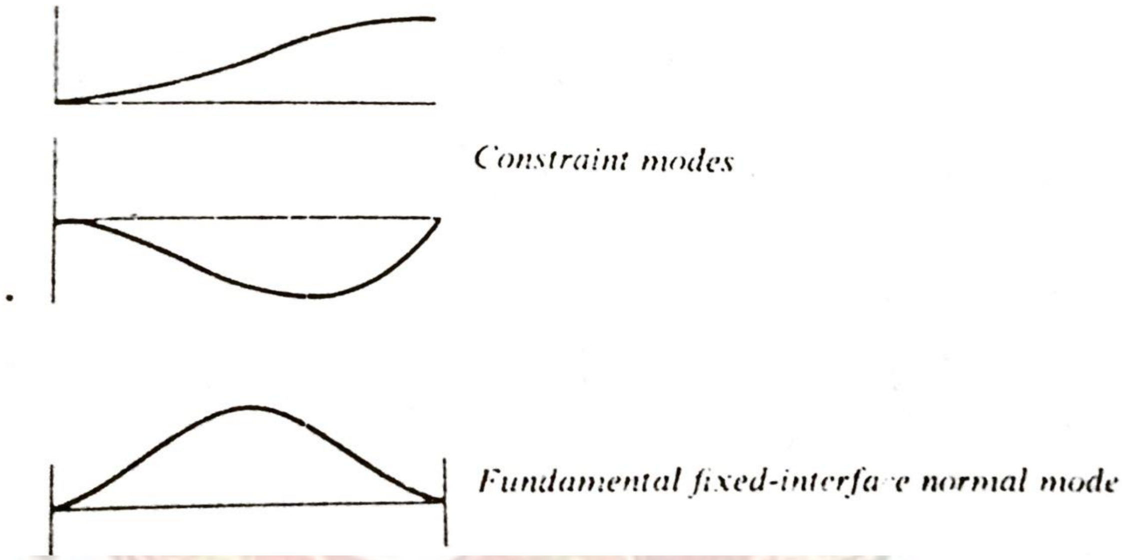


Fig. 7.6

The physical coordinates are divided into interior coordinates and juncture coordinates and thus

$$[u] = [u_i \; u_j]^T \tag{7.30}$$

The interior coordinates are related to kept modes. Juncture coordinates are related to constraint modes. Defining kept modes as $[\phi_k]$ and constraint nodes as $[\psi_c]$, we get

$$[u]= [\phi_k]\,[p\phi_k]+ [\psi_c]\,[p_c] \tag{7.31}$$

for each component.

Eq. (7.31) can also be written in the following partitioned form:

$$\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{bmatrix} \emptyset_{ik} & \varphi_{ic} \\ 0 & I \end{bmatrix} \begin{Bmatrix} p_k \\ p_c \end{Bmatrix} \tag{7.32}$$

It is to be noted that the normal modes are fixed-interface modes and $[p_c] \equiv [u_j]$.

The constraint modes are given by

$$[\varphi_{ic}] = -[k_{ii}]^{-1}[k_{ij}] \tag{7.33}$$

Now

$$[\mu]^{\alpha} = [\varphi]^{\alpha T}[m]^{\alpha}[\varphi]^{\alpha} \text{ and } [k]^{\alpha} = [\varphi]^{\alpha T}[k]^{\alpha}[\varphi]^{\alpha} \tag{7.34}$$

Writing full forms of $[m]^{\alpha}$ and $[\varphi]^{\alpha}$, we get

$$\begin{bmatrix} \mu_{kk} & \mu_{kc} \\ \mu_{ck} & \mu_{cc} \end{bmatrix} = \begin{bmatrix} \emptyset_{ik}^T & O^T \\ \varphi_{ic}^T & I \end{bmatrix}^{\alpha} \begin{bmatrix} m_{ii} & m_{ij} \\ m_{ji} & m_{jj} \end{bmatrix}^{\alpha} \begin{bmatrix} \emptyset_{ik} & \varphi_{ic} \\ 0 & 1 \end{bmatrix}^{\alpha} \tag{7.35}$$

Indicated multiplications are carried (superscript aon all matrices are dropped). We get

$$\mu_{kk} = I_{kk}$$

$$\mu_{kc} = \mu_{ck}^T = \emptyset_{ik}^T(m_{ii}\,\varphi_{ic} + m_{ij}) \tag{7.36}$$

$$\mu_{cc} = \varphi_{ic}^T(m_{ii}\,\varphi_{ic} + m_{ij}) + m_{ji}\varphi_{ic} + m_{jj}$$

It can be seen that an inertia coupling between normal mode coordinates and the constraint mode coordinates is present.

Similarly, [K]α can also be determined. Thus

$$[K]^{\alpha} = \begin{bmatrix} k_{kk} & k \\ k_{ck} & k_{kk} \end{bmatrix}^{\alpha} \tag{7.37}$$

$$k_{kk} = A_{kk}$$

$$k_{kc} = k_{ck} = 0 \tag{7.38}$$

$$k_{cc} = k_{jj} - k_{ji}k_{ii}^{-1}k_{ij}$$

It is evident that the stiffness matrix, expressed in component generalized modes, is diagonal.

Since and $[p]_c^\alpha = [u]_j^\alpha$ and $[p]_c^\beta = [u]_j^\beta$, interface displacement compatibility equations can be written as

$$[0\ 1\ 0\ \vdots\ -I]\left\{p_k^\alpha\ \ p_c^\alpha\ \ p_k^\beta\ \ \vdots\ \ p_c^\beta\right\}^T = 0 \qquad (7.39)$$

so

$$[C] = [0\ 1\ 0\ \vdots\ -I] \qquad (7.40)$$

where p_c^β is dependent coordinate.

Now, we determine [S] matrix. For this, we introduce a linear transformation [p] = [S][q]. Thus

$$\left\{p_k^\alpha\ \ p_c^\alpha\ \ p_k^\beta\ \ p_c^\beta\right\}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{p_k^\alpha\ p_k^\beta\ p_c^\alpha\right\}^T \qquad (7.41)$$

Now, we determine [M] and [K] with the help of the following transformations

$$[M] = [S]^T[p][S]$$
$$(7.42)$$
$$[Kl = [S]^T\ [K][S]$$

Now,

$$[M] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_{kk}^\alpha & \mu_{kc}^\alpha & 0 & 0 \\ \mu_{ck}^\alpha & \mu_{cc}^\alpha & 0 & 0 \\ 0 & 0 & \mu_{kk}^\beta & \mu_{kc}^\beta \\ 0 & 0 & \mu_{ck}^\beta & \mu_{cc}^\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (7.43)$$

Then,

$$[M] = \begin{bmatrix} M_{kk}^{\alpha} & 0 & M_{kc}^{\alpha} \\ 0 & M_{kk}^{\beta} & M_{kc}^{\beta} \\ M_{ck}^{\alpha} & M_{ck}^{\beta} & M_{cc} \end{bmatrix} \tag{7.44}$$

where

$$M_{kk}^{\alpha} = I_{kk}^{\alpha};\ M_{kk}^{\beta} = I_{kk}^{\beta};$$

$$M_{kc}^{\alpha} = (M_{ck}^{\alpha})^T = \mu_{kc}^{\alpha};\ M_{kc}^{\beta} = \left(M_{ck}^{\beta}\right)^T = \mu_{kc}^{\beta};\ M_{cc} = \mu_{cc}^{\alpha} + \mu_{cc}^{\beta} \tag{7.45}$$

The system stiffness matrix [K] synthesized similarly, is

$$[K] = \begin{bmatrix} K_{kk}^{\alpha} & 0 & 0 \\ 0 & K_{kk}^{\beta} & 0 \\ 0 & 0 & K_{cc} \end{bmatrix} \tag{7.46}$$

where

$$K_{kk}^{\alpha} = A_{kk}^{\alpha} \tag{7.47}$$

$$K_{kk}^{\beta} = A_{kk}^{\beta}$$

$$K_{cc} = k_{cc}^{\alpha} + k_{cc}^{\beta}$$

From the forgoing it should become clear that the system equations have only inertial coupling and since A_{kk}^{α} and A_{kk}^{β} are available from component eigen-problems. Only following matrices are needed for assembling the system matrices:

$\mu_{kc}^{\alpha}, \mu_{kc}^{\beta}, \mu_{cc}^{\alpha}, \mu_{cc}^{\beta}, k_{cc}^{\alpha}\ and\ k_{cc}^{\beta}.$

Vibration Synthesis of Complex Systems Utilizing "Building Block" Technique:

Building block approach is one of the most practical methods for the prediction of dynamic behaviour of complex structural/mechanical systems. It involves the division of a system into analyzable "Building Blocks" and analysis through experimental and analytical techniques. We can understand it through the study of the fig. 7.7.

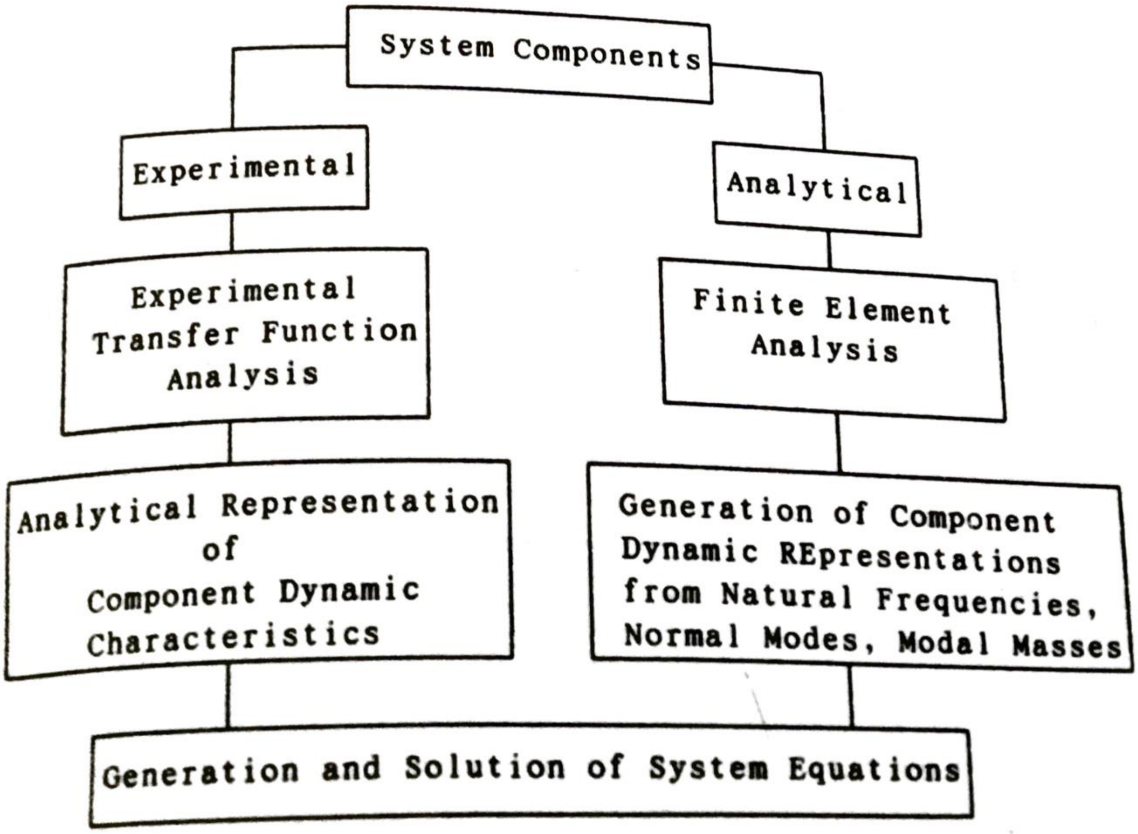


Fig. 7.7: General Approach to Building Block Concept

Building block method is basically a substructuring technique [52] which is a modification of the modal synthesis technique proposed by *Hurty [56]* and *Craig[64]*. The model synthesis technique and modal coordinates permit the development of a set of system equations which is substantially smaller than the number required for adequate description if only physical coordinates are used, it also allows for the description of components from experimental tesy data. It is very important in the case of complex mechanical/structural system since the dynamic characteristics of some components may be impossible to determine with analytical methods. Under this scheme, the system is divided into interconnected blocks(components) so that each block can be analyzed using either available digital computer routines, experimental tests or both. The total system response is obtained by appropriately coupling the dynamic characteristics of each component.

Vibration synthesis via the building block approach involves the following key elements:

1. Component identification
2. Assembling the system equations
3. Solution of system equations
4. Recovering detailed component response

The relationship of these four elements is shown in fig.7.8:

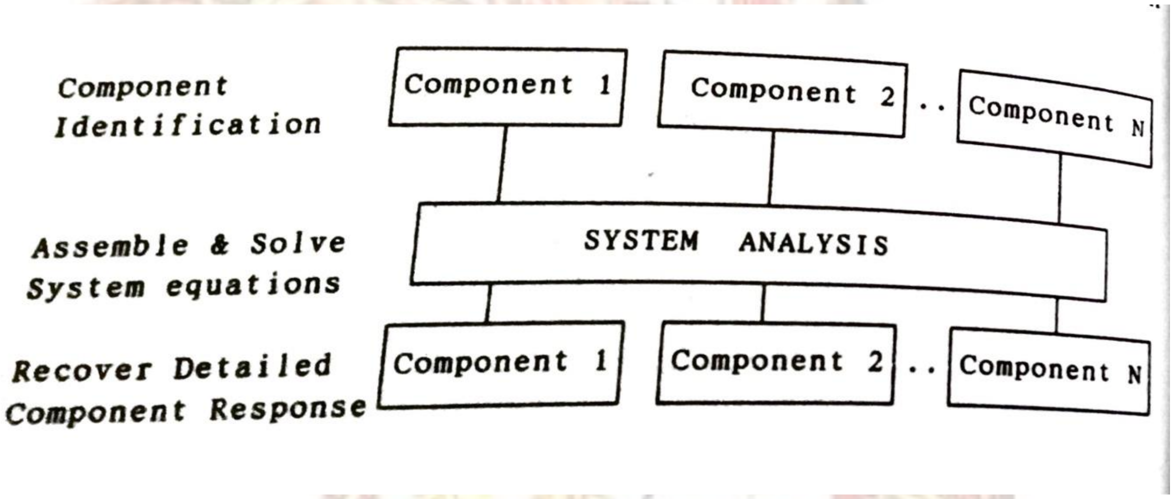


Fig. 7..8: Problem Flow via the Building Block Technique

The building block approach is similar to the design process in which major components (or subsystems) are often designed by different groups. This approach in the design permits component design to proceed as independently as possible with due consideration being given to the final coupling of subsystems to form the complete system. Groups responsible for the design of extremely components rely on experimental tests conducted to define dynamic behaviour, while other groups use analytical finite element investigations. The results of the dynamic analysis of each component can then be evaluated by the department responsible for the evaluation of total system dynamic performance.

EXTRACTION OF MODAL PARAMETERS FROM TEST DATA:

A flow logic diagram for writing a computer program in order to extract modal characteristics is shown below in fig. 7.9.

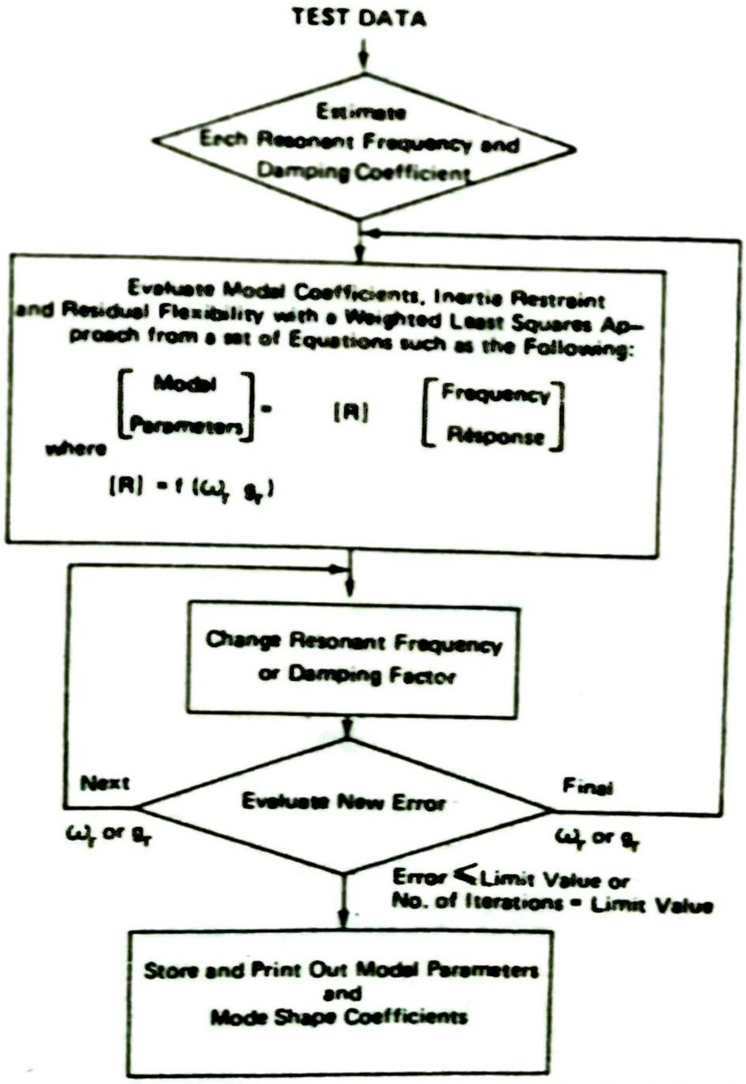


Fig.7.9 Flow Diagram for Extracting Modal Parameters

To completely define the dynamic characteristics of a component from an experimental point of view, the modal parameters must be determined. Suppose the equations of motion of this component assuming the presence of both viscous and hysteretic damping are

$$[M][\ddot{q}] + [C][\dot{q}] + j[D]\{q\} = \{F(t)\}$$

Appropriate testing equipment is to be used for the determination of the frequency response characteristics of this physical system. Transient, sinusoidal or random inputs can be applied. Various curve fitting techniques are then used to extract the required modal characteristics.

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