Chapter 3

A Difference Equation with Minima based Reaching Law for Discrete Sliding Mode Control

3.1 Introduction

This chapter introduces two reaching laws formulated on the framework of difference equations with minima for discrete-time sliding mode control. The objective here is to present an approach that strikes a balance between Gao's reaching law and Utkin's reaching law. These methods aim to address the shortcomings of both approaches, namely chattering in the former and excessively large control action in the latter. The proposed reaching laws for discrete-time sliding mode control are tailored for both unperturbed and perturbed systems. In the case of the unperturbed system, the switching variable converges to zero in finite time, while for the perturbed system, the variable remains in the vicinity of the switching manifold. The efficacy of the proposed methods is validated through an illustrative example involving a pendulum system subjected to matched type bounded perturbations. Simulation results underscore the effectiveness of the discussed reaching law-based methodologies.

This chapter is structured as follows: Section 3.2 offers preliminary information that will aid in understanding the key outcomes of the study. Section 3.3 presents the primary findings of this chapter. Section 3.4 validates the efficacy of the proposed reaching law based discrete sliding mode control with an example of pendulum system. Finally, Section

3.5 concludes this chapter.

3.2 Preliminaries

This section delves into pertinent background results pertaining to the design and structure of two crucial reaching laws. The aim is to provide a comprehensive understanding of the theoretical underpinnings and practical implications of these reaching laws in the context of control system design. By exploring the foundational aspects of these reaching laws, readers can gain insights into their significance and potential applications within the broader framework of control theory. The subsequent discussions will shed light on the intricacies of these reaching laws and their role in shaping the behavior of dynamical systems.

- In the literature, there are predominantly two schools of thought to design discrete sliding mode control:
  1. Switching based control
  2. Non-switching based control

- **Switching based control [25]:**
  This type of control is obtained from the direct discretization of switching based control laws for continuous-time cases. In this case, the system trajectory can't get arbitrarily close to the origin due to finite switching frequency.

- **Non-switching based control [22, 27]:**
  The fundamental idea behind this type of development is that discrete-time control is inherently discontinuous. The equivalent control based reaching law [22] results in no reaching phase but very high amount of control action.

3.2.1 On Gao's Reaching Law

$$s(k+1) = (1 - qT)s(k) - \epsilon\, T\, \text{sign}(s(k)) \tag{3.1}$$

where $q \in (0, 1)$, $\epsilon > 0$, $T > 0$, is the sampling time and $(1 - qT) > 0$ must hold which makes the choice of T quite restricted. An illustration of the evolution of sliding variable according to Gao's law is shown in Figure 3.1. One can observe the oscillatory behaviour of the trajectory from the graph. Moreover, the following remark points out another limitation of this reaching law.

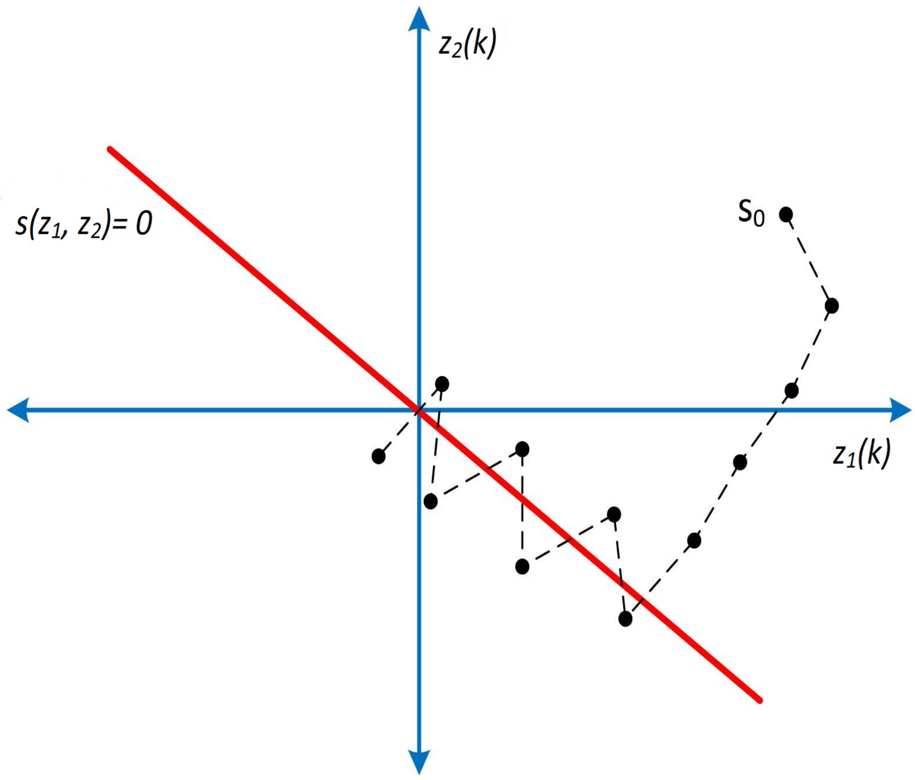

Figure 3.1: Evolution of sliding variable according to Gao's law

Remark 3.1 *The most of discrete-time sliding mode control methods [25, 28, 60], such as, in Gao's method, sliding variable oscillates about $s = 0$ instead of staying on it, since $s = 0$ may not fall on the integral multiple of sampling time T. This eventually leads to the chattering phenomenon.*

3.2.2 On Utkin's Reaching Law

$$s(k+1) = 0 \tag{3.2}$$

An illustration of evolution of sliding variable according to Utkin's law is shown in Figure 3.2. Although, this reaching law looks quite simple in its structure, however, this simple structure results in dead-beat type response. As a result, the amount of control effort required to execute this reaching law is very high.

The limitations observed in the structure of these two reaching laws motivate us to explore this area in order to craft a feasible solution to this problem.

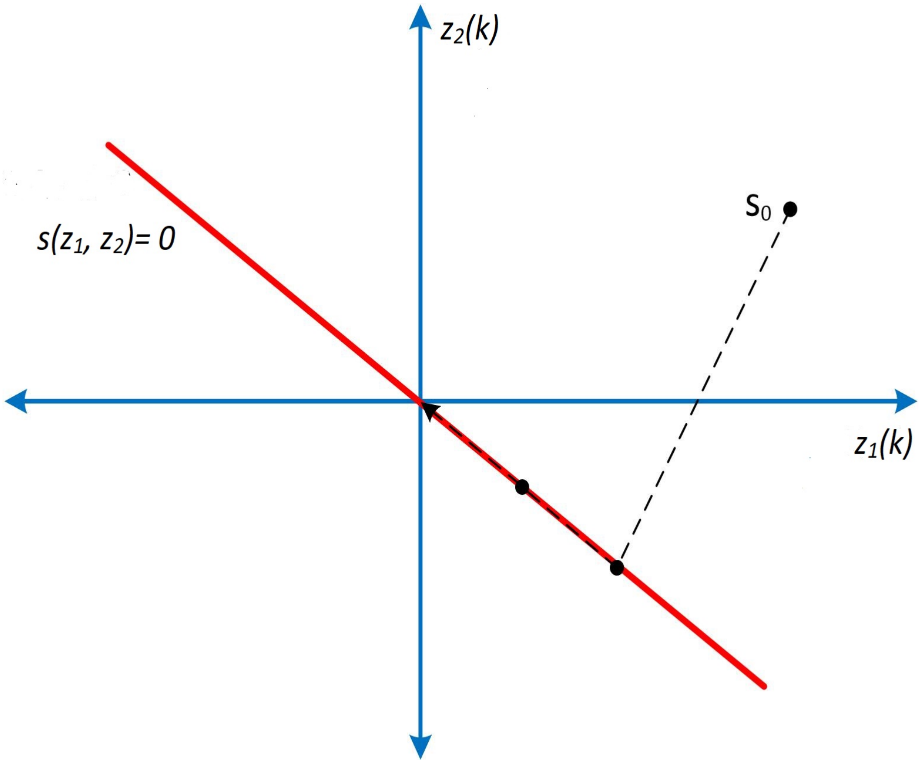

Figure 3.2: Evolution of sliding variable according to Utkin's law

3.3 Reaching Laws for Unperturbed and Perturbed Discrete-Time Systems

In this section, we propose two reaching laws for unperturbed and perturbed discrete-time systems. We provide the definitions of sliding mode and quasi-sliding mode along with the theorems showcasing the fundamental properties of the reaching laws.

3.3.1 RL1 for Unperturbed Discrete-Time Systems

Let us consider the following RL for unperturbed DTS

$$s(k+1) = s(k) - \text{sign}[s(k)] \min\{|s(k)|, \gamma\} \tag{3.3}$$

where $\gamma \in \mathbb{R}_+$ which is chosen by the designer. Let us consider the following DTS

$$z(k+1) = Az(k) + bu(k) \tag{3.4}$$

where $z \in D \subseteq \mathbb{R}^n$ with D as an open subset, $k \in \mathbb{Z}_{\geq 0}$, $u \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^{n\times 1}$. We define the switching function as $s(k) := \wp(z(k))$. The level set $\wp^{-1}(0) := \{z \in D :$

$\wp(z) = 0\}$ defines the sliding hyperplane which we assume to be sufficiently smooth. We further consider that $\wp(z(k)) = c^\top z(k)$ for our analysis. Consider the following switching function as

$$s(k) = c^\top z(k) \tag{3.5}$$

where $c \in \mathbb{R}^{n\times 1}$ with its elements chosen such that $c^\top b \neq 0$.

Definition 3.2 *We say that sliding mode exist on the level set $\wp^{-1}(0)$, if $s(k) = 0$.*

Remark 3.3 *Definition 3.2 has the resemblence with the ideal sliding mode which is defined for CTS. Compared with the existing definitions of the QSM [25, 27], here $s(k)$ perfectly lands on $\wp^{-1}(0)$ and remains there for all future time.*

Remark 3.4 *Definition 3.2 is different from the one proposed in [25], since the RL proposed does not need the system state to cross and recross the sliding hyperplane $\wp^{-1}(0)$ owing to the structure of the proposed RL. As a result, chattering is eliminated and hence needless control effort is avoided.*

Definition 3.5 *DTS (3.4) satisfies the reaching condition of the sliding mode iff for some $k \geq 0$, the following holds:*

$$|s(k)| > \gamma \implies |s(k+1)| < |s(k)| - \nu, \tag{3.6}$$

$$|s(k)| \leq \gamma \implies s(k+1) = 0 \tag{3.7}$$

where $\nu \in \mathbb{R}_+$, is sufficiently small.

To compute $u(k)$ such that (3.3) holds, we utilise (3.4) and (3.5) to compute

$$s(k+1) = c^\top Az(k) + c^\top bu(k) \tag{3.8}$$

On comparing (3.3) and (3.8), control law is obtained as $u(k) = (c^\top b)^{-1}\{-c^\top Az(k) + s(k) - \text{sign}[s(k)]\min\{|s(k)|, \gamma\}\}$. Next, we analyse the proposed RL and the properties it exhibits.

Theorem 3.6 *If in the RL (3.3), $|s(0)| > \gamma$, $\gamma > 0$, then for some $k \geq \mathcal{K}(s_0)$, the absolute value of the sliding function of the system ultimately becomes zero, where $\mathcal{K}(s_0) = \lceil \frac{|s_0|}{\gamma} \rceil$.*

Proof: We rewrite the RL proposed in (3.3): $s(k+1) = s(k) - \text{sign}[s(k)] \min\{|s(k)|, \gamma\}$. If we take the case when $|s(0)| \leq \gamma$, then $s(k+1) = s(k) - \text{sign}[s(k)]|s(k)| \implies s(k+1) = 0$ in just one sample instant, satisfying the conditions given in the Definition 3.5, with $\mathcal{K}(s_0) = 1$ time step. Further, for all $k \geq \mathcal{K}(s_0)$, $s(k) = 0$, depicting the existence of the sliding mode as per the Definition 3.2. We now move on to the case when $|s(0)| > \gamma \implies s(k+1) = s(k) - \text{sign}[s(k)]\gamma$. We can further write, $|s(k+1)| = |s(k) - \text{sign}[s(k)]\gamma|$. Since $|s(k)| > \gamma$, it follows that, $|s(k+1)| \leq |s(k)| - \gamma$. Further, it can be written as

$$\begin{aligned} |s(k)| &\leq |s(k-1)| - \gamma \\ &\leq |s(k-2)| - 2\gamma \\ &\vdots \\ &\leq |s(0)| - k\gamma \end{aligned} \tag{3.9}$$

If $(|s(0)| - k\gamma) \leq \gamma$, then from (3.9), $|s(k)| \leq \gamma \implies s(k+1) = 0$ for all $k \geq \lceil \frac{|s_0|}{\gamma} \rceil$ with the settling time function denoted as $\mathcal{K}(s_0) = \lceil \frac{|s_0|}{\gamma} \rceil$. Hence, the absolute value of $s(k)$ becomes zero after some finite time. □

Next, we move on to the case where system is subjected to matched type bounded perturbation.

3.3.2 RL1 for Perturbed Discrete-Time Systems

Consider the following RL for perturbed DTS

$$s(k+1) = s(k) - \text{sign}[s(k)] \min\{|s(k)|, \gamma\} + c^{\top} b \delta(k) \tag{3.10}$$

where $\delta(k)$ is the matched type bounded perturbation (MTBP) which satisfies $\underline{\delta} \leq \delta(k) \leq \bar{\delta}$, where $\underline{\delta}, \bar{\delta} \in \mathbb{R}$ are known constants. Additionally, the following holds: $\delta_0 = \frac{\underline{\delta}+\bar{\delta}}{2}$, is the mean of $\delta(k)$, and $\delta_d = \frac{\bar{\delta}-\underline{\delta}}{2}$, is the maximum deviation from the mean of $\delta(k)$. $\gamma \in \mathbb{R}_+$ is chosen such that $\gamma \geq \max\{|\underline{\delta}|, |\bar{\delta}|\}$. To analyse the properties of the mentioned RL, let us consider the following perturbed DTS

$$z(k+1) = Az(k) + bu(k) + b\delta(k) \tag{3.11}$$

where $z \in D \subseteq \mathbb{R}^n$ with D as an open subset, $k \in \mathbb{Z}_{\geq 0}$, $u \in \mathbb{R}$, $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^{n \times 1}$ and $\delta(k)$ is the matched type bounded perturbation. We define the switching function as $s(k) := \wp(z(k))$. The level set $\wp^{-1}(0) := \{z \in D : \wp(z) = 0\}$ defines the sliding hyperplane which we assume to be sufficiently smooth. We further consider that $\wp(z(k)) = c^\top z(k)$ for our analysis. Let us consider the $\wp(z(k))$ as in (3.5) with $c \in \mathbb{R}^{n\times 1}$ such that $c^\top b \neq 0$ and $|c^\top b| \leq 1$. For the analysis purpose, consider $\max\{|\underline{\delta}|, |\bar{\delta}|\} = \delta_m$, where $\delta_m \in \mathbb{R}_+$.

Definition 3.7 *We say that the QSM exist in the vicinity of the level set* $\wp^{-1}(0)$*, if* $|s(k)| \leq |c^\top b|\delta_m$.

Definition 3.8 *DTS* (3.11) *satisfies the reaching condition of the QSM in the neighbourhood of* $\wp^{-1}(0)$ *if and only if for some* $k \geq 0$*, the following holds:*

$$|s(k)| > \gamma \implies |s(k+1)| < |s(k)| - \nu, \tag{3.12}$$

$$|s(k)| \leq \gamma \implies 0 \leq |s(k+1)| \leq |c^\top b|\delta_m \tag{3.13}$$

where ν *is a sufficiently small positive constant.*

To compute $u(k)$ such that (3.10) holds, we utilise (3.11) and (3.5) to compute

$$s(k+1) = c^\top Az(k) + c^\top bu(k) + c^\top b\delta(k) \tag{3.14}$$

On comparing (3.10) and (3.14), control law is obtained as $u(k) = (c^\top b)^{-1}\{-c^\top Az(k) + s(k) - \text{sign}[s(k)]\min\{|s(k)|, \gamma\}\}$. Next, the proposed theorem aims to describe the properties of the aforementioned RL.

Theorem 3.9 *If in the RL* (3.10)*,* $|s(0)| > \gamma$*,* $\gamma > 0$*, and* $|c^\top b| \leq 1$*, then for some* $k \geq \mathcal{K}(s_0)$*, the absolute value of the sliding function of the system is ultimately bounded by* $|c^\top b|\delta_m$*, where* $\mathcal{K}(s_0) = \lceil \frac{|s_0| - |c^\top b|\delta_m}{\gamma - |c^\top b|\delta_m} \rceil$.

Proof: Rewriting the RL proposed in (3.10): $s(k+1) = s(k) - \text{sign}[s(k)]\min\{|s(k)|, \gamma\} + c^\top b\delta(k)$. If we take the case when $|s(0)| \leq \gamma$, then $s(k+1) = s(k) - \text{sign}[s(k)]|s(k)| + c^\top b\delta(k) \implies s(k+1) = c^\top b\delta(k)$ in just one sample instant. Further, it follows that $|s(k+1)| \leq |c^\top b|\delta_m$, satisfying the conditions given in the Definition 3.8, with $\mathcal{K}(s_0) = 1$ time step. Further, for all $k \geq \mathcal{K}(s_0)$, $|s(k)| \leq |c^\top b|\delta_m$, depicting the existence of the QSM as per the Definition 3.7. We now move on to the case when $|s(0)| > \gamma \implies s(k+1) = s(k) - \text{sign}[s(k)]\gamma + c^\top b\delta(k)$. We can further write, $|s(k+1)| = |s(k) - \text{sign}[s(k)]\gamma + c^\top b\delta(k)|$. Since $|s(k)| > \gamma$, it follows that, $|s(k+1)| \leq |s(k)| - \gamma + |c^\top b|\delta_m$. Further, it can be written as

$$
\begin{aligned}
|s(k)| &\leq |s(k-1)| - \gamma + |c^\top b|\delta_m \\
&\leq |s(k-2)| - 2(\gamma - |c^\top b|\delta_m) \\
&\vdots \\
&\leq |s(0)| - k(\gamma - |c^\top b|\delta_m)
\end{aligned}
\tag{3.15}
$$

If $(|s(0)| - k(\gamma - |c^\top b|\delta_m)) \leq \gamma$, then from (3.15), $|s(k)| \leq \gamma \implies |s(k+1)| \leq |c^\top b|\delta_m$ for all $k \geq \lceil \frac{|s_0| - |c^\top b|\delta_m}{\gamma - |c^\top b|\delta_m} \rceil$ with the settling time function denoted as $\mathcal{K}(s_0) = \lceil \frac{|s_0| - |c^\top b|\delta_m}{\gamma - |c^\top b|\delta_m} \rceil$. Hence, it follows that, absolute value of the sliding function is ultimately bounded by $|c^\top b|\delta_m$ after some finite time. This completes the proof. □

Remark 3.10 *It is important to note that the width of the ultimate band ($|s(k)| \leq |c^\top b|\delta_m \leq \delta_m$) can be significantly reduced by proper selection of vector c ensuring $|c^\top b| \leq 1$.*

3.3.3 RL2 for Unperturbed Discrete-Time System

Let us consider the following RL for unperturbed DTS

$$
s(k+1) = s(k) - \gamma \text{sign}[s(k)] \min \left\{ \frac{|s(k)|}{\gamma}, |s(k)|^{\varpi} \right\}
\tag{3.16}
$$

where $\gamma \in \mathbb{R}_+$ is a scalar constant which is chosen by the designer, $\varpi \in (0,1)$.

Remark 3.11 *In RL* (3.16)*, if γ is set to* 1 *and ϖ is kept very close to zero, then* (3.16) *behaves similar to RL* (3.3) *with $\gamma = 1$.*

Remark 3.12 *The interplay between γ and ϖ affects the convergence of the sliding variable as shown in Figure 3.3. The bending of curves closer to $s(k) = 0$, indicates the slower convegence, as can be seen for curves in blue and red for which γ and ϖ have small values. Keeping ϖ close to* 1 *and γ large will cause the faster convergence of the sliding variable. Moreover, the selection of γ and ϖ for different problems can be done based on the actuator saturation limit of these systems.*

Let us consider the DTS as in (3.4) and switching function as in (3.5). The existence of the sliding mode is defined as per the Definition 3.2.

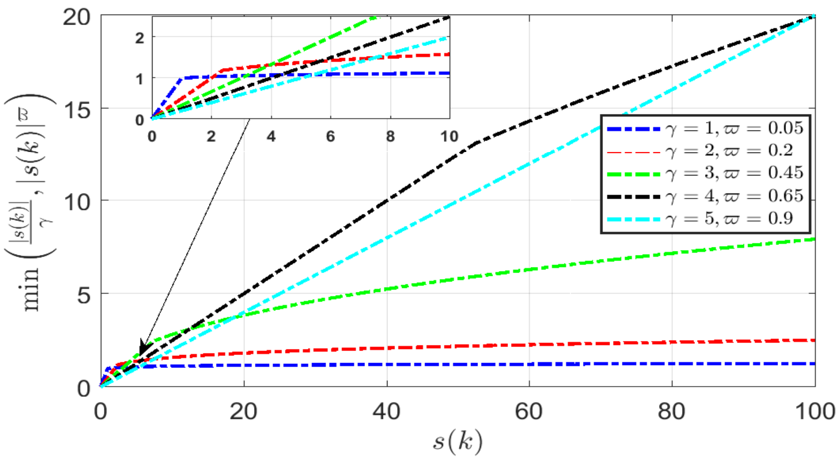

Figure 3.3: Effects of variation in γ and ϖ

Definition 3.13 *DTS* (3.4) *satisfies the reaching condition of the sliding mode if and only if for some* $k \geq 0$*, the following holds:*

$$\frac{|s(k)|}{\gamma} > |s(k)|^{\varpi} \implies |s(k+1)| < |s(k)| - \nu, \tag{3.17}$$

$$\frac{|s(k)|}{\gamma} \leq |s(k)|^{\varpi} \implies s(k+1) = 0 \tag{3.18}$$

where ν *is a sufficiently small positive constant.*

We utilise (3.4) and (3.5) to compute $u(k)$ such that (3.16) is satisfied,

$$s(k+1) = c^{\top}Az(k) + c^{\top}bu(k) \tag{3.19}$$

On comparing (3.16) and (3.19), required control law is obtained as, $u(k) = (c^{\top}b)^{-1}$ $\left\{-c^{\top}Az(k) + s(k) - \gamma \text{sign}[s(k)] \min\left\{\frac{|s(k)|}{\gamma}, |s(k)|^{\varpi}\right\}\right\}$. We analyse the proposed RL and the properties it exhibits.

Theorem 3.14 *If in the RL* (3.16)*,* $\frac{|s(0)|}{\gamma} > |s(0)|^{\varpi}$*,* $\gamma > 0$*,* $\varpi \in (0, 1)$*, then for some* $k \geq \mathcal{K}(s_0)$*, the absolute value of the sliding function of the system ultimately becomes zero, where* $\mathcal{K}(s_0) \leq \left\lceil \log_{[1-\gamma|s_0|^{\varpi-1}]} \frac{\gamma^{\frac{1}{1-\varpi}}}{|s_0|} \right\rceil + 1$*.*

Proof: We rewrite the RL proposed in (3.16): $s(k+1) = s(k) - \gamma \text{sign}[s(k)]$ $\min\left\{\frac{|s(k)|}{\gamma}, |s(k)|^{\varpi}\right\}$. If we take the case when $\frac{|s(0)|}{\gamma} \leq |s(0)|^{\varpi}$, then $s(k+1) = s(k) - \text{sign}[s(k)]|s(k)| \implies s(k+1) = 0$ in just one sample instant, satisfying the conditions given in the Definition 3.13, with $\mathcal{K}(s_0) = 1$ time step. Further, for all $k \geq \mathcal{K}(s_0)$, $s(k) = 0$, depicting the existence of the sliding mode as per the Definition 3.2. We now move on to the case when $\frac{|s(0)|}{\gamma} > |s(0)|^{\varpi} \implies s(k+1) = s(k) - \gamma \text{sign}[s(k)]|s(k)|^{\varpi}$. We can further write, $|s(k+1)| = |s(k) - \gamma \text{sign}[s(k)]|s(k)|^{\varpi}|$. Since $|s(k)| > \gamma|s(k)|^{\varpi}$, hence, $|s(k+1)| \leq |s(k)| - \gamma|s(k)|^{\varpi}$. Further,

$$\begin{aligned}
|s(k)| &\leq |s(k-1)|(1-\gamma|s(k-1)|^{\varpi-1}) \\
&\leq |s(k-2)|(1-\gamma|s(k-2)|^{\varpi-1})(1-\gamma|s(k-1)|^{\varpi-1}) \\
&\vdots \\
&\leq |s(0)|(1-\gamma|s(0)|^{\varpi-1})\cdots(1-\gamma|s(k-1)|^{\varpi-1})
\end{aligned}$$

Since, $\varpi \in (0,1)$, $(1-\gamma|s(k-1)|^{\varpi-1}) < (1-\gamma|s(k-2)|^{\varpi-1})$ and so on. Then,

$$|s(k)| \leq |s(0)|(1-\gamma|s(0)|^{\varpi-1})^k \tag{3.20}$$

If $\left|(|s(0)|(1-\gamma|s(0)|^{\varpi-1})^k)\right| \leq \gamma|s(k)|^{\varpi}$, then from (3.20), $|s(k)| \leq \gamma|s(k)|^{\varpi} \implies s(k+1) = 0$ for all $k \geq \left\lceil \log_{[1-\gamma|s_0|^{\varpi-1}]} \frac{\gamma^{\frac{1}{1-\varpi}}}{|s_0|} \right\rceil + 1$ with the settling time function denoted as $\mathcal{K}(s_0) \leq \left\lceil \log_{[1-\gamma|s_0|^{\varpi-1}]} \frac{\gamma^{\frac{1}{1-\varpi}}}{|s_0|} \right\rceil + 1$. Hence, it follows that, absolute value of the sliding function becomes zero after some finite time. This completes the proof. □

3.3.4 RL2 for Perturbed Discrete-Time Systems

Consider the following RL for perturbed DTS

$$s(k+1) = s(k) - \gamma \text{sign}[s(k)] \min\left\{\frac{|s(k)|}{\gamma}, |s(k)|^{\varpi}\right\} + c^{\top} b \delta(k) \tag{3.21}$$

where $\delta(k)$ is the matched type bounded perturbation which satisfies $\underline{\delta} \leq \delta(k) \leq \bar{\delta}$, where $\underline{\delta}, \bar{\delta} \in \mathbb{R}$ are known constants. Additionally, the following holds: $\delta_0 = \frac{\underline{\delta}+\bar{\delta}}{2}$, is the mean of $\delta(k)$, and $\delta_d = \frac{\bar{\delta}-\underline{\delta}}{2}$, is the maximum deviation from the mean of $\delta(k)$. $\gamma \in \mathbb{R}_+$ is a scalar constant which is chosen such that it satisfies $\gamma \geq \max\{|\underline{\delta}|, |\bar{\delta}|\}$. To analyse the properties of the mentioned RL, let us consider the perturbed DTS as in (3.11) and switching function as in (3.5). Let the existence of QSM is defined as per the Definition 3.7.

Definition 3.15 *DTS* (3.11) *satisfies the reaching condition of the QSM in the neighbourhood of* $\wp^{-1}(0)$ *if and only if for some* $k \geq 0$*, the following holds:*

$$\frac{|s(k)|}{\gamma} > |s(k)|^{\varpi} \implies |s(k+1)| < |s(k)| - \nu, \tag{3.22}$$

$$\frac{|s(k)|}{\gamma} \leq |s(k)|^{\varpi} \implies 0 \leq |s(k+1)| \leq |c^{\top} b| \delta_m \tag{3.23}$$

where ν is a sufficiently small positive constant.

To compute $u(k)$ such that (3.21) holds, we utilise (3.11) and (3.5) as, $u(k) = (c^\top b)^{-1} \{ -c^\top Az(k) + s(k) - \gamma \text{sign}[s(k)] \min \left\{ \frac{|s(k)|}{\gamma}, |s(k)|^{\varpi} \right\} \}$. Next theorem aims to describe the properties of the RL (3.21).

Theorem 3.16 *If in the RL* (3.21), $\frac{|s(0)|}{\gamma} > |s(0)|^{\varpi}$, $\gamma > 0$, $\varpi \in (0,1)$, *and* $|c^\top b| \leq 1$, *then for some* $k \geq \mathcal{K}(s_0)$, *the absolute value of the sliding function of the system is ultimately bounded by* $|c^\top b|\delta_m$, *where* $\mathcal{K}(s_0) \leq \left\lceil \log_{[1-\gamma|s_0|^{\varpi-1}+|c^\top b|\delta_m|s_0|^{-1}]} \frac{\gamma^{\frac{1}{1-\varpi}}}{|s_0|} \right\rceil + 1$.

Proof: Rewriting the RL proposed in (3.21): $s(k+1) = s(k) - \gamma \text{sign}[s(k)] \min \left\{ \frac{|s(k)|}{\gamma}, |s(k)|^{\varpi} \right\} + c^\top b\delta(k)$. If we take the case when $\frac{|s(0)|}{\gamma} \leq |s(0)|^{\varpi}$, then $s(k+1) = s(k) - \text{sign}[s(k)]|s(k)| + c^\top b\delta(k) \implies s(k+1) = c^\top b\delta(k)$ in just one sample instant. Further, it follows that $|s(k+1)| \leq |c^\top b|\delta_m$, satisfying the conditions given in the Definition 3.15, with $\mathcal{K}(s_0) = 1$ time step. Further, for all $k \geq \mathcal{K}(s_0)$, $|s(k)| \leq |c^\top b|\delta_m$, depicting the existence of the QSM as per the Definition 3.7. We now move on to the case when $\frac{|s(0)|}{\gamma} > |s(0)|^{\varpi} \implies s(k+1) = s(k) - \gamma \text{sign}[s(k)]|s(k)|^{\varpi} + c^\top b\delta(k)$. We can further write, $|s(k+1)| = |s(k) - \gamma \text{sign}[s(k)]|s(k)|^{\varpi} + c^\top b\delta(k)|$. Since $|s(k)| > |s(k)|^{\varpi}$, it follows that, $|s(k+1)| \leq |s(k)| - \gamma|s(k)|^{\varpi} + |c^\top b|\delta_m$. Further, it can be written as

$$\begin{aligned}
|s(k)| &\leq |s(k-1)|(1-\gamma|s(k-1)|^{\varpi-1} + |c^\top b|\delta_m|s(k-1)|^{-1}) \\
&\leq |s(k-2)|(1-\gamma|s(k-2)|^{\varpi-1} + |c^\top b|\delta_m|s(k-2)|^{-1}) \\
&(1-\gamma|s(k-1)|^{\varpi-1} + |c^\top b|\delta_m|s(k-1)|^{-1}) \\
&\vdots \\
&\leq |s(0)|(1-\gamma|s(0)|^{\varpi-1} + |c^\top b|\delta_m|s(0)|^{-1}) \cdots \\
&(1-\gamma|s(k-1)|^{\varpi-1} + |c^\top b|\delta_m|s(k-1)|^{-1})
\end{aligned} \tag{3.24}$$

Since, $\varpi \in (0,1)$, $(1-\gamma|s(k-1)|^{\varpi-1} + |c^\top b|\delta_m|s(k-1)|^{-1}) < (1-\gamma|s(k-2)|^{\varpi-1} + |c^\top b|\delta_m|s(k-2)|^{-1})$ and so on. Thus, it follows

$$|s(k)| \leq |s(0)|(1-\gamma|s(0)|^{\varpi-1} + |c^\top b|\delta_m|s(0)|^{-1})^k \tag{3.25}$$

If $\left||s(0)|(1-\gamma|s(0)|^{\varpi-1} + |c^\top b|\delta_m|s(0)|^{-1})^k\right| \leq \gamma|s(k)|^{\varpi}$, then from (3.25), $|s(k)| \leq \gamma|s(k)|^{\varpi}$ $\implies |s(k+1)| \leq |c^\top b|\delta_m$ for all $k \geq \left\lceil \log_{[1-\gamma|s_0|^{\varpi-1}+|c^\top b|\delta_m|s_0|^{-1}]} \frac{\gamma^{\frac{1}{1-\varpi}}}{|s_0|} \right\rceil + 1$, with the settling time function denoted as $\mathcal{K}(s_0) \leq \left\lceil \log_{[1-\gamma|s_0|^{\varpi-1}+|c^\top b|\delta_m|s_0|^{-1}]} \frac{\gamma^{\frac{1}{1-\varpi}}}{|s_0|} \right\rceil + 1$. Hence, it follows that, absolute value of the sliding function is ultimately bounded by $|c^\top b|\delta_m$ after some finite time. This completes the proof. $\square$

Remark 3.17 *The key difference between RL1 and RL2 is the number of design parameters, which is more in the latter. Additionally, in RL2, both arguments of minimum function are continuous functions which makes the evolution of sliding variable comparatively more smooth.*

3.4 Illustrative Example

In this section, we design the proposed RL based control law for a perturbed pendulum system. Consider the following system

$$\begin{aligned} z_1(k+1) &= z_1(k) + z_2(k) \\ z_2(k+1) &= z_2(k) - \frac{mgl}{2J}\sin(z_1(k)) - \frac{B}{J}z_2(k) + \frac{1}{J}u(k) + \delta(k) \end{aligned} \tag{3.26}$$

where $z_1(k)$ is the pendulum angle from the mean position, $z_2(k)$ is the angular velocity, m is the mass of the pendulum, g is the acceleration due to gravity, l is the length of the pendulum, J is the inertia of the pendulum arm, B is the friction coefficient and $u(k)$ is the control input. The bounded perturbation $\delta(k)$ is considered as 0.1 $\sin(0.3k)$. The control law is designed based on RL1 and RL2. For the simulation purpose, the system parameters are taken as follows: $m = 1.1kg$, $l = 1m$, $g = 9.81\frac{m}{s^2}$, $B = 0.18\frac{kg\cdot m}{s^2}$. The system initial conditions and design parameters are taken as $z_1(0) = 5$, $z_2(0) = 2$, $\gamma = 6$, $c_1 = 0.3$, $c_2 = 1$ and $\alpha = 0.5$. Figures 3.4- 3.7, pertain to RL1. The evolution of the system states for the unperturbed case can be seen in Figure 3.4, while the sliding variable and control input can be seen from Figure 3.5. One can see that sliding variable as well as control input becomes zero in some finite time. For the perturbed case, the evolution of states, sliding variable and control input can be seen in Figure 3.6 and Figure 3.7. Due to the presence of perturbation, the states and sliding variable remain within a bound for all time, as a consequence, the control input is also non-zero. Figures 3.8- 3.11 pertain to RL2. Similar to the RL1, the states, sliding variable and control input become zero as seen from Figure 3.8 and Figure 3.9. For the perturbed case (Figure 3.10 and Figure 3.11), states and sliding variable remain in an invariant set while the control remains non-zero.

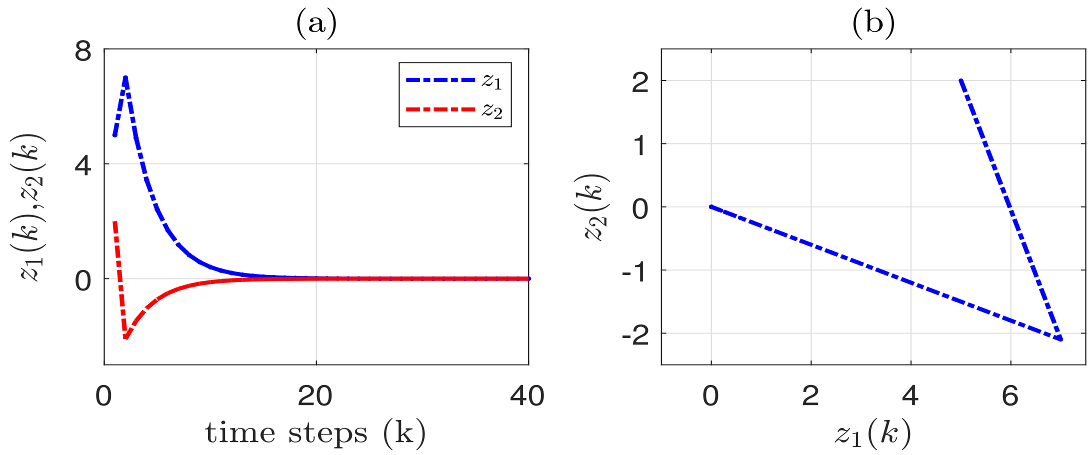

Figure 3.4: States and phase portrait with RL1, for unperturbed DTS

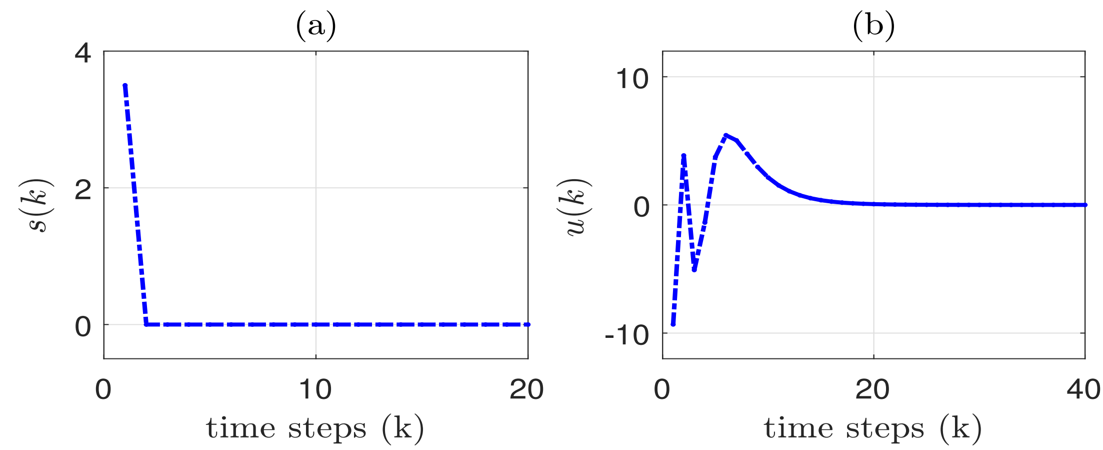

Figure 3.5: Sliding variable and control input with RL1, for unperturbed DTS

3.5 Conclusion

In this chapter, an in-depth exploration of reaching laws for discrete variable structure systems based on the difference equation with minima was undertaken. This new approach seeks to overcome the limitations inherent in both Gao's reaching law and Utkin's equivalent control-based approach. The proposed reaching laws aim to address two critical issues: the complete elimination of chattering and the avoidance of large control

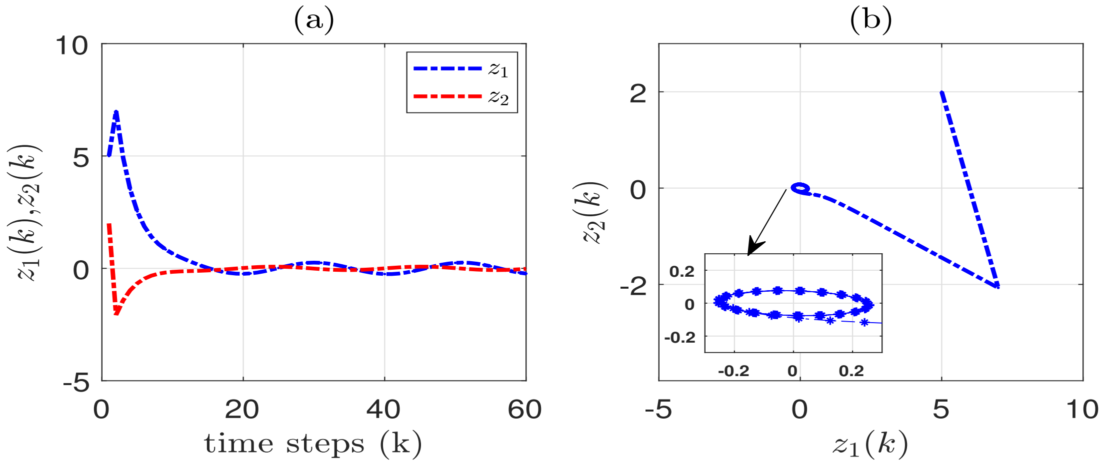

Figure 3.6: States and phase portrait with RL1, for perturbed DTS

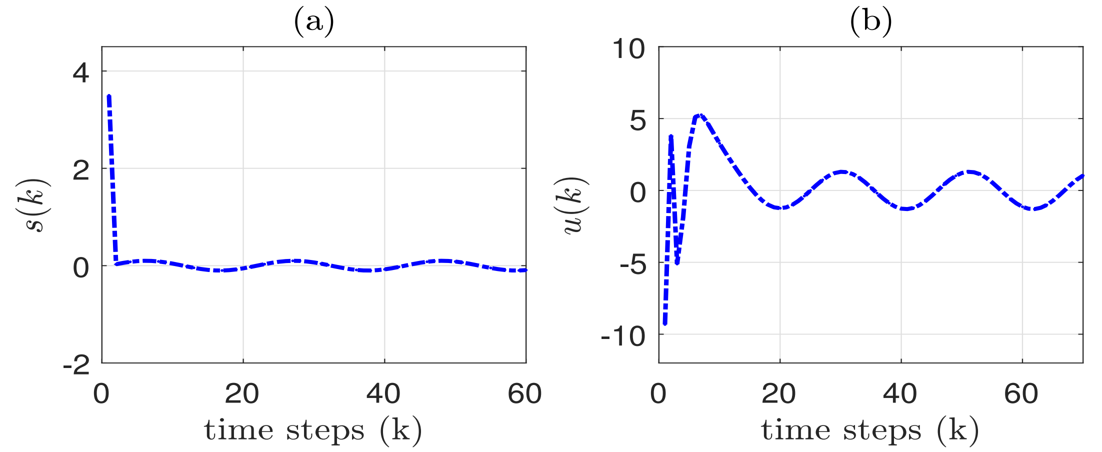

Figure 3.7: Sliding variable and control input with RL1, for perturbed DTS

actions. To illustrate the effectiveness of these reaching laws, simulation results are presented, utilizing the example of a pendulum system. The simulations encompass scenarios with both unperturbed and perturbed conditions, providing a comprehensive evaluation of the proposed methods. The results obtained through simulation serve as a compelling demonstration of the efficacy and robustness of the introduced reaching laws.

In the upcoming chapter, the scope of reaching laws based on the difference equation with minima is expanded to include a variable gain introduced through the proportional

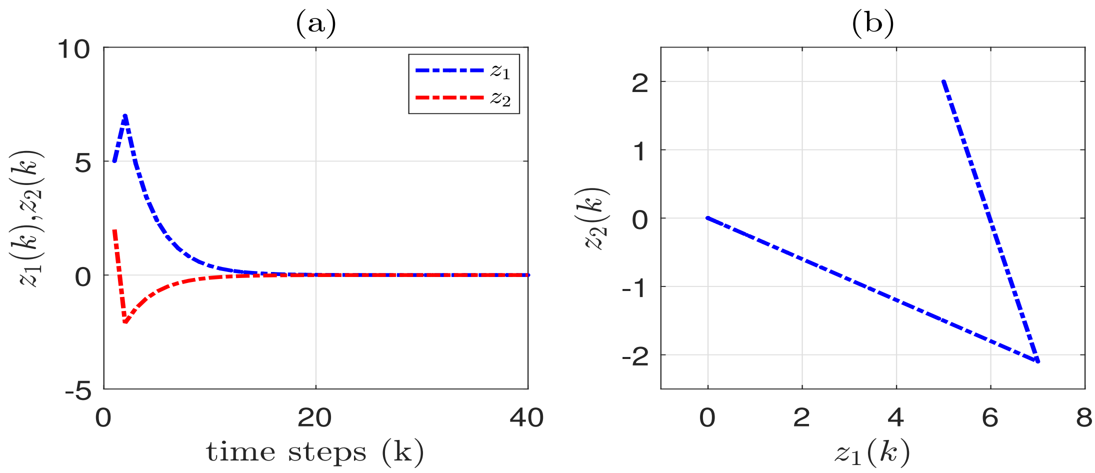

Figure 3.8: States and phase portrait with RL2, for unperturbed DTS

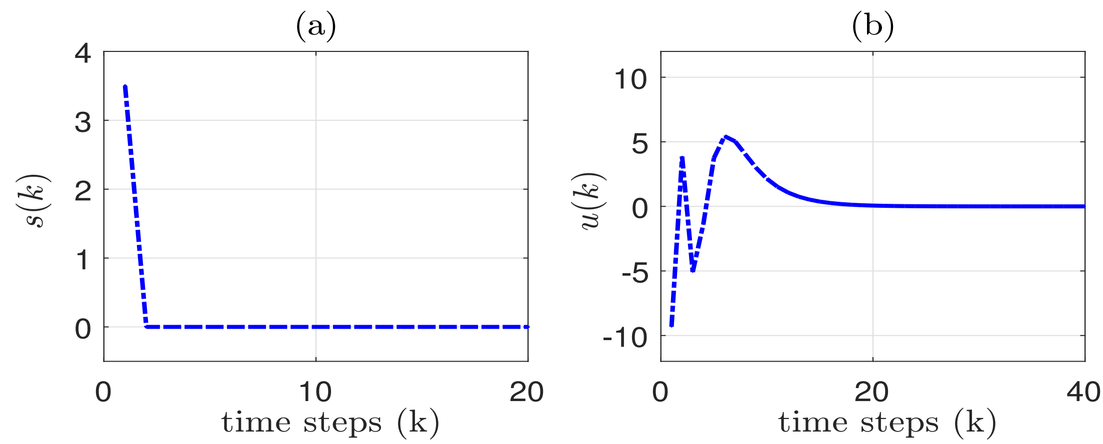

Figure 3.9: Sliding variable and control input with RL2, for unperturbed DTS

term, utilizing a rate-regulatory function. This extension aims to address the challenge of slow convergence rates encountered in the present approach.

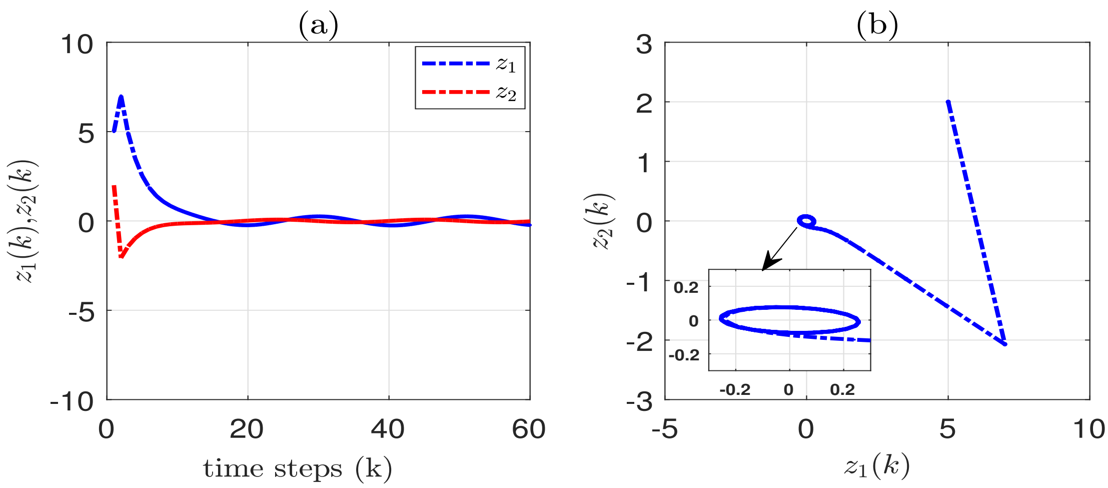

Figure 3.10: States and phase portrait with RL2, for perturbed DTS

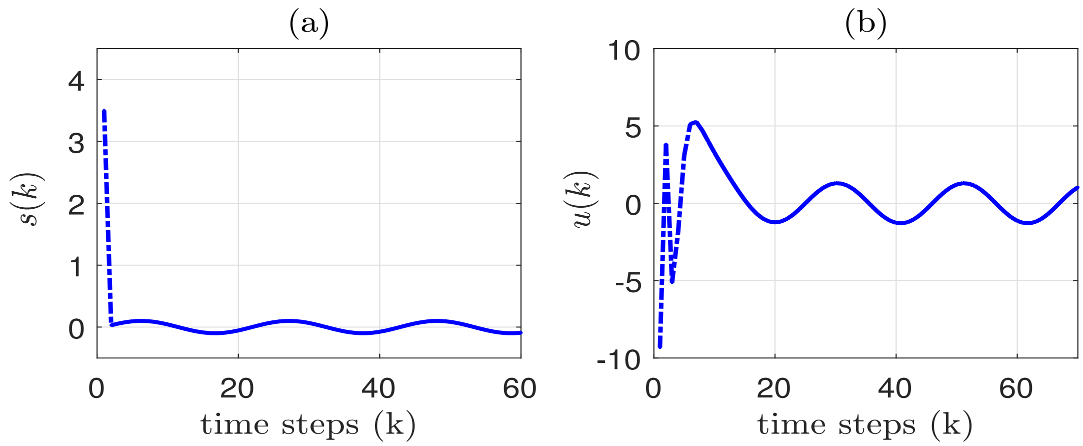

Figure 3.11: Sliding variable and control input with RL2, for perturbed DTS