THERMOELASTIC RESPONSES DUE TO DIFFERENT HEAT SOURCES UNDER NON-CLASSICAL HEAT CONDUCTION

Thesis Submitted in Partial Fulfillment For the Award of Degree

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by

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I, **Robin Vikram Singh**, certify that the work embodied in this thesis is my own bonafide work and carried out by me under the supervision of **Prof. Santwana Mukhopadhyay** from July, 2017 to December, 2023 at the **Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi**. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not willfully copied any other's work, paragraphs, text, data, results, etc., reported in journals, books, magazines, reports dissertations, theses, etc., or available at websites and have not included them in this thesis and have not cited as my own work.

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ROBIN VIKRAM SINGH

ABBREVIATIONS

DPL	Dual-Phase-Lag
ETE	Extended Thermoelasticity
GN	Green-Naghdi
LS	Lord-Shulman
TPL	Three-Phase-Lag
GL	Green-Lindsay
MGL	Modified Green-Lindsay
TRDTT	Temperature-Rate Dependent Thermoelasticity Theory
CTE	Classical Coupled Thermoelasticity
MGT	Moore-Gibson-Thompson
QMGT	Quintanilla-Moore-Gibson-Thompson

LIST OF FIGURES

LIST OF SYMBOLS

σ_{ij}	Component of stress tensor
u_i	Displacement vector
e_{ij}	Component of strain tensor
q_i	Heat flux vector,
T_0	Uniform reference temperature
T	Temperature
θ	Temperature variation from the uniform reference temperature
$\nu_{,i}$	Thermal displacement
b_i	Body force per unit mass
Q	Heat source per unit mass
S	Entropy per unit mass
s	Laplace transform parameter
ρ	Mass density
$K_{ij}/\ K$	Thermal conductivity tensor/ constant
$K^*_{ij}/\ K^*$	Thermal conductivity rate tensor/ rate constant,
C_{ijkl}	Elasticity tensor

λ, μ	Lame's constants of material,
γ_{ij}	Thermoelasticity tensor
$\gamma = (3\lambda + 2\mu)\alpha_t$	Thermoelasticity constant
α_t	Linear thermal expansion coefficient of the material
c_E	Specific heat at constant strain
τ_θ	Phase-lag of temperature gradient
τ_q	Phase-lag of heat-flux vector
τ_1 and τ_2	Thermal relaxation parameters
$\boldsymbol{x} = (x_1, x_2, x_3)/(x, y, z)$	Cartesian coordinates
t	Physical time
$\boldsymbol{\nabla}$	Gradient operator
∇^2/Δ	Laplacian operator
δ	Dirac delta
δ_{ij}	Kronecker delta

Note: Throughout the thesis, the subscripted comma notations are used to denote the partial derivatives with respect to the space variables. The over-headed dots denote partial derivatives with respect to time variable, t. Subscripts i, j, k, l take the values 1, 2, 3 and summation is implied by index repetition.

PREFACE

The subject “coupled thermoelasticity” explores the mutual interactions between the thermal and mechanical fields in elastic materials subjected to thermomechanical loading effects. The theory of coupled thermoelasticity, therefore, provides a framework for understanding how temperature changes influence the mechanical behavior of solids and structures and vice versa. It is particularly important in engineering, materials science, and geophysics, where accurate predictions of the thermal and mechanical responses of materials and structures are essential. Constitutive equations in thermoelasticity describe the relationship between stress, strain, and temperature. These equations incorporate material properties such as thermal expansion coefficient, elastic moduli, and thermal conductivity. The first studies on the topic were conducted using the concept of "uncoupled theory of thermoelasticity," which made the simplistic assumption that the impact of strain and stresses on the temperature field could be avoided. The heat conduction equation for uncoupled thermoelasticity turns out impractical without the inclusion of an elasticity element. This is because the mechanical loading of an elastic body results in strain, which in turn affects the temperature field. Furthermore, the conventional heat conduction equation of this uncoupled theory is a parabolic type partial differential equation that yields an infinite thermal wave transmission velocity, hence contradicting observed physical facts. Therefore, there has been a significant focus on coupled thermoelasticity in recent years, mostly owing to its extensive use in the fields of science and technology. Biot (1956) made a significant advancement in the field by introducing the concept of coupled thermoelasticity. Nevertheless, despite a successful elimination of the first drawback of the uncoupled theory, the presence of a parabolic-type partial differential equation for heat conduction still persists, hence giving rise to the infinite velocity of the thermal wave. Research interest in this area, therefore, concentrated towards the modification of Biot's theory. In recent years, there has been introduced an increasing variety of generalized models in the field of thermoelasticity. The primary objective of these models is to fit mathematical models in a realistic way to explain various practical problems.

At its core, generalized Thermoelasticity incorporates additional parameters and considerations, offering a more realistic depiction of material response. The versatility of generalized thermoelasticity is reflected in its diverse range of applications. From the microscopic scale of nanostructured materials to the macroscopic scale of seismic events, these theories find relevance in understanding the coupled thermal and mechanical responses. Applications of this field extend to fields such as biomedical engineering, aerospace, materials science, and beyond. In structural engineering, generalized thermoelasticity offers wide applications to analyze the impact of temperature changes on the performance of materials. Meanwhile, in biomedical applications, it plays a vital role in understanding the thermal and mechanical responses of biological tissues during therapeutic interventions and diagnostic procedures. Out of some generalized coupled thermoelastic theories, five models are well studied namely, Lord-Shulman model (1967), Green-Lindsay model (1972), Green-Naghdi models (1991a; 1992; 1993), dual-phase lag (Tzou (1995b)) and Three-phase lag model (RoyChoudhuri (2007a)), which are extensively used in modeling and simulation of physical phenomena. The Lord-Shulman model (LS) is based on Cattaneo-Vernotte heat conduction model, and it involves one thermal relaxation time parameter in heat flux related constitutive equation. Green-Lindsay model (also called as temperature rate-dependent model) is based on entropy balance inequality, and here two relaxation time parameters are introduced in the constitutive equations that include the temperature rate terms in addition. Further, in 1990-95 Green and Naghdi advocated an alternate form of generalized thermoelasticity theory by introducing new constitutive variable, thermal displacement in Fourier's law. Tzou independently introduced the dual phase-lag heat conduction model, which is a modification of Fourier's law by considering time delay terms in temperature gradient and heat flux. This model explains the non-equilibrium thermodynamic transition and impact of microscopic structure in heat conduction process by macroscopic formulation. Later on, Chandrasekhariah (1998) developed the thermoelasticity theory based on this dual phase-lag heat conduction model. Yu et al. (2018) have introduced a modification to the Green-Lindsay thermoelastic model. This modification involves the inclusion of a strain-rate component, which effectively removes the discontinuity in the displacement field, as reported by several researchers in Green-Lindsay model. The constitutive equations in strain rate-dependent thermoelasticity involve terms that couple the strain rate, temperature rate change, and stress. These equations take into account the fact that the material response depends not only on temperature changes but also on how quickly the deformation is applied. The effect of strain rate on thermoelasticity is an important consideration in understanding the material behavior under different loading conditions. The response of a material to mechanical deformation and temperature changes can be influenced by the rate at which the deformation occurs. Higher strain rates can result in a more rapid conversion of mechanical work into heat due to the thermal inertia of the material. Faster strain rates can lead to higher temperature rises within the material due to the conversion of mechanical energy into heat. In some materials, especially polymers or certain metals, high strain rates can induce phase changes that affect the overall thermoelastic behavior. For example, phase transitions such as melting or solidification can be influenced by the rate of deformation. However, the strain rate can also influence failure mechanisms in a material. For instance, at high strain rates, failure modes such as dynamic fracture or fragmentation may become more prevalent, impacting the overall thermoelastic behavior in a significant way. Understanding the interplay between strain rate and thermoelasticity is crucial in various engineering applications, such as impact analysis, crash simulations, and high-speed machining, where materials experience rapid and dynamic loading conditions.

The theory of Green and Naghdi is divided into three parts that can describe a wider class of problems of coupled thermoelasticity. These three parts of their theory correspond to three different models, which are subsequently called as GN-I (Green and Naghdi (1991a)), GN-II (Green and Naghdi (1992)) and GN-III (Green and Naghdi (1993)) models. Out of the three theories developed by Green and Naghdi, the GN-I model is equivalent to the CTE model when one considers the linearized form of this model. The temperature and the gradient of temperature are considered as the independent thermal variables in the theories of Green and Naghdi. For GN-II and GN-III models, thermal displacement has a great role. In GN-II model, it has been deduced that there is no energy dissipation and this model has been investigated to predict the finite propagation of thermal signals. Therefore, this theory is referred to as the thermoelasticity theory without energy dissipation. Moreover, the temperature, the gradient of the temperature, and the gradient of thermal displacement are taken as the independent thermal variables in the GN-III model. This model includes both GN-I and GN-II as limiting cases. The exponential decay of solutions has been obtained in this case. However, this theory is reported to predict the instantaneous propagation of thermal waves and has the same drawback as the conventional theory. To overcome the apparent shortcoming inherent in this GN-III model, the concept of a thermal relaxation parameter is introduced in the heat conduction law of the GN-III model by Quintanilla (2019), who introduced a totally innovative theory of thermoelasticity which is known as the Moore-Gibson-Thompson thermoelasticity theory or QMGT theory.

The QMGT and MGL generalized thermoelasticity theories have been introduced very recently and yet to get attention. From a mathematical perspective, these theories are as simple as LS and GL models, which are developed to overcome the apparent drawback of Biot's theory. The present study is focused on identifying some important aspects of these theories and aims to explore the responses of various types of heat sources and thermomechanical loading in thermoelastic media under these models. The work is carried out to understand the significance of applying these two thermoelasticity theories to investigate the thermoelastic interactions due to heat sources and hence attempts are also made to compare the results of these theories with the corresponding results found in existing literature. This thesis is divided into three parts on this basis. The first part discusses MGL model, which includes Chapters 2-4, whereas the second part (Chapter 5) is based on application of the model in the study of skin tissue under thermal ablation under MGL thermoelasticity theory. Further, Chapter 6, the last part of the thesis, includes a detailed study of a thermoelastic problem under the Quintanilla-Moore-Gibson thermoelasticity model in the presence of continuous line heat source. Therefore, this work aims to understand the nature of physical fields when the conventional theory is altered using either modified heat conduction law or other reformed constitutive relations.

The outline of the thesis is as follows:

Chapter 1 provides an introduction to the subject related to the thesis. This work provides an overview of the historical progression of the thermoelasticity theory, followed by a comprehensive review of relevant publications in the field. Finally, it ends up with the objective of the thesis.

Chapters 2-5 emphasize the features of the recently developed generalized thermoelasticity theory by Yu et al. (2018). This theory is referred to as modified Green-Lindsay (MGL) theory. With the help of extended thermodynamics, authors have established this thermoelastic model by including the strain-rate and temperature-rate terms in the constitutive relations. This model is also an attempt to remove the discontinuity in the displacement field under temperature rate-dependent thermoelasticity theory as reported by several researchers. This theory is also referred to as the strain and temperature rate-dependent thermoelasticity theory. **Chapter 2** investigates the MGL model in a detailed way to understand the nature of coupling effects of thermoelastic interactions on a linear, homogeneous and isotropic unbounded elastic medium with a cylindrical cavity when the boundary is subjected to thermal shock. To study this problem, we merge the governing equations of the strain and temperature-rate dependent theory of thermoelasticity with two other generalized thermoelasticity theories (namely GL and LS theories) by unified governing equations. In order to solve the problem, we apply the Laplace transform technique to the governing equations. We execute inverse Laplace transform by using short-time approximation method and obtain the expressions for displacement, temperature and stress fields. We enumerate the point of discontinuity and analyse the solution of the field variables separately in the context of three models. We observe that the analytical results for different field variables predicted by MGL model show an infinite speed of disturbance.

Chapter 3, consisting of two subchapters, further deals with the study of MGL theory by considering two different problems of thermoelastic interactions due to the presence of different heat sources and provides the analytical as well as numerical results of the problems. A detailed analysis of the results predicted by this model is presented**. Subchapter 3.1** investigates the effect of line heat source on a linear, homogeneous and isotropic unbounded elastic medium, which is situated at its center (origin). The main motive here is to investigate a detailed analysis of the modified Green Lindsay Model in the presence of thermal line heat source by finding short-time approximated solutions of the field variables. A significant disagreement is observed in the prediction by the present MGL thermoelastic theory as compared to the other existing theories. **Subchapter 3.2** further attempts to investigate the MGL theory and solves a thermoelastic problem where an unbounded medium with a cylindrical cavity is exposed to an external moving heat source along the radial direction in the absence of body force. The inner boundary of cavity is maintained at constant temperature in the absence of traction force along the radial direction on inner surface. This problem is solved using Laplace transformation and the final results for field variables are obtained numerically using Stehfest method (1970) for inversion of Laplace transforms. The predictions are analyzed via graphical results to mark the effects of the relaxation time parameters, which is characteristic of the present generalized thermoelasticity theory. A prominent impact of these parameters is observed in all the physical fields, i.e., displacement, temperature and stress fields. The observations for MGL model are compared graphically with the outcomes of LS and GL models to notice the differences in results. Further, the impact of velocity of heat source is examined.

The aim of **Chapter 4** is to develop a mathematical model to investigate the stresses and temperature that are induced layer-by-layer within the triple-layered skin tissue during a thermal loading by employing the Modified Green-Lindsay thermoelastic model. The present problem is formulated in a unified way to derive the governing equations and constitutive relations under the MGL model and DPL thermoelastic model that involves two phase-lag times. We apply Laplace transform technique and then we adopt Stehfest algorithm (1970) to calculate the temperature, displacement and thermal stress distributions in the skin tissue. The MGL model predicts significantly different temperature and thermal stress distribution in tissues in comparison to DPL heat conduction model.

In **Chapter 5**, we establish the boundary integral equations formulation for the homogenous isotropic thermoelastic medium having mixed type of thermal and mechanical boundary conditions in the context of strain and temperature rate-dependent thermoelasticity theory (MGL theory). The formulation of boundary integral equations (BIE) plays important role to solve coupled thermoelastic problems numerically, especially when analytical solution of the problem is difficult to obtain. To derive the BIE formulation, firstly some important results on this new thermoelasticity theory are established in this chapter. We obtain the fundamental solutions in the Laplace transform domain when the body is subjected to two different situations: one is for concentrated heat source and another for body force in a particular direction. Accordingly, we obtain a reciprocal relation between the field variables for these two systems of causes. Further, the integral equation formulation of field variables is carried out in terms of boundary conditions by using this reciprocity relation. Lastly, we illustrate the implementation of our BIE formulation and discuss the aspects of numerical implementation through boundary element method.

The goal of **Chapter 6** is to investigate the dynamics of the solutions for another new thermoelasticity theory, namely the QMGT (2019) theory for linear, isotropic, elastic, and unbounded medium due to the presence of continuous line heat source. In this chapter, we enumerate the points of discontinuity, velocity of thermal and elastic disturbances and analyse the dependency of field variables on various physical parameters under QMGT theory. The finite speed predictions for elastic as well as thermal disturbances by the present thermoelastic model is identified analytically. Furthermore, the damping nature of elastic and thermal waves are also shown analytically. The results of present case are compared with the corresponding results of other thermoelastic models (LS, GL, MGL and GN). We observe that in QMGT model, displacement field exhibits continuity. However, temperature and stress components suffer infinite singularity at the position of heat source and discontinuity at both the wavefronts (thermal and elastic). The analytical results are verified with the numerical results of the present problem for a suitable material and it has been shown with graphical results that the field variables have no effect after thermal wavefront, which verifies the correctness of our analytical results and reveals that QMGT model show finite domain of influence of the disturbance at any particular time. Similarity and dissimilarity in prediction of this model with other existing models (LS, GL, MGL and GN) are highlighted in detail.

Chapter 7 covers a comprehensive analysis of the current study by highlighting the significant points of investigation. Further, it concludes with an exploration of possibilities for future research within the relevant disciplines.